Structural Dynamics
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Lecture - 21
Equations of Motion for Multi Degree of Freedom Structures

Hello, the last lectures that we discussed have all been on single degree of freedom structure, whether they were real structures; for example, the one story one way frame that we looked at was the single degree of freedom with rigid beam of course. And then we looked at generalize single degree of freedom, where if it was rigid body assemblage, it was a true single degree of freedom system. And if it was a flexible, we saw that by considering only one unknown and defining shape functions, we could define that as generalize single degree of freedom systems. From today, what I am going to be starting of is by looking at systems which cannot be classified a single degree of freedom systems. And therefore these are known, what are known as multi degree of freedom systems.

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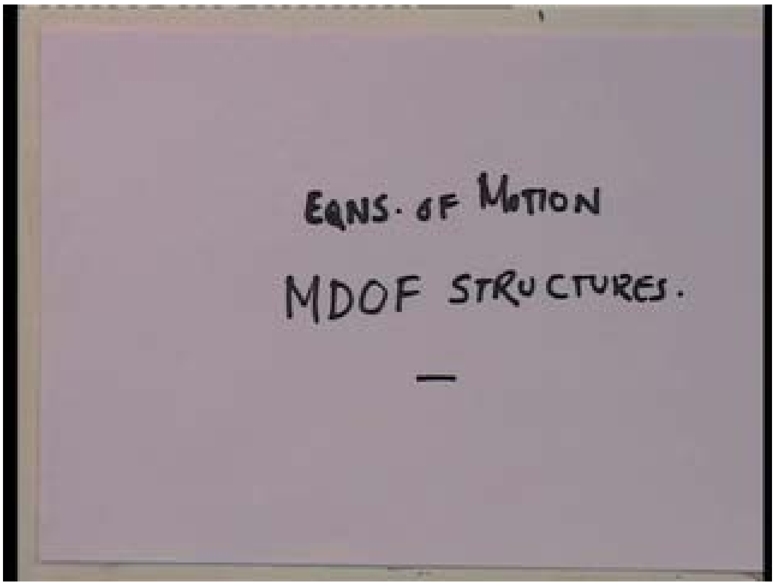

So, what I am going to be looking at from today onwards for the next few lectures is going to be and today especially, we are going to looking at equations of motion for multi degree of freedom structures. So, let us start looking at one of those because as you see that, what is a multi degree of freedom. Let us first define that, before we start looking at equations of equilibrium etcetera, and so let us look at, what multi degree of freedom system looks like, let me take this problem.

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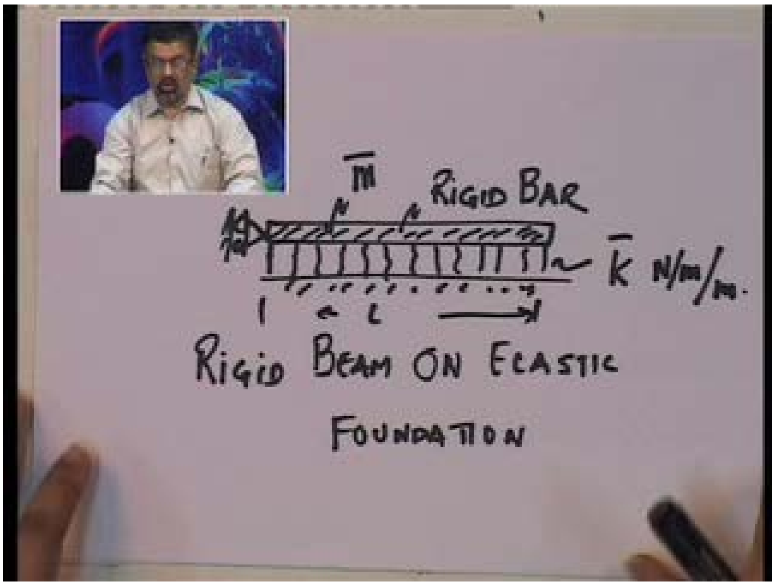

So, what we have is a rigid bar, which cannot move in this direction, it is restrained, so this is a rigid bar and what I am going to do is, this one only ensures that this one does not move in this direction. The basic things that we looking at are here and let me take a situation, where I have... So, this is a rigid beam on elastic foundation and let say that, the sub grade modulus is given as K Newton per meter, that is like the sub grade modulus.

So, this is like a spring constant per meter, so this is spring constant in a per meter sense, we also look at this having m bar, which is mass per unit length. So, this is the stiffness per unit length and this is the mass per unit length. So, this is the classical rigid beam on elastic foundation problem and let us just assume, that the length of this bar is L.

So now, let us see how this problem needs to be solved, so having defined the problem, I am going to again now look at this bar. And now the only thing is that, since we know that this bar is not going to move in this direction, I am not putting that thing, I am just saying it is restrained from moving in this direction, it is only allowed to move in this direction. If you look at this bar, now it cannot move in this direction, so it can only move in this plane.

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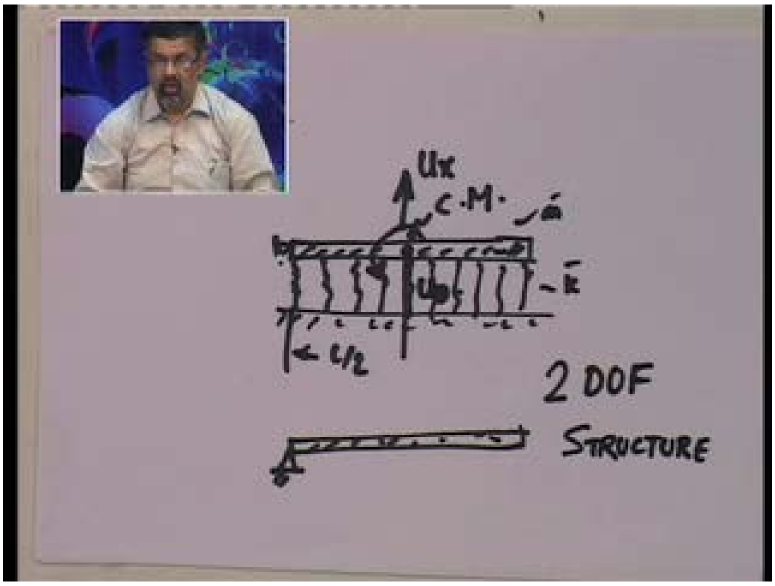

When we look at this, what are the possible degrees of freedom of this, now it is obvious that, this motion is a degree of freedom what else. Actually we look at it, this bar can rotate also I mean, if you look at it, it could move like this, it could move like this. So, if it moves like this then what happens, essentially means that, this beam has two degrees of freedom.

Note that, this motion is not allowed, remember that this is not allowed, so if we look at it, I can define this structure as a two degree of freedom structure by saying that, look I am going to take, see this is the rigid beam with uniformly distributed mass, the mass m bar is a constant. So, there the center of mass of this bar will be where it will be there at the center, which is L by 2. So, let me take the center of mass of the bar and let me say that, this is the center of mass.

So, I am going to define two degrees of freedom at the center of mass u x and u theta so now, how is this different from the problem that we have looked at last time. If we looked at last time, we are seeing that, the bar was like this, remember the previous problem that we looked at rigid bar. So, it could only rotate about this point now, since this is allowed to move up and down, it not only rotate, it can also move up and down. So therefore, there are two degrees of freedom for this structure, so if we have two degrees of freedom, so I have k star and m star and I have defined my degrees of freedom at the center of mass.

