Structural Dynamics

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Lecture - 2
Inverse Power Method

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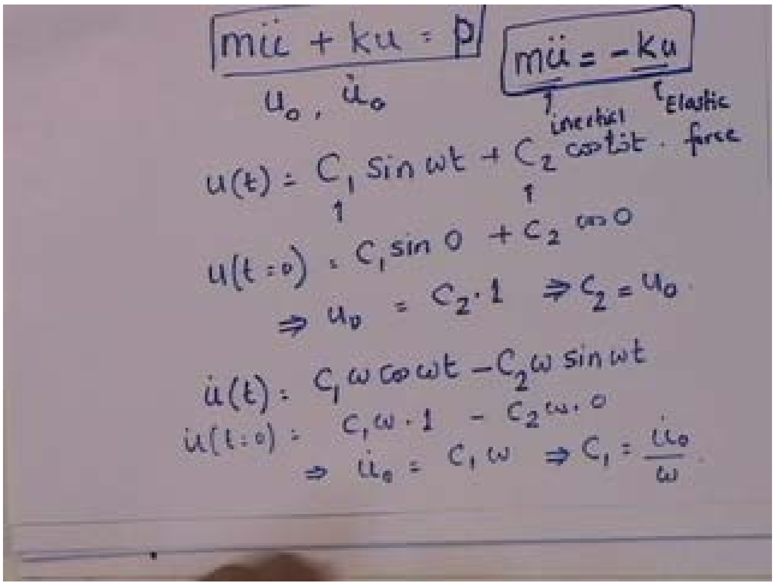

Hello again, just to review what we had done yesterday. We had tried to solve the free vibration equations subject to the initial conditions, displacement at time t equal to 0 and velocity at time t equal to 0. And we saw that the solution to this equation turns out to be equal to u of t is equal to C 1 sin omega t plus C 2 cosine omega t, where C 1 and C 2 are real constants. And these real constants can be obtained from these initial conditions, in this particular way, put u at t equal to 0, if you substitute that this becomes sin 0 plus C 2 cosine 0, which is equal to C 2 into 1, which implies that u 0 is equal to C 2 into 1, which implies that C 2 is equal to u 0.

Now, how do we compute, so we have already got C 2 is equal to u 0, how do we get C 1? For C 1 we need to differentiate u of t. If you differentiate u of t, you get u dot of t is equal to C 1 omega cosine omega t minus C 2 omega sin omega t. Again substituting u dot at time t equal to 0, we get C 1 omega into 1 minus C 2 omega into 0, which implies that u dot 0 is equal to C 1 omega, which implies that C 1 is equal to u dot upon omega.

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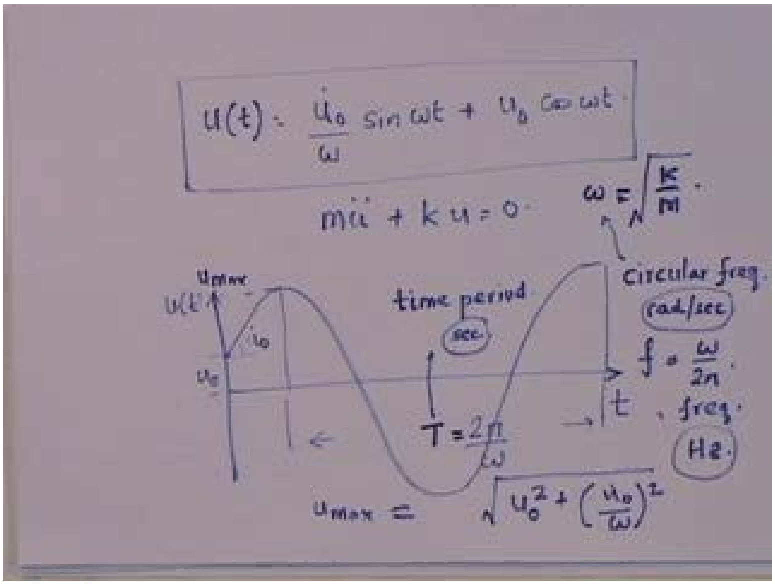

And therefore, the final solution to this equation turns out to be equal to u of t is equal to u dot omega sin omega t plus u 0 cosine omega t, this is the complete solution to this equation. If you look at this term, what does it look like, how does it vary with time t? t u of t, if you look at it, it starts at u 0. So, this is u 0 where, if you look at this time t, it is equal to 2 pi upon omega, so if you look at it and over here as we know, this is u dot 0. So, if you look at the amplitude, the peak amplitude which is u max, u max is equal to u 0 squared plus u dot omega whole squared.

