**Structural Dynamics**
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**Lecture - 20**
**Generalized Single Degree of Freedom System Equation of Motion and Free Vibrations**

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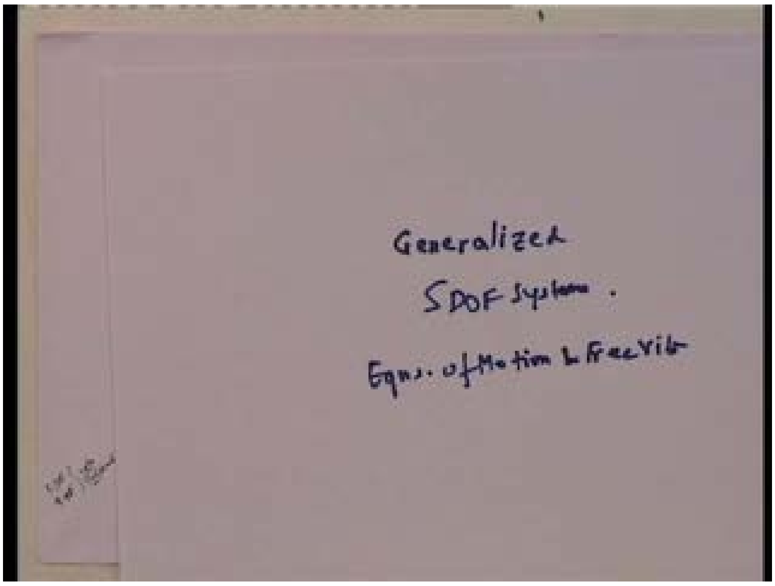


Hello again, we continuing look at Generalized Single Degree of Freedom Systems, both. Now, we are looking at equations of motion and free vibration, and let us just look back at the problem that we were doing in the last lecture. And that was with z 0 as the degree of freedom, and of course with an assume shape psi x and you know towards the end of the lecture, I was actually solving the problem with psi x is equal to cosine pi x upon 2 L. This was the problem that we were solving and of course because without saying that this is equal to 2 L upon pi, and here we get 2 L upon pi, we are differentiating it, so this is pi upon 2 L, the whole squared cosine pi x upon 2 L.

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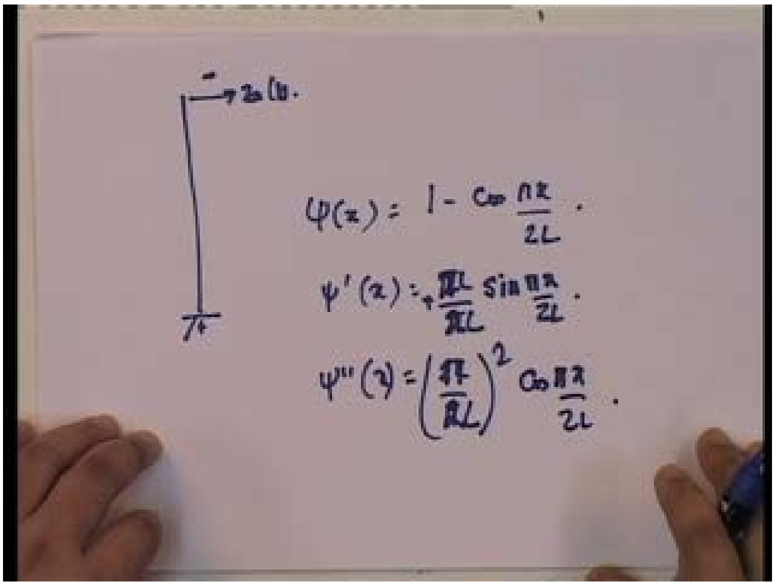

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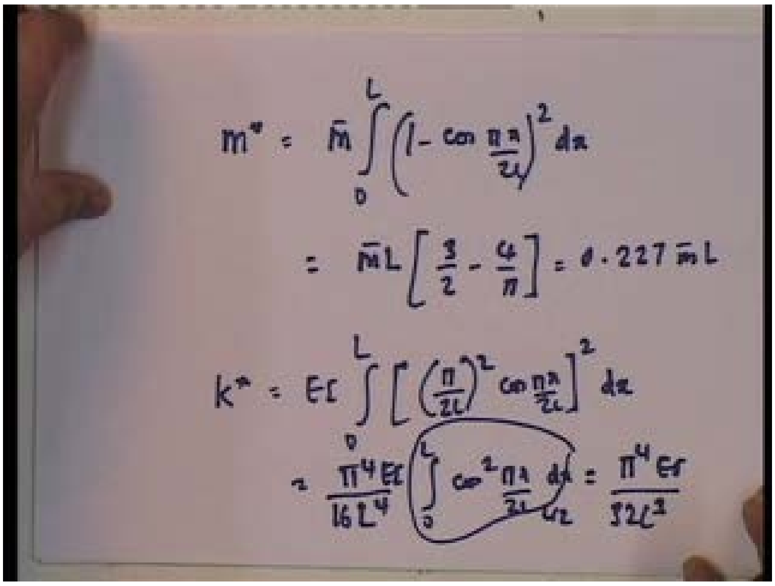

And so with that we saw that, if you took the m star, m star was equal to m bar into 0 upon L 1 minus cosine pi x upon 2 L the whole squared and we saw that, this was equal to m bar L 3 by 2 upon 4 by pi and these is equal to 0.227 m bar L. So, we already gone this, now k star was equal to E I 0 to L, now this is equal to pi upon 2 L the whole square cosine pi x upon 2 L the whole squared d x. So, if I take this the whole these become pi 4th by 4 squared, so there is 16 E I and L squared of 4th 0 to L cosine squared pi x d x. And this is equal to we saw was that these was equal to this term becomes L upon 2, you already seen that, so this becomes pi 4th E I upon 32 L cubed, because this term is L by 2, this we already see from here size, we wanted to do this.