Now, there are various ways, as per as we know that there are two unknown displacement quantities, that can define the motion of this bar, we call it two degree of freedom structure. So now, once we have this two degree of freedom what happens, let us see how we can solve this particular problem. If we look at this, what you have over here is, let us look at it, so now I am going just do this as a line.

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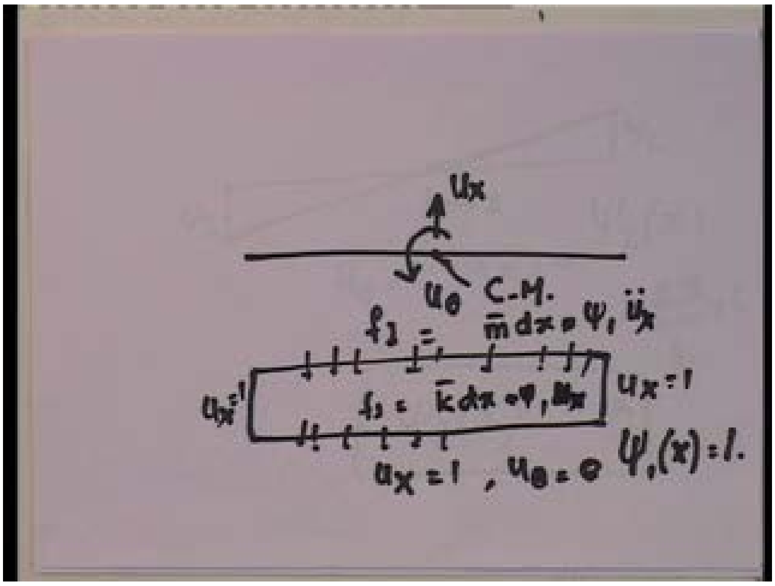

So, I have u x and u theta defined at the center of mass, why is the geometric center of mass, the mass is uniformly distributed. So obviously, the center of mass is the geometric center, so these are my two degrees of freedom. So now, if this two degrees of freedom what do I do, I look at what displacements do I get for each degree of freedom. For u x, if u x is equal to 1 and u theta equal 0 then what do we have. Since there is no rotation it just means that, you have a u x rigid body displacement, at every point this is u x so, that is u x equal to 1 and u theta equal to 0.

Let us look at the second degree of freedom, what happens when we put the second degree of freedom, what we get is the following, u theta equal to 1, u x equal to 0 that means, the center cannot go up and down, it can only rotate. So, when it rotates, remember small displacements, so this if u theta is equal to 1, this is L by 2 and this is L by 2 here so, that becomes my two. So, if I look at this, this is equal to 1, this is equal to 1 and therefore, if I look at zi 1 x is equal to 1 and what is zi 2 x, this shape is zi 2 x.

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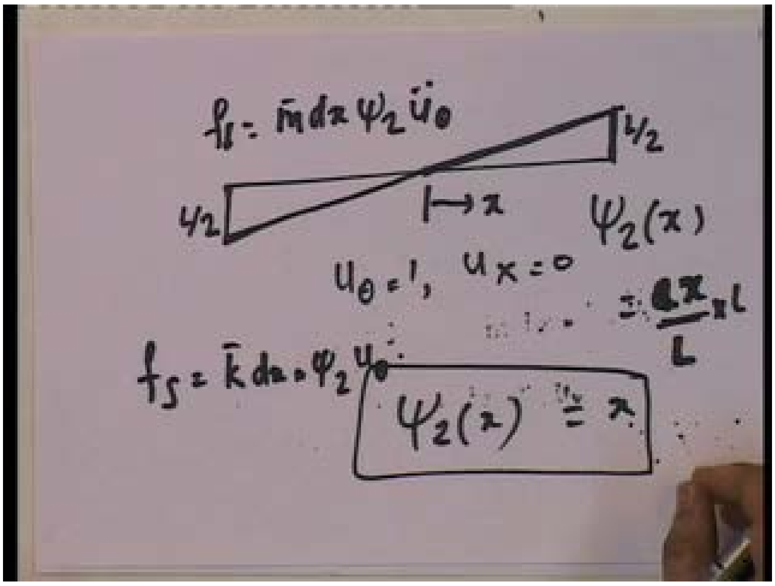

If we look at zi 2 x, I do not need to go through the what zi 2 x is but if I take x from here then zi 2 is equal to 2 x square. It is u theta equal to 1, so it will be x into L into L so, this is zi 2 x is equal to x, so these are my this. So now, let us look at this, now let us look at a situation, so how do I solve this problem. Remember, how I solve this problem, earlier what did we say, given this displacement, I find out the forces acting on the structure.

So, given this displacement, what are the forces acting on the structure, I have one which is, one is the f I which in this particular case is equal to m bar d x. F I is inertial fort per unit length and that is equal to m bar d x, that is equal to this into zi 1 into what, u x double dot, that is my f I. Similarly, I have my f s and that is equal to k bar d x into zi 1 into u x. So, those are the forces here and what are the forces due to this displacement, similarly f I is equal to m bar d x into zi 2 into u theta double dot and f s is equal to k bar d x into zi 2 into u theta.

So, this is the forces in this particular one and these are the forces in that one so now, what I want to do is, I want to find out… Now understand that, remember this that, how will this equation look like. This equation I mean, let me first write down what an equation looks like and the equation looks like this.

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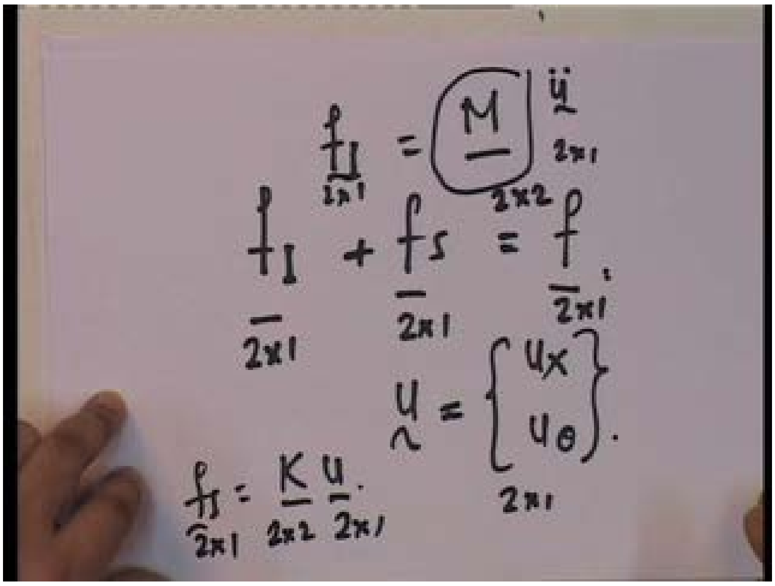

The vector of f I plus the vector of f s is equal to the vector of loads, that is what we have, f I plus f s is equal to f. And if I really look at this and what are the degrees of freedom, the degrees of freedom in this particular case u is equal to u x and u theta. So now, if u is equal to u x u theta, what is f I equal to, by taking analogy f I is equal to, look at this units, this is going to be equal to 2 by 1, this is going to be 2 by 1, for this two degree of freedom this is going to be a 2 by 1, this is a 2 by 1, so this is a 2 by 1 and here I have the u which is 2 by 1.