This is u max, this is the amplitude of the displacement and the amplitude of the displacement is given in terms of the initial displacement and the initial velocity with the ratio of omega. Now, there are certain things that are important to look at over here, remember that omega as we had defined, it was given as root K over m, this is known as the circular frequency of the structure and it is given in radiance per second. This is known as the time period of the structure and it is given in seconds.

We have another term defined as f, which is omega upon 2 pi, which is known as the frequency and is given in hertz. So, this is defined in seconds, the circular frequency defined radiance per second and f is given in hertz. Now, one important point over here is that, omega which is strictly the circular frequency is also known as a frequency. I mean in other words, omega and f are both frequencies, omega is the frequency given in radiance per second and f is the frequency given in hertz and t is the time period of the structure.

Now, these are dynamic characteristics of the structure and note that, this frequency is related to the square root of the stiffness upon the mass of the structure. So, in other words, think of a structure that is exceedingly rigid, what do we mean by rigid, rigid implies that K tends to infinity. If K tends to infinity then from this equation you will see that, if K tends to infinity omega tends to infinity.

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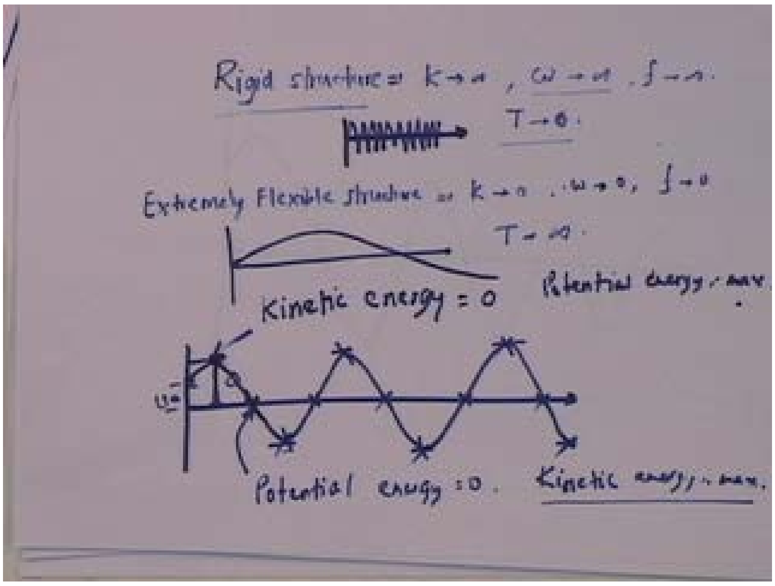

So, in other words, if you have rigid structure, this implies K tends to infinity, which implies omega tends to infinity, which implies f tends to infinity and which implies, T tends to 0. If you have an extremely flexible structure, which implies K tends to 0, which implies omega tends to 0, which implies f tends to 0 and implies T tends to infinity. So, in other words, it is interesting that, if you have a rigid structure, it characterized by zero time period or infinite frequency.

In other words, what we are saying is, for the rigid structure when it vibrates, it vibrates in this fashion. And an extremely flexible structure will vibrate where, the time period is almost in other words, if you look at, this does not have a frequency. If you have an extremely flexible structure, you have an extremely flexible structure here. And if you look at this structure, if it is extremely flexible, will actually collapse, if it collapses what it basically means is that, it never comes back.

If it never comes back, so in other words, if you look at this kind of thing and I give you the initial displacement, it is starts vibrating and from this point through this point, back to this point and back here is one time period, do you understand. Start here, given initial displacement, it goes here and comes back here, this is one cycle or we could also say that, if you look at this, if you given initial displacement comes here, this ((Refer Time: 10:17)) this this, from here to here, here and back again is one time period.