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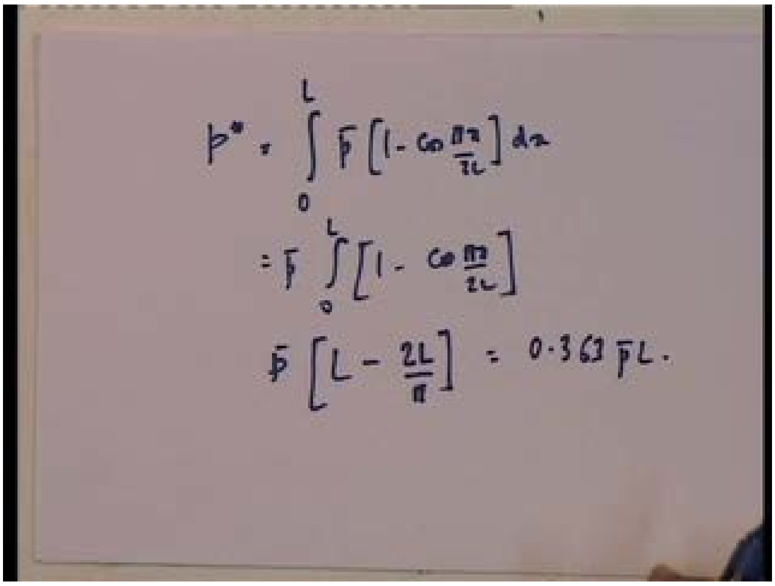


And of course, we saw that p star, which is equal to 0 to L p bar into 1 minus cosine pi x upon 2 L d x, 0 to L 1 minus cosine pi x upon 2 L, this becomes equal to p bar L into 2 L upon pi. So, this becomes equal to 0.363 p bar l, so that is what we get and having that, now let me just a put this together in 1 context, and that is what do we get?

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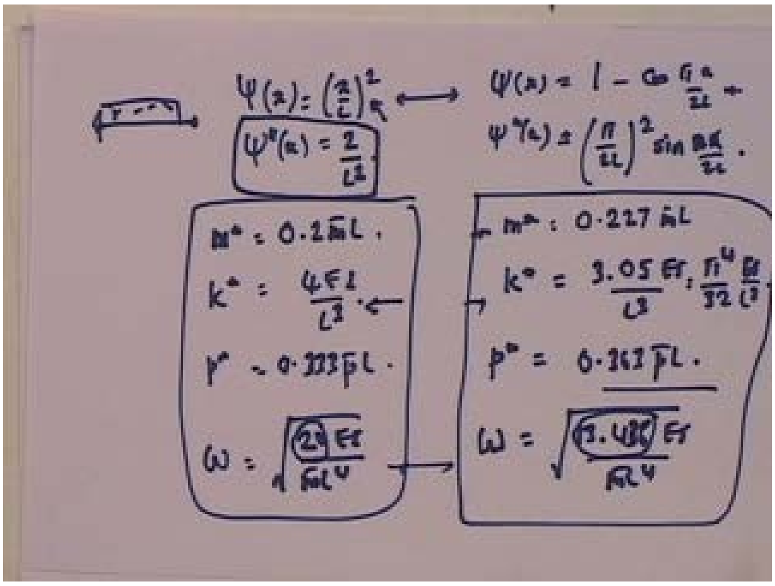

If I take psi x equal to x upon L, this becomes psi prime becomes nothing but 2 x, so psi prime becomes 2 x upon L, so this become 2 upon L squared and psi prime into 1 minus cosine pi x by 2 L psi double prime becomes equal to pi by 2 L the whole squared sin pi x upon 2 l. And for this the m star is equal to 0.2 m bar L, k star is equal to 4 E I by L and p star is equal to 0.333 p bar L. For this m star is equal to 0.227 m bar L, k star is equal to 3.05 E I upon L cubed, because we got pi 4th upon 32 E I upon L cubed. So, pi fourth upon 32 becomes this and p star becomes equal to 0.36363.

So, now, let us look at this particular values, I mean you see what is the difference between this and here, we got omega equals to 20 E I upon m bar L 4th and this omega turns out to be equal to 13.436 E I upon m bar L 4 th, so note this. So, let us look at this and let us look at this very interesting things that we see, if you look at this gives omega equal to 20 E I upon, the forgot E I upon m bar, because E I upon m bar L 4th exist in both.

In this case, this is 20, this is 13.436, this is 0.2, this is 0.227, this is 4, this is 3, this is 0.33 0.363 and why is there such a huge difference well, if you look at this these assumes that, the curve which is a constant. And curve, which a constant implies that, it is a constant moment problem. Now, note that a constant moment is not a very accurate estimation constant curve, which constant moment, because this is the problem.

These is what the load is subject to constant moment would mean, that moment over here and here are going to be the same and in a cantilever with that you obviously, know that is not the case, the moment is 0 at this end and is a certain value at this end, so it cannot be constant. So obviously, this although it satisfies the geometric boundary condition, it violates the basic curvature and the moment relationship over here in this particular case.

So obviously, this is a forced understand, that when we force system to behave in a particular manner, in which it not suppose to a what happens is the k star goes up, because you are restraining your constraining the structure to deflect the particular way. When you constraint it you are k star value goes up significantly, if the k star value goes up significantly omega becomes much larger than, it should be, where if you look at this particular 1, if you look this the psi prime, let us put you know psi over here, psi you know x equal to L, if you put x equal to L, you get what you get sin pi by 2. So, this becomes just this value and if you put x equal to 0 over here, you get this equal to 0, so curve which a is 0 at 1 end, so there is the variation of curve, which and since it in cooperates the variation of curve which a, you see k star, you look at other values m star only 10 percent variation, p star again only about 10 percent variation, k star almost 33 percent variation this is 33 percent higher.

And which is why, if you look at this is almost root 24 point something in this about 3 point something almost 25 percent error in omega bar. So, therefore, this is closer to the real variation and this is of course, very, very construct, I would live it to do it with the other specific the 1 that, I have done and you will see that in that particular case, if you take 3 x upon L squared, you will get that m star becomes about 0.226 m bar L, k star is going to be 3 E I by L squared L cubed. And so therefore, the value of that omega is going to be about 13.4 very close to this particular value that you get, that is the beauty of this method. Now, the question then becomes is that well how do we get an appropriate psi x.