So, relate inertial forces to the acceleration vector, this is going to be the mass and that is going to be a 2 by 2. Similarly, for f s is going to be equal to k into u where, k if since this is also 2 by 1, this is a 2 by 1, this is a 2 by 2. So, essentially, remember in the generalize single degree of freedom, we have to find out m star k star and p star.

Here, what we are doing is, we have to find out M matrix, K matrix and the P vector, P vector is the load vector, and so these are the things that you have to find out in this particular case. So, if you really look at it how will this equation look, now this is what we need to find out. And note that, these are not property matrices, this is mass matrix, this is stiffness matrix, this is load vector and none of all of these correspond to u vector. I will explain this to you corresponding to this particular problem, that we are solving over here.

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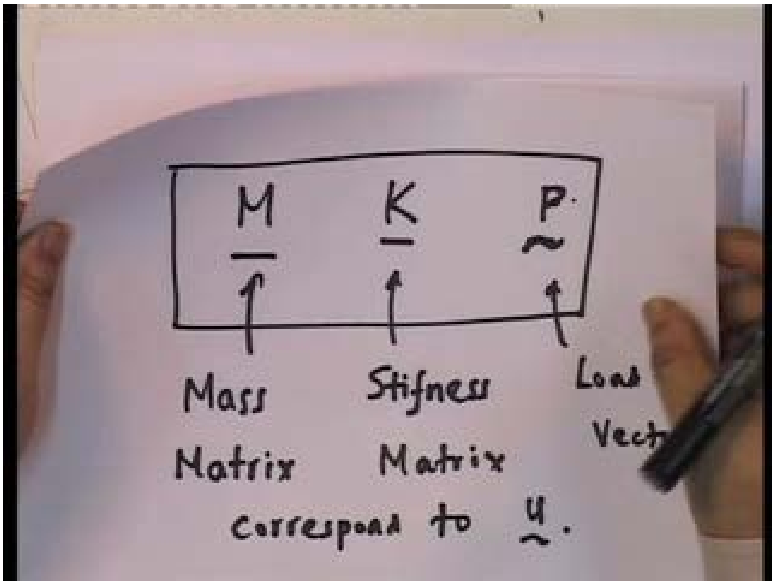

So, look back at this and that is, that if you really look at this particular problem, I have m f s, f I, here I have f I 1, f s 1 and this is f I 2, f s 2. This f I 1 is due to, if you look at this f I 1, this is due to the u x displacement and your f s 1 is also due to the u x displacement and f I 2 and f s 2 are due to the u theta displacement. So, if you really look at this, this is going to be equal to u L, so having done that, now how do I solve this particular problem. So therefore, if you really look at it, if I give it any arbitrary displacement, what is f I equal to, f I is going to be equal to sum of the both.

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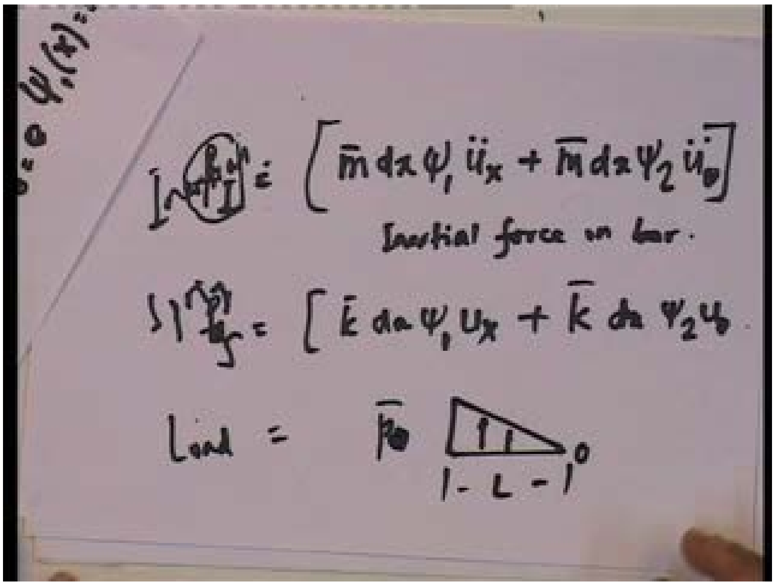

So, f I is going to be equal to I mean, this is not f I, this is the mass per unit I mean, inertial force per unit length, that is going to be equal to m bar d x y 1 u x dot plus m bar d x into zi 2 u 2 theta. So, this is the mass, this is the inertial force on bar similarly, so this is not your… this is like the inertial force on the bar. Similarly, the resistant force is going to be equal to k bar d x zi 1 u x plus k bar d x zi 2 u theta, so that is my f s. So, over the infinitesimal element, these are the forces, the inertial force and the spring force, over head infinitesimal length.

So now, what we are trying to find out, we are trying to find out and let us put some load on it, otherwise what do we do, let me put some load on it and I will say that, look I am going to put a triangular load where, this is going to be p bar 0. So, this is the intensity at this point and of course, at this point it is 0. So, it is a triangular loading and if we have triangular loading, so therefore, we also have that load on that and on top of that, we have load which is given by how, it is given by let us look at it.

So, it is going to be p bar, if x is defined from the center, so load will be, let me not de label the point, the load will be in this fashion, 0 over the length L and p naught at the end. So, these are all the loads that you have on the structure, so now, what are we trying to find out, remember that we are trying to find out, if you look at this the old one that I had given. If you look at it, these that you have are actually corresponding to the degrees of freedom u.

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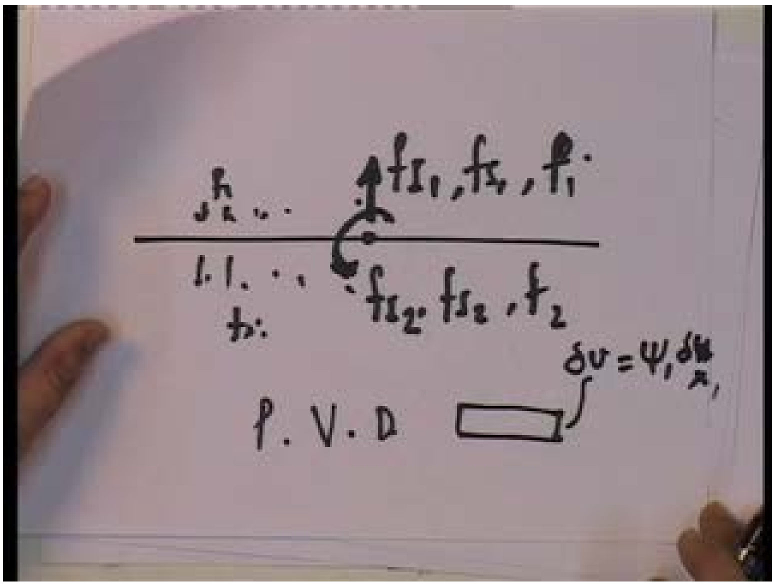

So, these are the equal inertial forces corresponding, so from all these forces, I have to find out the equivalent inertial force, so equivalent spring force and the equivalent this thing, defined at which point.While corresponding the degrees of freedom means, I want to find out, what I want to find out is this. I want to find out that, given this bar, given all this loads I have the inertial forces, I have the spring forces. I have the load, I want to find out what is f I 1, f I 2 and similarly f s 1, f s 2 and the load f, I want to find out f 1, f 2. So, these are what I want to find out corresponding to; so these are the force directions I given in this fashion.