So, if you are saying that, it is time period is infinity that means, it has 0 frequency, what we have is essentially, it never comes back. Whereas, if you have an extremely rigid structure, it is vibrating so fast, that you cannot see it. That is, T equal to 0, it is practically not vibrating because it is extremely rigid, it is not vibrating. And therefore, it has infinite frequency, because not vibrating, so it is actually vibrating with infinite frequency.

Later on, during this course, we will actually take you to a lab and show you what I am talking about right now. So, coming back to this, in a sense if we solve this equation what we get is, this kind of a variation. So, if you look at it if I draw this in a longer time stamp so here, what you are getting is that, this ((Refer Time: 11:56)) this this this this this are all the same. So, this is essentially, because if you look at this, this is just a harmonic function.

So, harmonic function implies that, the peak and this peak is given by this, this will remain for all times. In other words, if we go back to what I am looking at, which is it is vibrating in this way, it will keep vibrating. Once I give it a initial displacement and it starts vibrating, it will keep vibrating, that is what this equation then this is what happens and it keeps going on and on ((Refer Time: 12:43)) and on and on and on and on. So, once I vibrated it, it will keep vibrating for ever now, in the real world, this does not happen because you must have vibrated systems and you have seen that, they start vibrating and then they slowly decay and they slowly come back.

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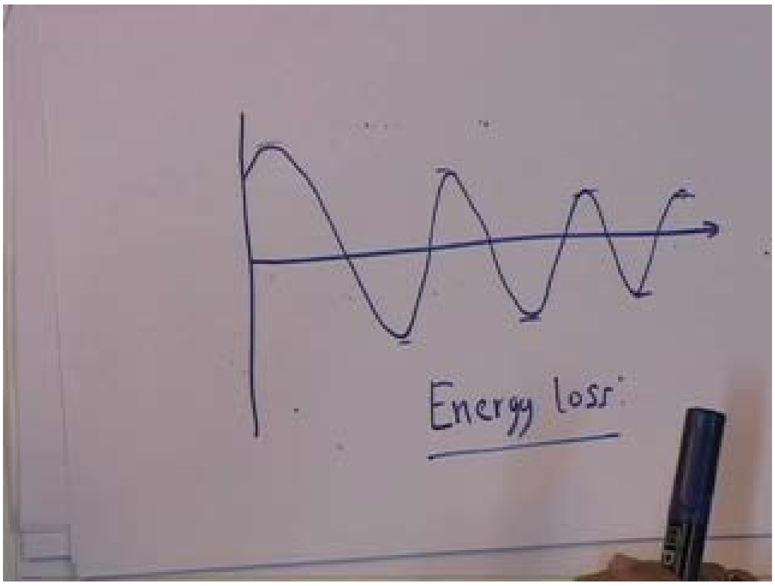

So, in other words, in the real world what we see is, we have a decaying amplitude, we have amplitudes continuously decaying. So, how do we model this particular kind of behaviour obviously, this behaviour is not the same as this behaviour and so therefore, the equation of motion is not this. Because, if it is this then the solution is this, which then becomes this and it does not reflect the real world situation.

In a real world situation what is happening, if you look at this, this is actually what is known as a classical situation, in which we have the situation where, this equation can be written as m u double dot is equal to minus K u. So, what we have is, we have equilibrium between the inertial force and the elastic force, this is the elastic force in the columns and this is the inertial force. Now, what happens essentially is that, we have equilibrium between the inertial force and the elastic force.

And what we have over here in this kind of situation is, when it is at this point, if you look at it, the velocity is 0. This system actually represents a conservative system where, at this point, since velocity is equal to 0, kinetic energy is 0 and potential energy is maximum, at this point potential energy is equal to 0, kinetic energy is maximum. And so therefore, what you have is a constant energy, which is given by the initial displacement and initial velocity and what you have is, the energy input by this system.

Then, essentially becomes the energy that supplied into the system at any given time anywhere here, there is a certain amount of potential energy plus kinetic energy, which is equal to the input energy. At this point, the kinetic energy is 0 because the velocity 0 but the potential energy is maximum because amplitude of u is maximum. So, potential energy is maximum, at this point since the displacement is 0, potential energy is equal to 0 but you have the maximum velocity and so kinetic energy is a maximum.