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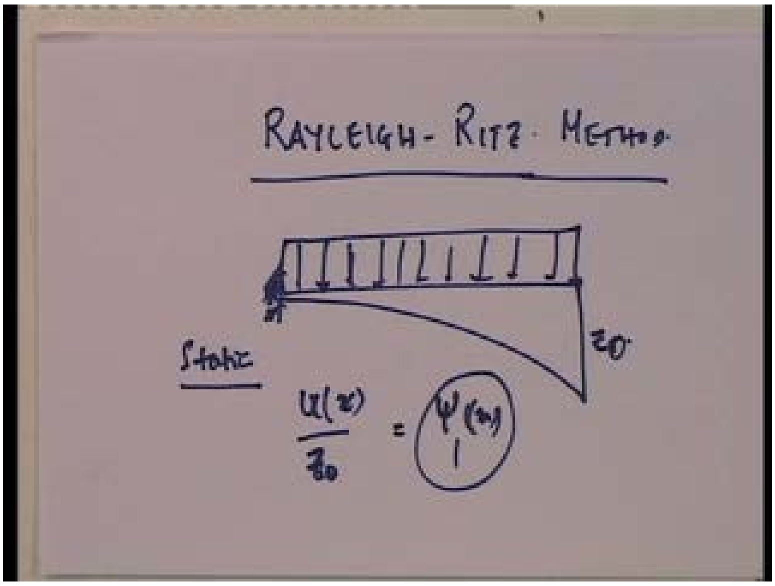


Now this method, the method that, I am going to be talking about right, now is actually known as the Rayleigh-Ritz method and the way this is done is that the psi x is found by putting u d l on the system. And finding out the deflected shape and whatever is the z 0, the v x upon z is 0 gives me psi x, this is of course, done with in static sense. So, in other words given a particular, let say that you have u d l, you know cantilever you apply u d l on it and find out the deflected shape and this u d l this is the static problem, you can find out v of x, I am u of x and u x upon z 0 is a first estimate of psi x.

If you use this particular psi x, it is almost clear that, it will satisfy not only the geometric boundary conditions, it will also establish the equilibrium boundary conditions, for this u d l case. So, in other words, it does approximate the relationship and you see what happen is that why does this give reasonable close result, because it satisfies both equilibrium and geometric boundary conditions, so the psi that, you get is much much closer to the real psi, that you will get under vibration.

So, that is all there is to it in this particular case and then once you get, so this is the way to get psi x of course, the other approach that, we have which is a Ritz vector approach, you can use any psi that satisfies the geometric boundary conditions. A more appropriate psi is this one where, you actually apply a u d l on the structure that, you have and you find out the static displacement and divide that by the displacement at the end, because that is the degree of the freedom and that gives your first order approximation of your psi x.

Later on, we will see that this procedure is going to be used, in a multi degree of freedom sense to actually you know iterate 2 the true psi x, that you have for a particular system. So, that much, so therefore, what we have now is we have looked at real body assemblages, which are in a way they are true single degree of freedom systems, the only thing is that because you cannot appropriately there are many different ways are defining a single degree, I mean the degree of freedom.

So, we take that degree of freedom as any generalize displacement for example, if you think back, I took rotation and I also took at displacement at a particular point, but they are related to each other. So, those Ritz body assemblages that, I will look at earlier where through single degree of freedom problem, now, the last lecture, in this lecture, what I am taking about or not true single degree of freedom systems, they are actually continuous systems and we are making approximations.

And the approximation that, we are making is by take assuming that, the displaced shape and specific you know in a particular at any instance of time is given by a shape function into the generalized displacement z 0 of t. If you look at mathematical what we actually done is we taken this particular mathematically, physically what we say we already talk about, mathematically what we done is we taken u x of t and made it into psi x z 0 of t.

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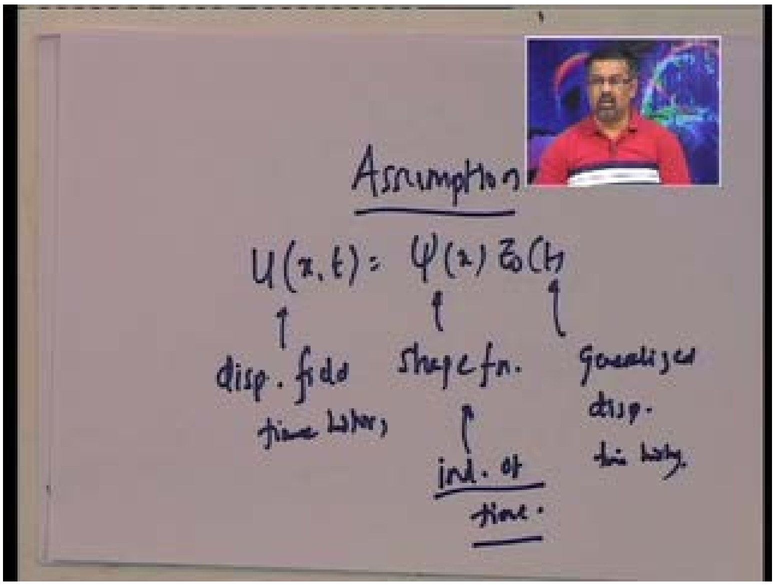


Where this represents the shape function and this represents and this represents displacement field time history and this is the generalized displacement time history. If you look at this mathematically, what have I be done, we taken up function, which is the function of x and t and we separated them out into 2 functions 1 a function of x and a another function of t. So, this is known separable functions and in this particular case, we are assuming note very, very this is an assumption, that the shape function is independent of time the shape in dependent of time and this is the generalized displacement time history.

So, that if you look at a situation where, we look at curvature, this curvature is actually del squared u by del x squared of u, this is equal to if I look at u double dot x of t acceleration, that is given by. So, the time derivative and the special derivative are separated and that is the beauty of this approach note of course, that psi x is and assumed shape function. However, if psi x is derived in the way that, we done where, we are said is that is apply a u d and find out the displace shape and normalized that with respect to the you know the displacement, you will get the shape function, if you use that psi x, you will get an estimation of k star m star p star and omega, which are fairly close to the real value. Of course, in real m star k star and p star do not do not exist, but you know, if you where to look at the solution for a particular p and you even look at omega, which is the free vibration.