So now, if I want to find out the equivalent force suppose, I want to find out f 1, how do I find out f I 1. Remember that, to find out a real force and using the principles of virtual displacement what do I do, corresponding to that force I give a virtual displacement. If I give this, what is the virtual displacement, the virtual displacement is equal to, at any point the virtual displacement is given by delta v is equal to zi 1 delta u x, if you look at it, delta u x. So therefore, what happens is, if I want to find out f I 1 that is, the inertial force only, so I have to only take the inertial forces. F I 1 becomes the following, f I 1 becomes f I 1 into delta u x, that is the work done by the equivalent inertial force.

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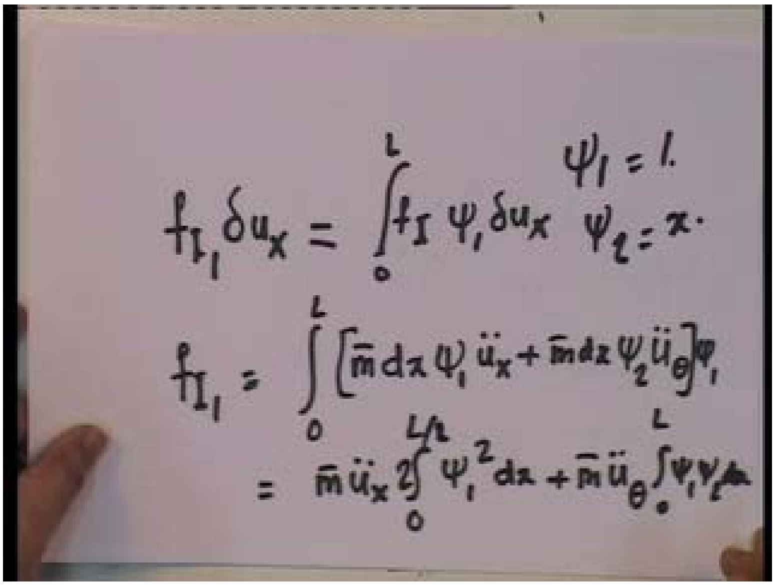

And this has to be equal to, let me look at the f I now, f I which is the mass, the inertial force per unit length into zi 1 delta u x and integrated over the whole length. So, this is what this is equal to so therefore, delta x exists on both sides. So, f I 1 is equal to, let me substitute f I, what was f I 1, it was equal to m bar d x zi 1 into u double dot x plus m bar d x zi 2 u naught. So, this is my f 1, all into zi 1 into that is all, zi 1 because del u x del u x have been and this entire thing is to be integrated from 0 to L.

So, if I have put this together what does this become, if you look at f I 1 this becomes the following. If you look at this, this becomes m bar I can take outside and u, this also I can take out, so this becomes m bar u double dot 0 to L zi 1 squared d x plus m bar u theta, 0 to L zi 1 zi 2 d x. You see this, this just comes out from there, and so if I rewrite this and what was zi 1. Remember, zi 1 was equal to 1 and xi 2 was equal to x, 0 to L by 2 and then 2 so this basically would become 2 L by 2.

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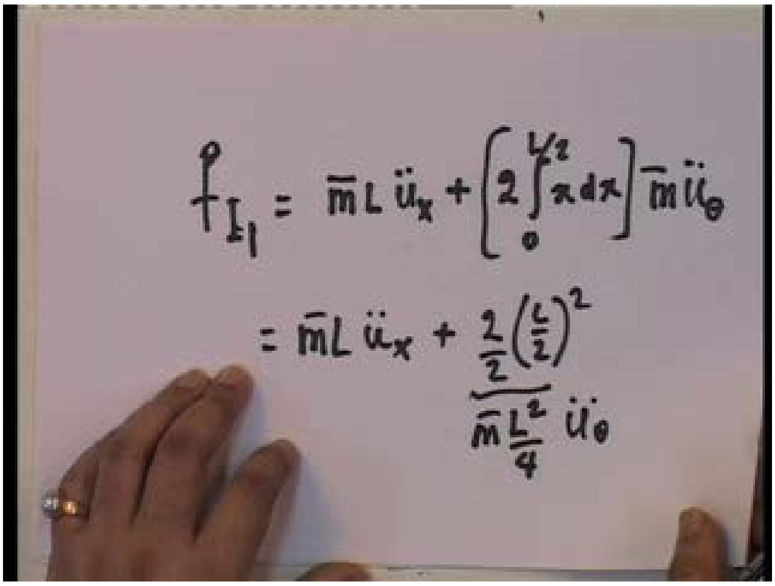

And so, if I do that what happens, I get the following, f I 1 is equal to m bar, so I have a zi, zi 1 squared is essentially what, I got it equal to zi 1 was equal to m bar L into u x double dot and then plus now I have over here 1 into x and 1 into minus x. So, I have both 0 to L, so I can do it two times 0 to L x d x, so this becomes x squared by 2. So, this basically becomes x d x twice 0 to L by 2 into m bar u theta double dot. So, this is equal to m bar L u x plus if you look at, this becomes x squared by 2, so this becomes essentially 2 into L by 2 square divided by 2.

So, this basically becomes L squared by 4, so this thing becomes L square by 4 x squared upon 2, so that is the thing and that is into, so it becomes m bar L u theta. This is my f I 1 similarly, I can find out f I 2, for f I 2 how do I do, I give it a initial displacement. So, I want to find out, let us go back here, I want to find out f I 2, so what do I do for finding out f I 2, I give it which is delta v is equal to zi 2 in del 2 delta u theta, that is del be.

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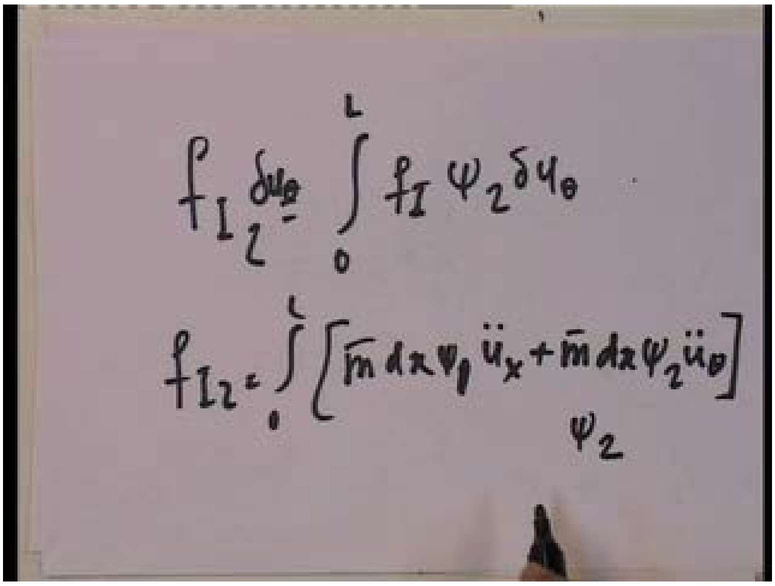

If I go through that procedure, I am going through the entire thing, what does it become, let us look at it, I get f I 2 is equal to integration 0 to L I mean, I am just denoting 0 to L as over the length is going to be del u theta. This is going to be equal to f I zi 2 delta u theta, so if you look at this, f I 2 is equal to again nominally given in that way, m bar d x zi 1 u x plus m bar d x zi 2 u double dot theta into zi 2, so that is my f I 2.