And same way at these points potential energy is 0, kinetic energy is maximum, at these points kinetic energy is 0 and potential energy is a maximum. So, in other words, we are not losing any energy in the system and we know that, a real structure always has an energy loss system built into it. If you look at this particular system, what does it show, that you input a certain energy that is fine, that energy shows up at this point, kinetic energy 0, potential energy maximum because it is a maximum displacement and 0 velocity.

And what you have ultimately is the fact that, this energy since it is starts dissipating, there is some energy loss in this structure, which we are not modeling by taking this equation. This equation is not complete because it does not modeled you know, it only says that, energy once input is only changing between kinetic and potential energy and that is all. So therefore, the energy loss in the system is not modeled now, how do we model the energy loss.

Now, this is a fundamental problem that you have in structural dynamics, the energy loss mechanism is not obvious. Now, since it is not obvious, we do not know how to model it so what people have done is that, they have postulated and this is, I want to be very, very clear about it that, nobody still and the stands what the energy loss mechanism in a structural system is. We know that, once it is starts vibrating, the energy dissipates over a period of time till it comes back to rest.

So, we know that, there is an energy loss system but even today, we still do not know what that energy loss mechanism is. So, once we do not know, we starts postulating because there is an energy loss and as long as you cannot model that energy loss in your equations of motion then all your responses, etcetera that you get for dynamic loading is going to be erroneous. Because the only equation that you have here is this equation and if you put this p here what happens, that solution does not…

If we get this solution for a loading p, it still does not incorporate the energy loss in the system. So, the u that we get from that is going to be erroneous so it is very very important for us to be able to model the energy loss. So, how do we model the energy loss now here, I must add that, there are various models for energy loss in the system and this energy loss is actually classified as damping.

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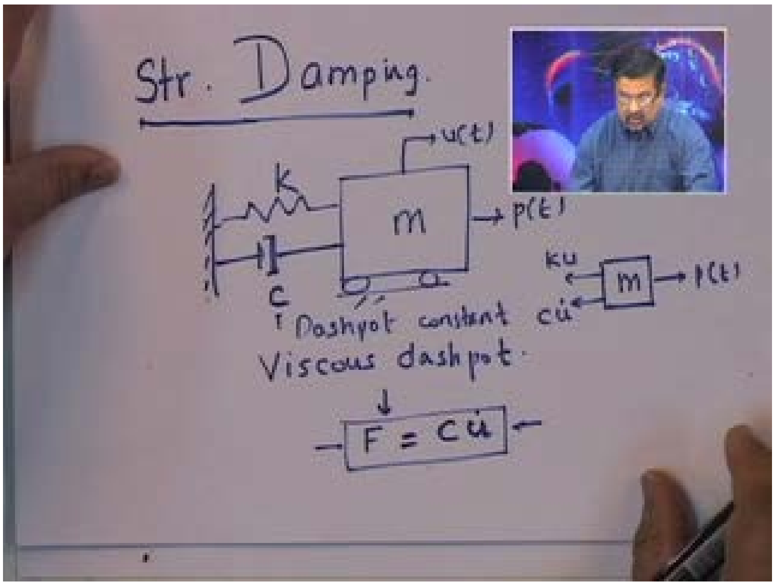

So, what we say is, this energy loss is by damping because what you are essentially doing is, you are essentially damping out the vibration of a structure. Since you have damping out the vibration of the structure, this is called this energy loss mechanism is called damping. So, how do we model structural damping, this is the major question now, when people looked at real life problem and realize that, there was a structural damping in the system.

You see the basic equation that we looked at, you saw that the equation m u double dot plus K u is equal to p is the same as for the mechanical vibration system. Remember, I just showed this, and this equation and our structural equation turns out to be the same. So, since the equation of the system remains the same, the first thing that we include damping is, we take the mechanical equivalent. The mechanical equivalent is a viscous dash pot, a viscous dash pot with dash pot constant C. What is this C? The C is such that, the force in the viscous dash pot is equal to C times u dot. So, this dash pot constant and this is a characteristic of the viscous dash pot is such that, F is equal to C u dot. And if I take the free body of the rigid mass then what we get is p of t K u C u.