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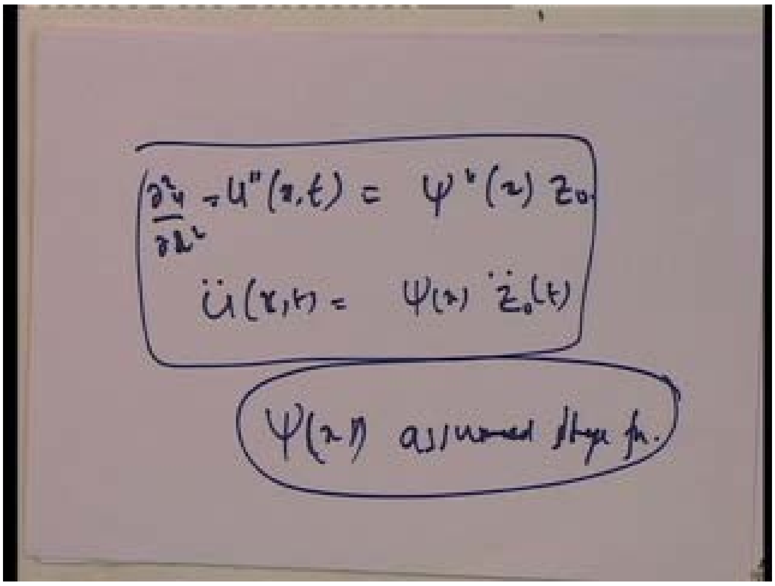

So, if you look at both free and force vibration by taking this generalize single degree of freedom system with the psi x obtain through the Rayleigh Ritz procedure then by enlarge, you are solution both for free vibration force vibration are reasonable accurate. And note that what have, we done we taken a very complex structure made it into a generalized single degree of freedom and the you know whatever response and free vibration response, we get are fairly close to you each other, how much better can you get.

So, now, let us look at solving using this and solving some problems, in which now you see up to now we have looked at what have looked, first at rigid body assemblage. And then I have look at something, which is no rigid body just a deformable body where the deformation only the flexural deformation are concerned and we solve that particular problem.

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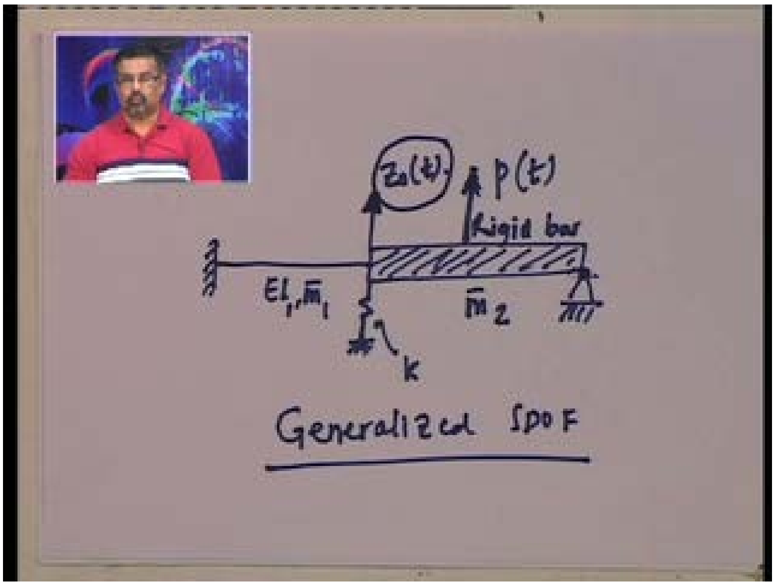


So, now, I am going to try to solve, it using let us look at a particular situation and I am putting on this problem, only as a specific kind of a problem and let me put apart from that, I have E I m bar m bar 1 E I 1, this one of course, is a rigid bar this is the pin. And so this one, I am saying is m bar 2, let me look at this and call this as k. So, this a problem, now if you look at this particular problem, this is this has a flexible part and it has a rigid part and there are some external.

So, you see you got this resistance and you also got an external k, now how would I solve this particular problem, let us look at this now. This is problem that, I need to look at very, very carefully and what I will do is I will do the following, I will consider z naught t, let me some load on it, so we put some p t is a concentrated load acting at this point. So, I will consider this as my generalized, now this is a generalized, this is the generalized single degree of freedom, because I defined this as my now, how would I solve this problem, well again the question goes back to the same concept and that is put z 0 equal to 1 and find out your displacement.

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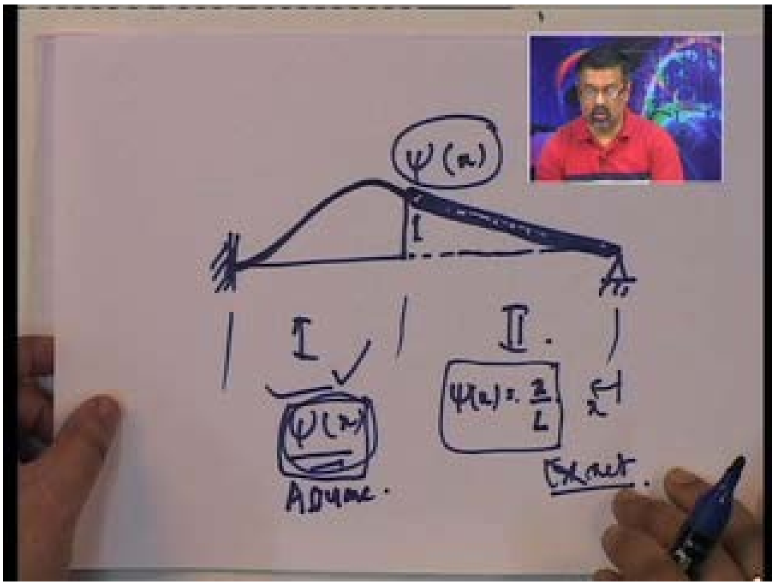


So, let us look at this here is equal to 1, this is this way this is fixed and here you have continuity between the 2, in other words this is continuously connected no pin here, if you do not have a pin here, then how does this go this is a rigid bar, so this rigid bar is going to go straight. Now, here note that this goes like this and comes like this, so this end is my displacement pattern for unit and if you this is my psi x, the psi x, now I can take it either way, I can consider these 2 separately.