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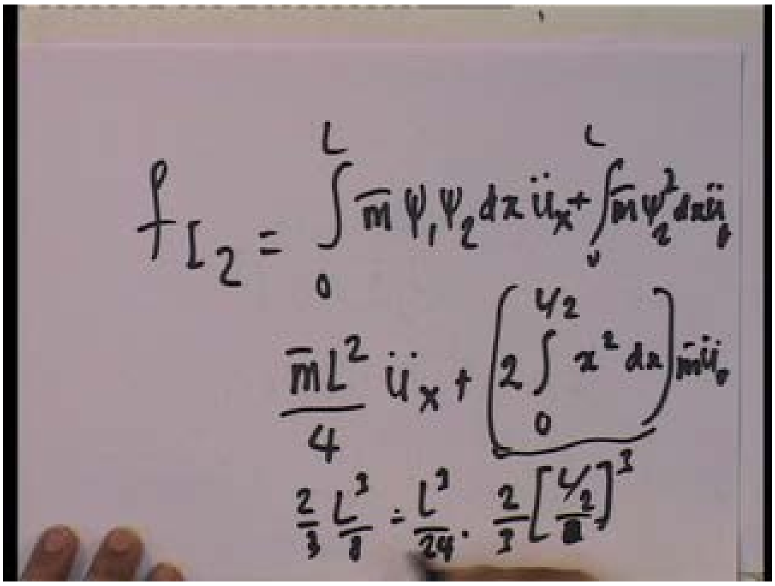

So, again what we have is that, this becomes equal to the following, it becomes f I 2 is equal to 0 to L m bar zi 1 zi 2 d x into u x double dot plus, 0 to L m bar zi 2 squared d x u double dot theta. So, if I look at this, this essentially becomes the same as m bar zi 1 zi 2, so this is m bar L square by 4 into u theta x. And on this one, you have 2 times 0 to L by 2, now zi squared is x squared d x and of course, you have m bar into u theta double dot here.

So, what is this, this is equal to 2 times into L by 2 cubed upon x cube by 3, so this becomes 2 by 3 cube L by 2. Then if I have, this becomes equal to two thirds of L cubed by 8, so this becomes L cube by 24, is that correct. This is L by 2, so 2 by 3, so 2 into 8 12, so this becomes L cube by 12.

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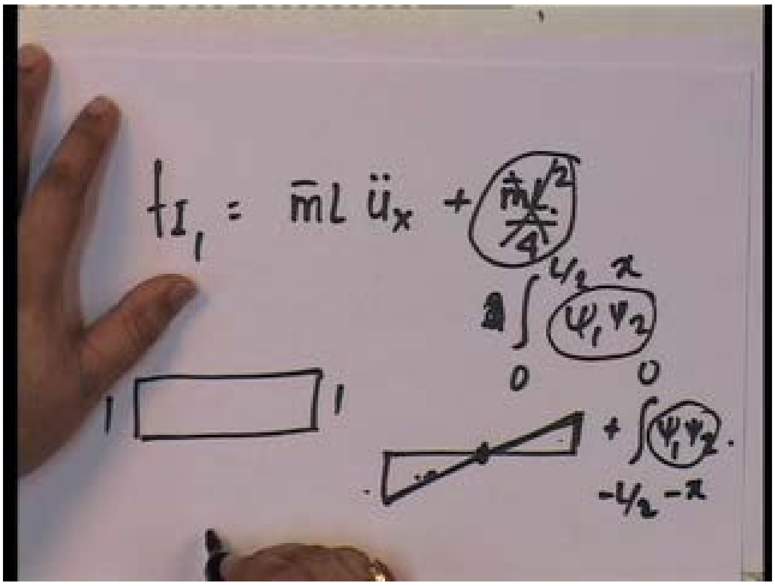

So, ultimately what does my final thing look like, f I 1 looks like it is equal to, f I 1 is equal to m bar L into u x double dot plus m bar L square by 4 made a mistake here, it is something that I did not point out. This one is wrong, we say that this was equal to 2 times 0 to L by 2 zi 1 zi 2, it is not correct if you look at zi 1, zi 1 is a constant 1 over the whole length. And if you look at it, zi is in this fashion, it is plus on this side and its minus on this side.

So, actually it is not twice, it is 0 to L by 2 zi 1 zi 2 plus, minus L by 2 to 0 of zi 1 zi 2, if you really look at it, when you go from minus L by 2, so this becomes x. So, if you look at it, this entire thing essentially becomes x and minus x that was a mistake I made, that it is really minus L by 2 to 0. So, if I really look at it, this one zi 1, this one is equal to x and this one is equal to minus x. So, if you really look at it, the whole thing becomes squared there is no doubt about it, this is minus x, x square upon 2. So, when you do this, this instead of being m bar L upon square, so let me redo the thing I have done it incorrectly. So, let me just redo this all, we do this particular problem.

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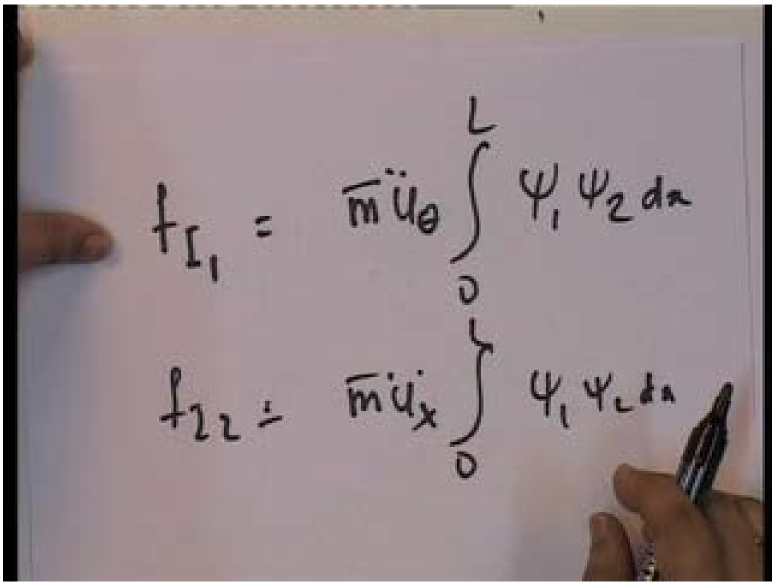

So, let us go back to f I 1 I mean, forget about this, there was just problem where we were doing m bar u theta in the first one in f I 1, which had 0 to L, zi 1 zi 2 d x and in f I 2, we had m bar u 0 to L zi 1 zi 2 d x. So, this was the term that we are looking at, this over L what is it, so if I redo this particular one zi 1 zi 2 d x, this is the one that I am looking at over the whole length. If I look at this you will see that, this is equal to the following, you see here this is x.

When x is been taken from here to here and here to here, so if I take 0 to L, this notional 0 to x zi 1 zi 2 d x becomes the following. The first one going from minus L by 2 to 0, because x goes from here to here, x starts from here. So, this is 0 to L by 2 x d x plus, 0 to L by 2 x d x if I look at this, this is equal to and this is where, the problem of trying to double becomes the problem, this is the actual problem. If you really look at it, this becomes x squared by 2 minus L 0 plus x square by 2 0 to L by 2 if I look at this, this becomes 0 minus this. So, this becomes then L squared upon 8, but minus, because this minus this, so this become minus and this one over here is L squared upon 8, so this one is 0.