So, I can consider these as 1 and this as 2 for 2, I know my psi x, since it is a rigid bar my psi x is what, if I start my x from here for 2 my psi x is equal to x upon L, it is obvious, it is this is not an approximation, this is exact. So, if you look at it, if you look at your psi x for a rigid bar, the psi x is exact, for this, we need to get a psi x, now let me assume that, I know this you know this is not obvious, we will see later on I am not right now solving this particular problem.

But, let us assume that, well I can find out how would I find out this psi x well I would put u d L on this one and under the u d l this would also go up and therefore, you would get all kinds of things. But, here the point then becomes is that, I know this psi assume or use the Rayleigh-Ritz approach to get this psi x, it is a known this psi x is known. So, if I know this psi x then what do I have, let us look at what are the things that, I have note that this is very interesting, because I can now write this as a problem, which is of the following nature.

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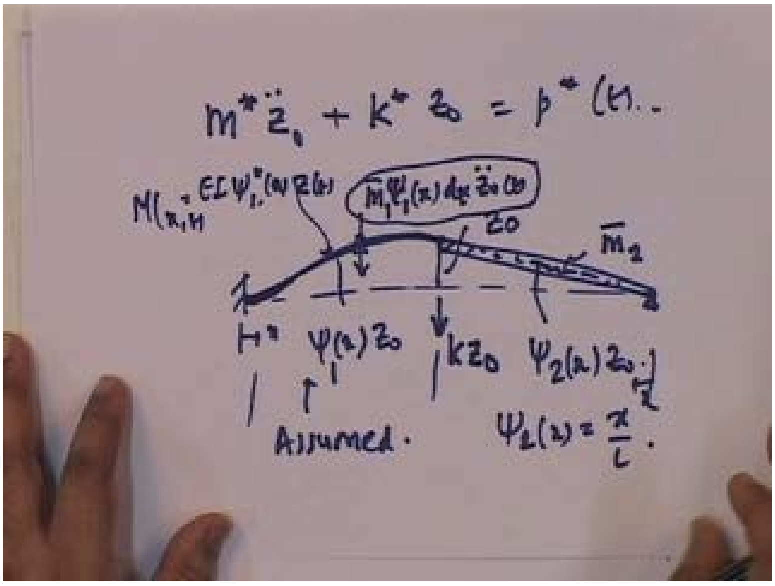


And the nature is this, that note that this is going to have the problem m 0 z plus k 0 is equal to p star, now to get this all I need to do is write down find out all the forces, that are acting on this body and based on that, you know derive the equations. So, and then use the virtual displacement pattern to solve the particular problem, so if I do this then I have the following well let me just draw this.

So, what do I do I give it z 0 right and if I give z 0 over here then what do I get, this one becomes psi x into z 0 and then psi 1 x and here, this is psi 2 x into z 0 where, psi 2 psi know is x upon L this is exact, this is assumed. So, if I have these kind of loads then now what are the things that, I am going to have following this m star 2 giving rise to inertial forces, I am going to have these spring give rise to what k z 0 load.

I am going to have this mass over here mass undergoing inertial forces in addition, I will have the moment over here also I mean in know the flexural deformation occurring over here and due to the flexural deformation it will give rise to moments. So, those are all the forces that, I have in the system and what are those forces equal to well, let us look at this in this particular case, I will take x from here to here.

So, this is the first part and this over take x see, these are 2 different bodies, I am going to separating out the 2 bodies and then since I connected to each other when you write down the virtual work equation all of them will be connected to each other right. So, if I put it this way and this is psi 1 x corresponding to the mass, I will have m bar 1 into psi 1 d x, if you look at this m bar into d x m bar 1 d x is the mass per unit length, I mean you know infinitesimal length into psi 1 x into z naught is the displacement and if I put acceleration, that will be the acceleration of this point. So, that is this what would be the moment at any particular point, the moment at any particular point would be E I right, then 9 x into z of t, because E I into this gives me the curvature and E I into the curvature is the moment is equal to the moment x of t. So, I have now figured out all the forces that, this body is being subjected to and if I look at all these forces then let us look at what I get the following, I am going to give it a virtual displacement.

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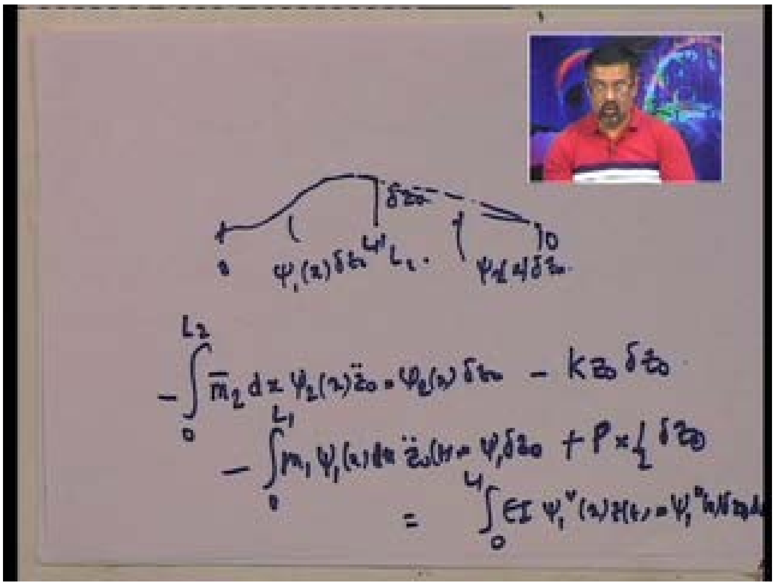

Again virtual displacement, I am going to give del z 0 as my virtual displacement note that, this and this are going to be identical this is going to be psi 1 x into del 0, this is going to be psi 2 x into del g 0. And so if I look at the work done by the internal external forces note, that in terms of our internal forces over here, you know this is undergoing reverse. So, if I put all of them down, I will get what, I will get m 2 m bar 2 d x into psi 2 z naught double dot, this is the force multiplied by the displacement, which is equal to psi 2 x del g and this integrated from 0 to L 2, which is 0 here L 2 here, that gives me the work done and that is of course, negative work done, because the it always opposes the moment. So, that is negative, that is the external work, then I have the k 0, so k 0, so that is also going to do negative work and that is equal to k 0 into del 0.

Because, it is at that particular point itself right, what else then I have minus m 1 psi 1 x d x z naught t multiplied by psi 1 del z naught from 0 to L 1 where, this is 0 this is L 1. So, that these are the external forces and on top of that I have the load, which the load is going to be it is a concentrated load and that is acting at mid span. So, that is going to be equal to half into del g naught, this is the work done by the load half del z naught, because that is exactly the at that particular point.

And this is the external work done is equal to the internal work done, internal work done is going to be equal to E I psi 1 prime x z t, this is m x and into, we seen that the virtual relative rotation and that virtual relative rotation is given by psi 1 double prime del g 0 zeta x, this going from 0 to L 1. So, if I put all of these terms and then know that del g 0, which shows up in all of them is arbitrary then I can rewrite this equation, in the following from and that is going to be equal to I am going to take all these minus terms on this side.

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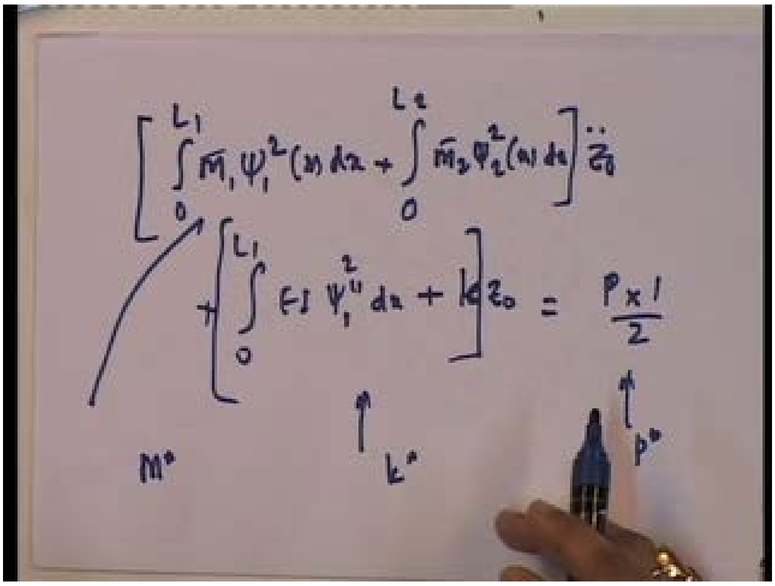

So, this is going to look like this 0 to L m 1 psi 1 squared x d x plus, so its 0 to L 1, 0 to L 2, m 2 into psi 2 squared d x into z double dot plus 0 to L 1 E I psi 1 whole squared into d x plus k z k into z 0. So, these are z 0 plus is equal to p into half and del z 0 disappears, so you see what is m star, m star is this, k star is this, and p star is this. So, you see ultimately, if you really look at it all that happens is that, all you need to do is you need to now find out, you do not need to do anything in m star, you find out, which ever m mass per unit length you multiply that by the corresponding shape function squared and then all of them have to be added to each other. Then if we look at k star 1 aspect is distributed flexibility will have E I psi i double prime squared and you know specific flexibility, which is specific springs.

Those are given actually by k here, its 1 squared, because it was k was applied in that particular point, but in a way it is k psi squared the way we derived it. So, you see the point then becomes the following that, if I have even a combination of structures, then I have a situation where, I do not have any specific issues associated with finding out m star k star and p star. I will now solve a specific problem by using without even going through first principles and using this idea, I will solve a specific problem let me take.

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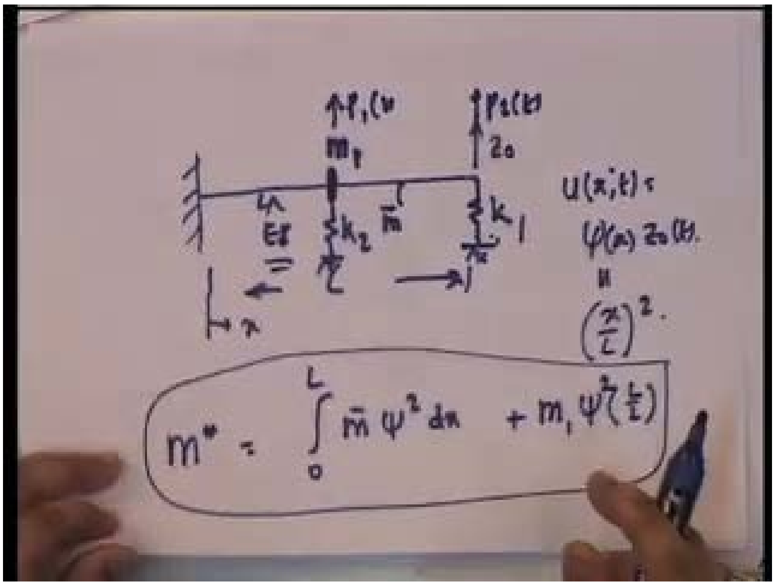

So, this is a problem, in which I have a mass, a point mass given by m 1 and I have a distributed mass over the entire bar, this is a flexible bar. So, this is a flexible bar given its flexural rigidity is given by E I, m bar is its mass per unit length and it is a constant, it is a uniform bar length L and at the end, I have a and my and let me also put another, I will put it here. This mass is at L by 2 and I will put another, this is k 1, I will put this as k 2 let us see what happens and I am going to put it the following, I will put psi. So, I am going to put a displacement this is z 0 u x of t, where x is defined from here.

U x of t is given by some psi x z naught t and psi x, I am going to take it equal to let us just take, I mean I know this is in a proper, let us just take this, because this is the either reasonably good, we know that, it is in a bad approximation, but for solution purpose let us write now take this as this. So, this is the problem and let me try to get m star k star and t star, without going through the procedure of having 2 you know, go from first principles, I am going to just derive them, the way we derive them.

Now, m star let us look at m star, there is a distributed part the distributed part is going to be 0 to L m bar psi squared d x, that is the distributed part, we know that this plus there is 1, which is given over here, which is m 1, which is going to be equal to psi at that particular L over 2 squared. You see mass into whatever the acceleration that point, so that is given by psi squared, I mean you can I mean 1 part psi gives, you the acceleration the other one gives you know when you write down the principle virtually, you get that, we have already gone through that. So, that means, this is my m star what is my k star and by the way, I forgot to put a load right, I put the load over here, there is 1 load acting here, p 1 of t and there is another load acting here p 2 of t. So, now let us go back.