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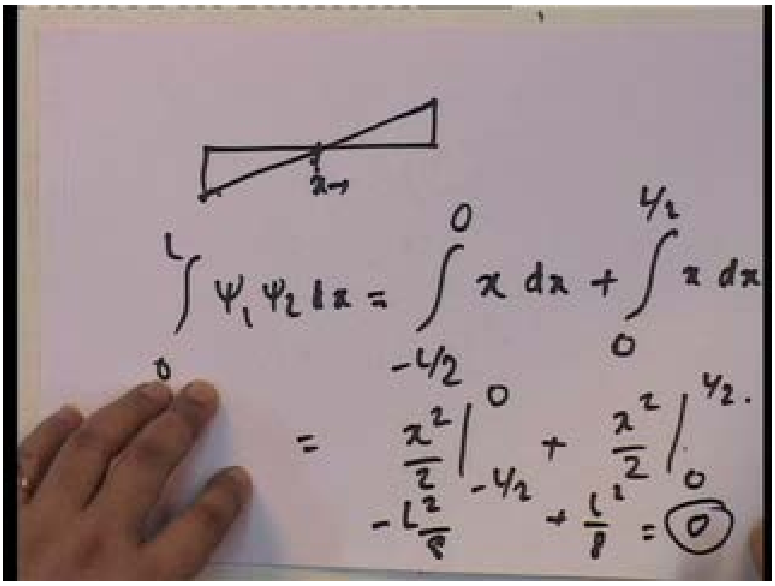

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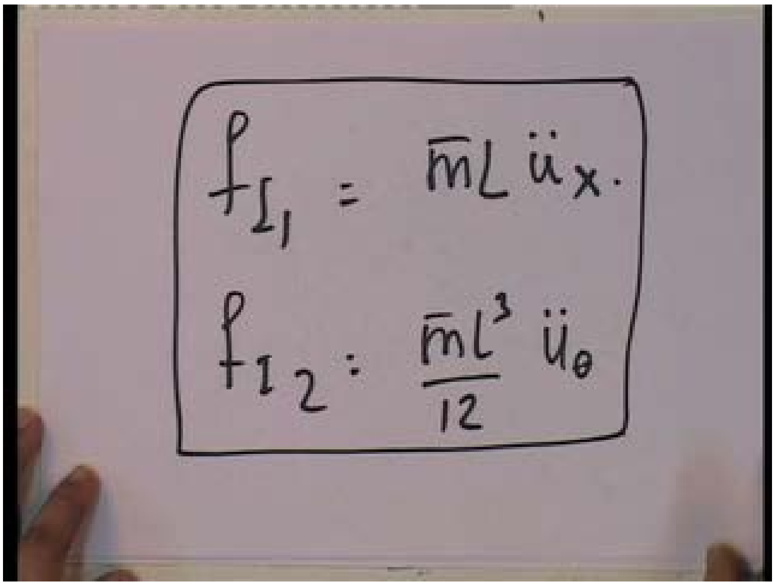

So, actually if you look at it, so that is the reason why, if I rewrite f I 1 and f I 2 what we get are the following. F I 1 is equal to m bar L into u x plus 0 u double dot theta and f I 2 is m bar L cube by 12 u theta, because again the u x 1 over here is 0, so that is f I 1 plus and f I 2. So now, let us look at f s, what was f s equal to, the f s if you really look at it was equal to…

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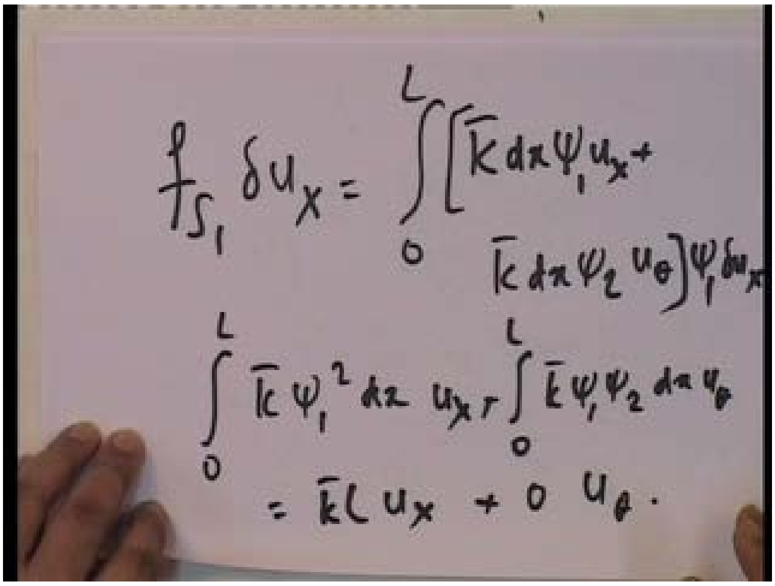

So now, If I have to put in the same way, f s 1 into del u x is equal to integrated over the whole length of what, of k bar into d x into zi 1 u x plus k bar d x zi 2 into u theta. So, these are the, this is the total force and this infinitesimal into del u, which is equal to zi 1 del u x, because that is del v, this is the particular infinitesimal element. So, again the problem then becomes the following, 0 to L k bar zi 1 squared d x into u x plus, 0 to L k bar zi 1 zi 2 d x into u theta. And since we have already derived these what do I get, this one turns out to be k bar L u x plus 0 into u theta, that is f s 1.

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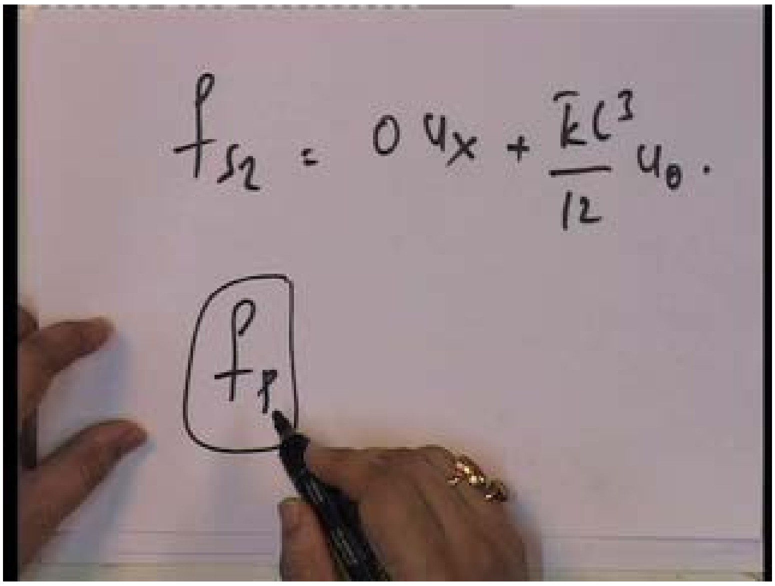

In similar fashion, we will get that f s 2 in similar, I do not want to go through the process, you get it equal to 0 into x plus k bar L cubed upon 12 into u theta. Now finally, what do you want to do, we want to put down the, what happens to f p, now I want to find out f p 1.

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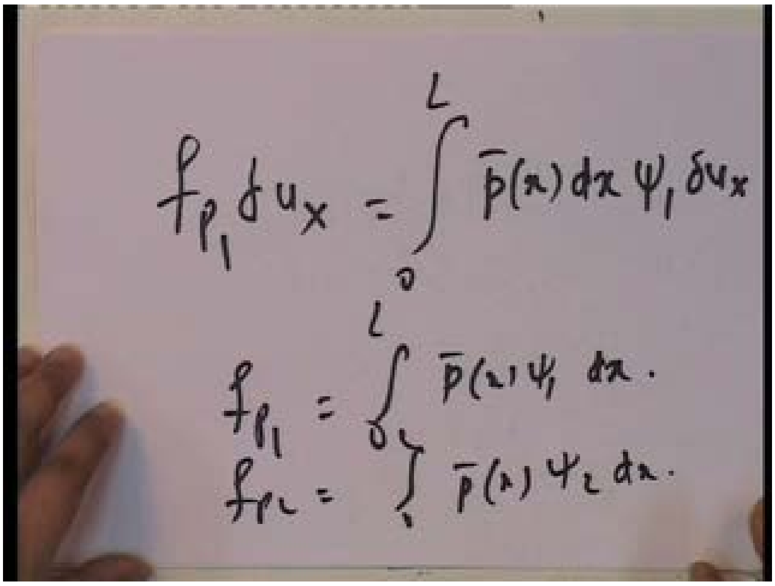

So, for f p 1 what do I do, I put for f p 1 into del u x is equal to now, I have p bar x, that is the intensity at a particular point x d x, that gives me the load into zi 1 del u x and this integrated over the whole length, that is what my f p 1. So, f p 1 is equal to over the whole length p bar x into zi 1 into d x and similarly, we will get f p 2 is equal to 0 p bar zi 2 d x. So, these are our equivalent loads, so how do I get this for this triangular load, remember the load that I had was given in this format, it was given in the load was like this.

So, if I look at this with my x starting from here what do I have, this is p bar 0, so this is p bar 0 by 2. So, what I have over here is, p bar x actually is equal to let us look at this, if this is 0, it is equal to p bar 0 1 minus x over L. This is my p bar why, let us see when x equal to ((Refer Time: 45:22)) 2 x by L, x equal to L over 2, p bar x is equal to put x equal to L by 2 1 minus 1 p bar x is 0. If p bar x is minus L over 2, put minus L over 2 over here you get 1 plus 1, it is equal to p bar 0. So, this indeed is my p bar x so now, zi 1 is equal to 1 and zi 2 is equal to x.

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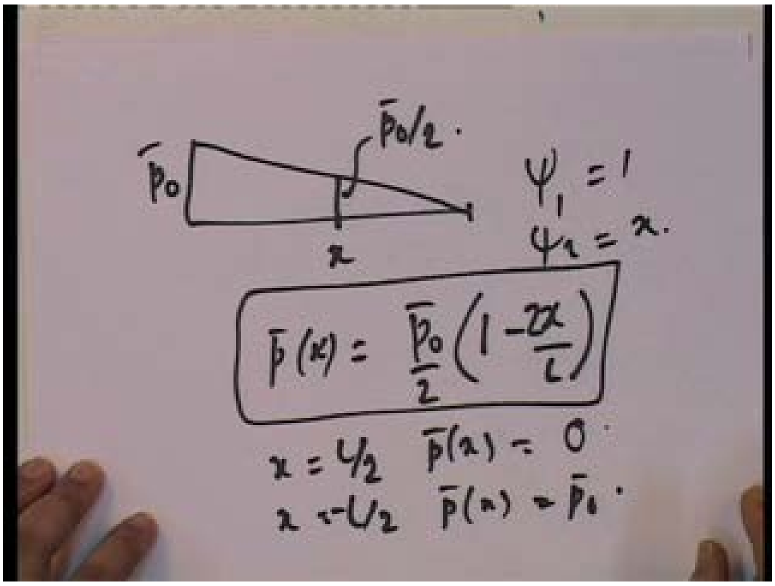

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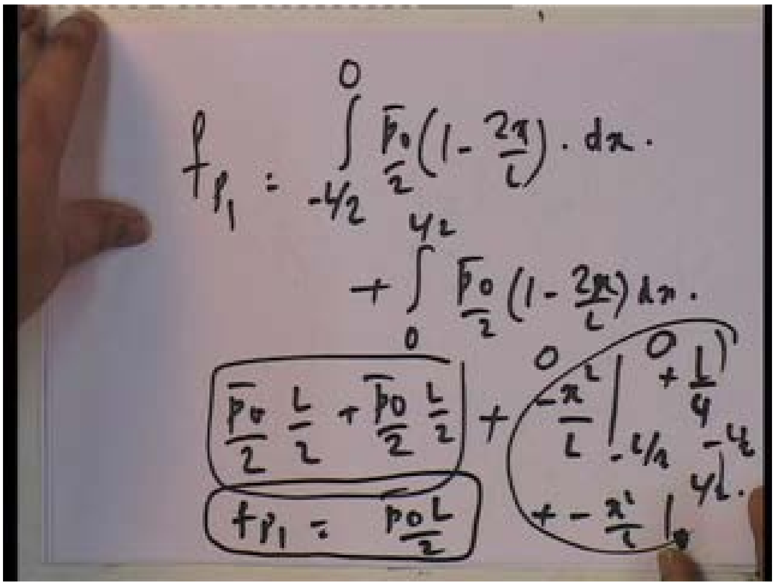

So, if I do this, what is my f p 1 equal to, my f p 1 is equal to integral, I will do it formally, 0 to this p naught upon 2 into 1 minus 2 x L into 1 into d x plus, 0 to L over 2 1 minus 2 x upon L d x. So, if I look at it what do I get, I get over here this becomes p naught by 2, that is the first one. So, this becomes x, so this becomes minus L by 2, so this becomes L by 2, because 0 minus of minus L by 2, so this becomes this. From this one and from this one, it becomes p naught upon 2 into L by 2.

So, that is the first one then the next one comes out from here, so this is the minus 2 x by L, so if I do minus 2 x by L, when I integrate that what do I get, plus so, I will do 2 x by L minus, so this becomes minus x squared upon L 2 x. So, it becomes x squared, so the first one is minus to 0 and plus minus x by L, 0 to L by 2. So, if I look at this one what do I get, this one turns out to be 0 minus, so this becomes plus, so it becomes plus L by 4 this one and this one becomes L squared, so this becomes minus L by 4. So, this whole thing becomes 0, so if I look at f p 1, my f p 1 turns out to be equal to p naught L by 2, that is my f p 1. What is my f p 2, f p 2 will become nothing but let us put it down now.