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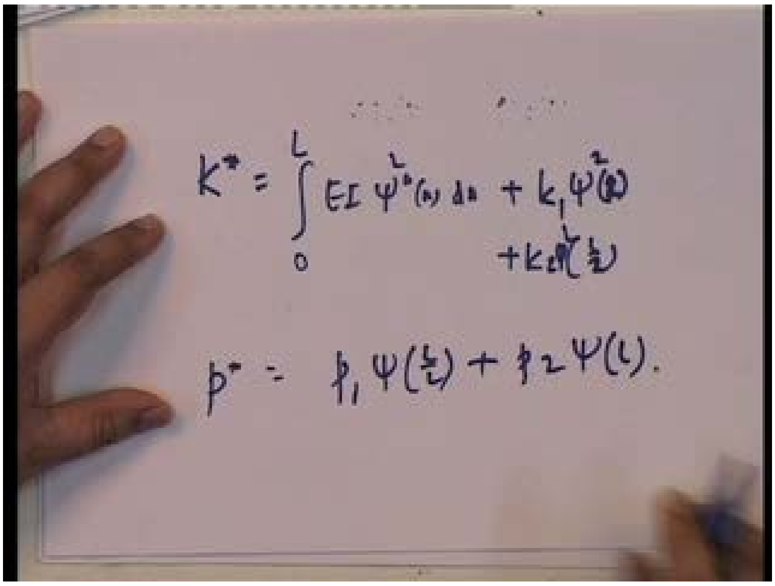


So, k star there are 2 parts, one part is the distributed flexibility given by E I and another one is these 2 concentrated. So, if I do that the distributed part becomes 0 to L E I psi double prime x squared d x, that is the distributed part in the k star and then I have k 1 into psi at L squared plus k 2 at psi at L by 2 squared right. I mean it is very simple and we have already derived this, so this is the k 1 is at L and k 2 is at L by 2, so that is all.

That is k star and what is p star well p star, there are 2 concentrated loads, so concentrated loads are going to be p 1 into psi 1 at L by 2 plus p 2 into psi at L, so that is all there is to it, this is m star k star. And now, let us derive this with psi equal to x upon L whole squared, so if you put psi x upon L the whole squared then what do we get, let me put down first, what do I get as my, so psi x turns out to be equal to x upon L squared.

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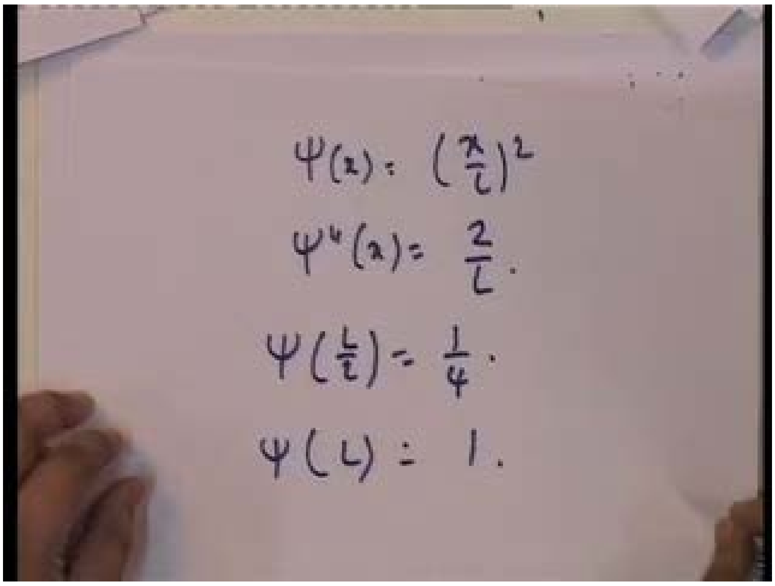

So, psi double prime x is equal to 2 upon L then psi at L upon 2 well L upon 2 1 half the whole squared, its 1 4th psi at L is equal to 1. So, I have got all the parameters over here defined and we already done, this particular you know for a constant 1, we have got what we get 0.2 m bar L right. So, m star, if I look at it then m star turns out to be equal to m bar 0 to L x by L 4th L plus m 1 into psi squared, so that was 1 1 4th. So, 1 4th squared, so this is equal to we know 0.2 m bar L plus m 1 upon 16, so depending on what your m bar and L is that is your m star, we have derived it, because this is known, this is known, this is known, we can find out what m star is.

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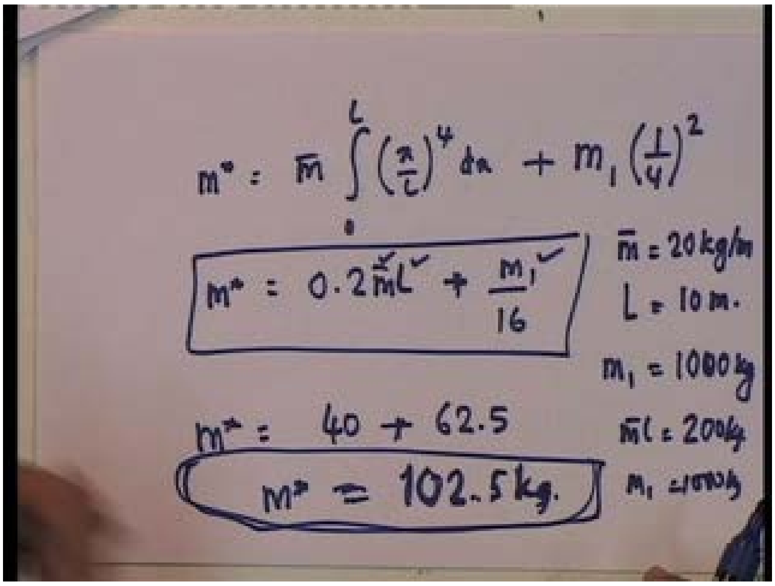


I will I will put down some values, let us put on some values, let me put m bar is equal to 20 k g per meter, let us take L is equal to 10 meters and let us take m 1 is equal to 1000 k g. So, total mass of the bar is 200 k g's and this point mass is 1000 k g's, so if we do this m star is equal to 200 into 0.2, that is 40 plus 1000 upon 16, so that is equal to 62.5. So, m star is equal to 102.5 k g, see I have m bar L, I have 200 k g's there and I have a m 1, which is 1000 k g's, I have almost 1200 k g's of total mass. But, the participating mass in that particular is just 102.5 k g corresponding to the end one.

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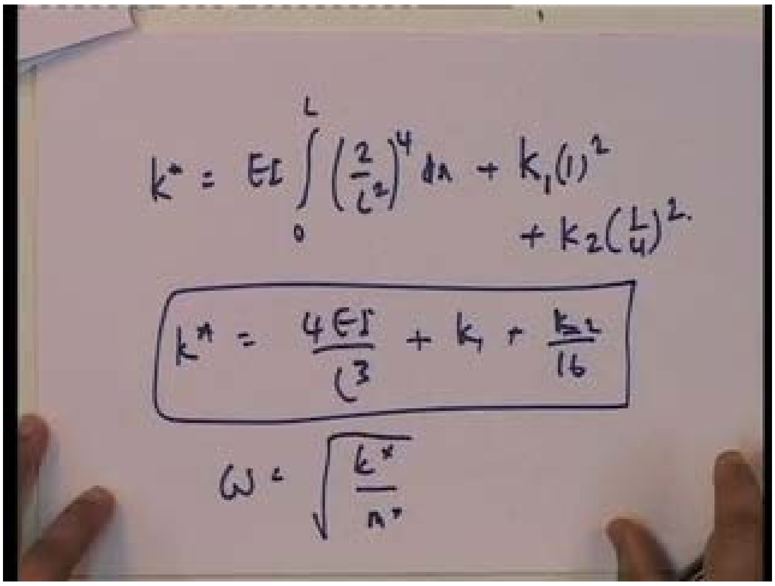

So, that is what my m star comes out to be equal to then what is my k star going to be turning out to be equal to, if you look at it is going to be equal to E I and then 0 to L 2 by L d x plus k 1 into 1 squared plus k 2 into 1 4th squared. So, therefore, this is equal to 4 E I upon L cubed plus k 1 plus k 2 by 16, so this is my k star. And so now, you know my omega is going to be equal to k star upon m star into omega and finally, what is my p star equal to well, I have 2 loads and those 2 loads, if I look at them, they are equal to the following.

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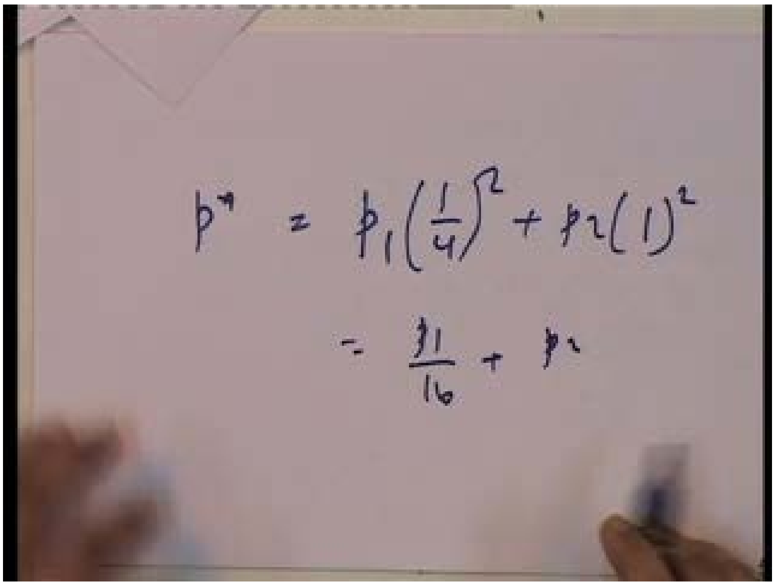


Let me go back, I want to show you the specific problem that, we had that, I had considered and that is this particular problem right. So, this was the particular problem p 1 was at the half point, p 2 was at the second point. So, it becomes equal to p 1 into 1 4th squared plus p 2 into 1 squared. So, p 1 upon 16 plus p 2 that is my p star, these are absolutely elegant ways, so therefore, I give you any problem where, it is a mixture of distributed mass.

If its distributed mass then the mass per unit length into psi squared d x integrated over the length over, which the distributed is provides the contribution of the distributed mass to m star. If I have fixed masses, point masses then the position of that point into the displacement in the shape function squared gives me the contribution of that point mass to the m star now.

If you look at k star there are 2 paths, there are distributed flexibility, if you distributed flexibility where, the flexural rigidity is given by E I then the distributed flexibility is going to be E I psi double prime squared d x 0 to L where, L is the length over, which the distributed flexibility exists. If I have fixed flexibility, you know fixed springs, then it is the spring constant into the displacement at that particular point in the shape function squared, that is the contribution to k star.

And how do I get p star well, if I have distribute d loads then and where the distributed load intensity is given by p bar x of t, then p star the contribution of the distributed load is going to be integration over the length over, which the distributed load exists, p bar x of t into psi of x d x, that is the contribution of the distributed load. If there is a concentrated load well then it is going to be equal to p into the shape function value at that, particular point squared, that is the contribution to p star.

Suppose, there was a moment applied how would I will do that, well moment into what the work done is z prime, so it is moment applied moment into psi prime at that particular point where, the constant moment is applied. So, you see this is the elegant way that, we can work with generalized single degree of freedom and get m star, k star and p star, earlier I looked at c star as you know for specific by giving viscous dashpot.

But, we know that in real systems, we never really compute c star, what we do is, we assume well we compute, we measure psi where, psi is equal to c star upon 2 m star omega, that is psi. So, all we do is define psi, so we do not really look at c star. So, we looking at m star, k star and p star and once you have that and you have the generalized degree of freedom your equation of un-damped equation of motion become m star z 0 double dot plus k star into z 0 plus p star, which is the function of time.

And this is all that we solved, this is a single degree of freedom problem and when you solve that, you got it. So, just to say that, we have looked at true signal degree of freedom and generalized single degree of freedom systems at this particular point, but of course, once you have the equation of motion of a single degree of freedom, then the solution process is what we have looked at for the last many lectures. So, I am done now, with single degree of freedom system problems, from next time onwards, we shall start looking at multi degree of freedom system problems.

Thank you very much, bye.