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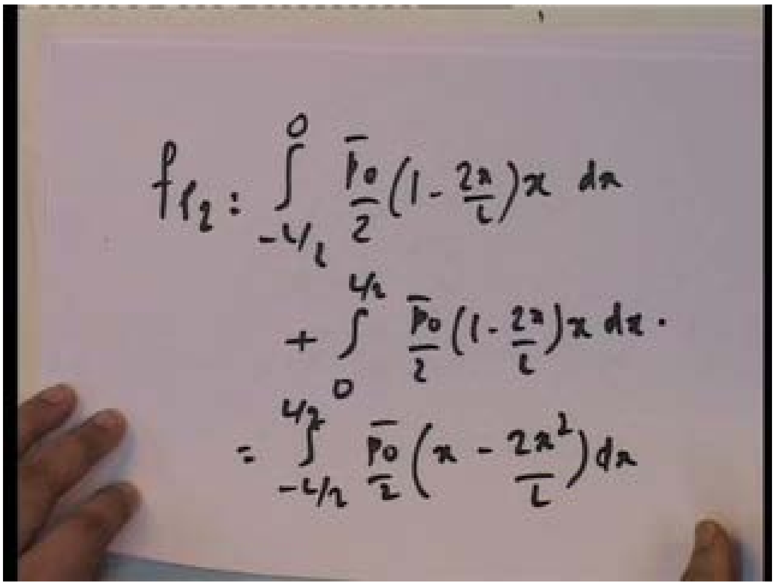

So, f p 2 is going to be equal to minus L upon 2 to 0 p bar 2 1 minus 2 x into x d x plus, 0 to L by 2 1 minus 2 x upon L x d x. So, if you look at this, since these two are identical what I am really saying is, minus L upon 2 to L upon 2 of p naught 2 x minus 2 x squared upon L d x. Let us go through this integration and if you get this integration, what you get is the following.

F p 2 is equal to, so this becomes then so this is my limits and the inside one becomes p bar upon 2 into x squared upon 2 minus the p naught bar, 2 and 2 cancelled, so this becomes x cubed by 3 L and this is the integration. So, if you look at this, so this is minus L and this becomes 0 and this is the only one that is left. So, this goes minus p naught, so this becomes L by 2, 3 by L and then plus p naught minus L upon 2 cubed upon 3 L. So, if you look at this, this becomes L cubed by 8 and this is again minus L cube by 8, so L cube by 24. So, this becomes then minus p naught L squared, because the L L cubed cancels, upon this is 8 and 8, 24. So, this becomes 12, so that is my f p 2, so if I rewrite this…

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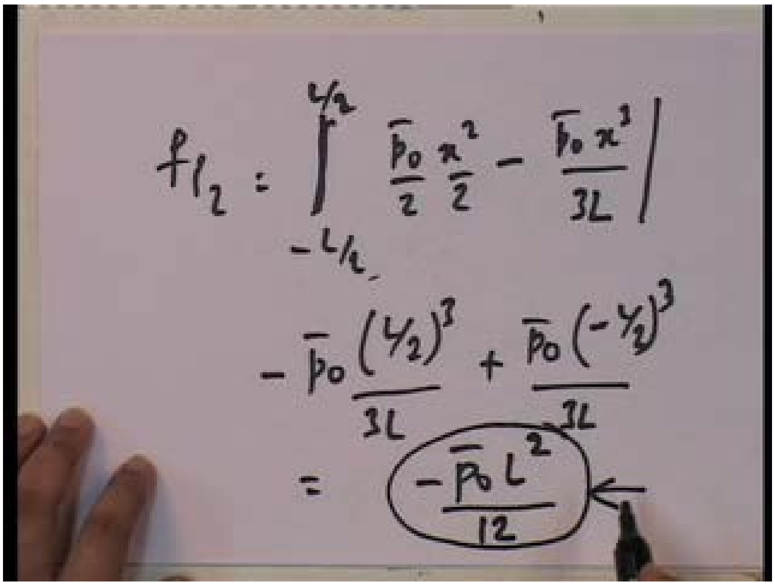

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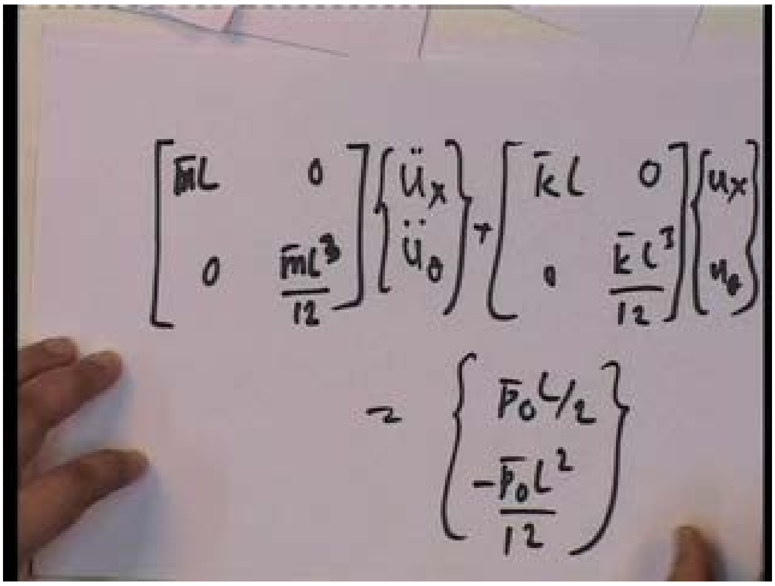

So therefore, the equations of motion for this particular one then becomes what, it becomes the following. I will rewrite this now, it comes out as m bar L 0 0 m bar L cubed by 12, u x double dot u theta double dot plus k bar L 0 0 k bar L cubed by 12, u x u theta is equal to, what did we get f p 1 equal to, f p 1 was equal to p naught L over 2 and this is equal to p naught L squared by 12. So, if you look at this, this actually if you look at this, these are uncoupled equations and what the two equations become are the following.

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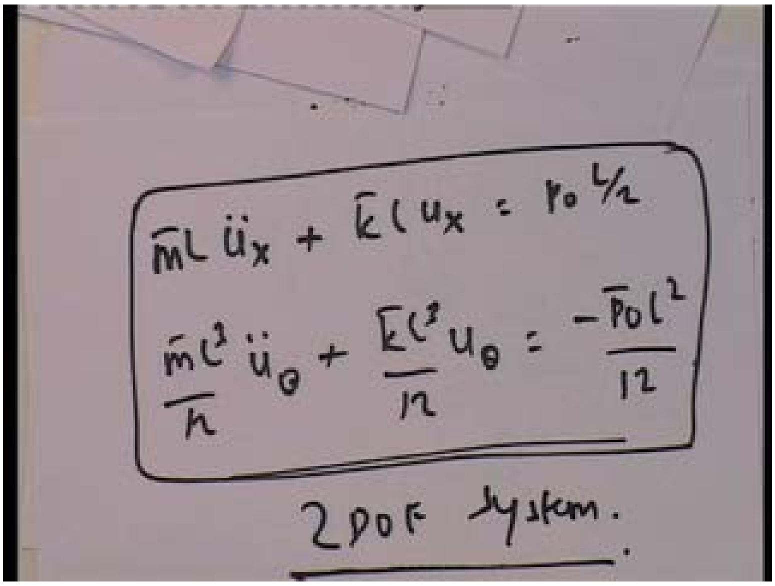

The equation become this, m bar L into u x double dot plus k bar L into u x is equal to p naught L and the other one becomes m L cubed u theta double dot plus k bar L cubed by 12, u theta is equal to minus p bar L squared upon 12. So, you see it is a multi degree of freedom, but because of the special nature of, how I have defined my mass and stiffness, it is so happens that, this entire equation becomes uncoupled, but it still two degree of freedom system. So, this is how, we find out the equations of motion of a multi degree of freedom system.

Of course, I have illustrated this by taking a specific example, now in the next class, I am going to continue with this and I will show you, how we can derive certain important things. And then we will move on to looking at, still going to look at a rigid bar, but after that, we will move on to flexile bars and see, how we can get those kind of equations.

Thank you very much, bye.