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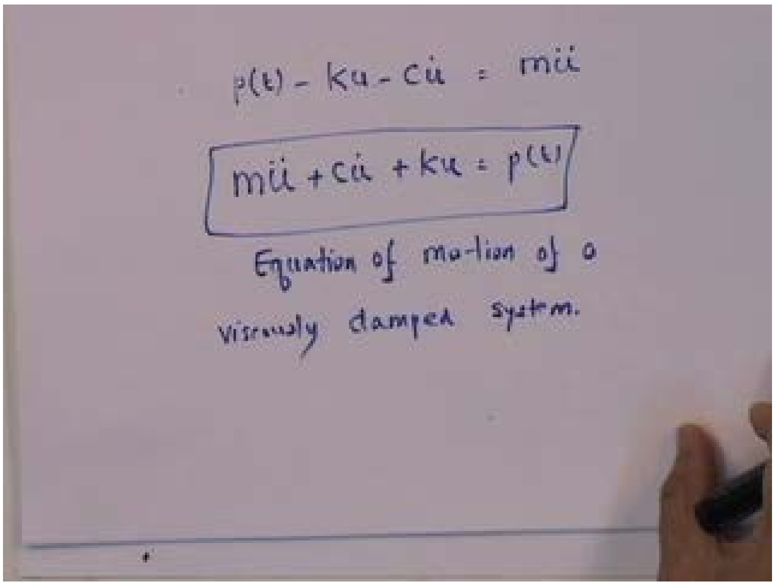

And therefore, again using d’alembert’s principle what we get is, the net force which is p of t minus K u minus C u dot is equal to mass into acceleration. I am putting this equation in appropriate terms, we get it as m u double dot plus C u dot plus K u is equal to p of t and this becomes, what is known as the equation of motion of a damped should I say, viscously damped mechanical system. So, this is the equation of a viscously damped mechanical system, which is given by this, so this is the mechanical system.

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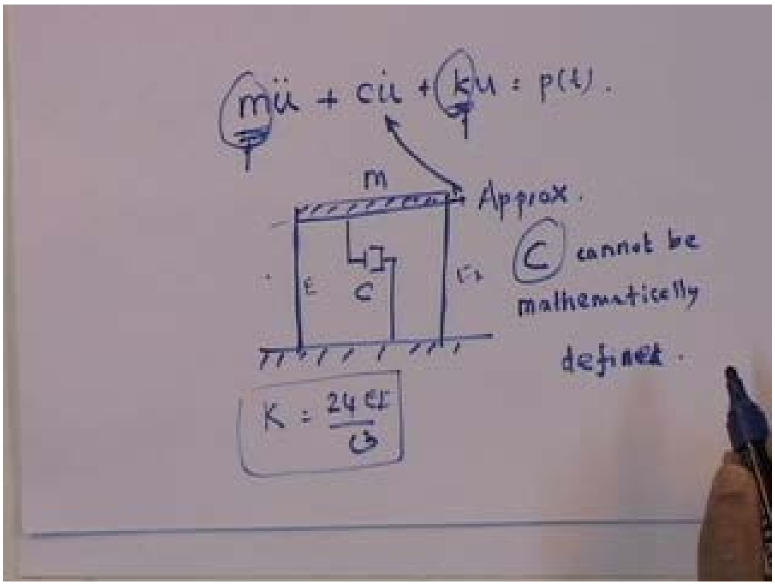

So, since this is the equation, what structure engineers said is, let us just take a situation where, we mathematically have the same system. We will look at the one story one way frame with the rigid beam and the floor mass m and these being the two columns. What they said was that, the energy loss in the system is actually modeled by a mechanical dash pot. Now, in reality, this dash pot does not exists because after all you have, you ever seen a single story building with a dash pot in between no, it is a single story building with columns, beams and the floor, etcetera that is what we have in our structural systems.

And so therefore, this is just a mathematical manner of representing the energy loss that you see in the structural systems. So, understand that, this is an approximation and although in a mechanical system the C is a dash pot constant, here C cannot be mathematically defined. So, understand that, m is the mass we can actually get it, k is the stiffness we can get this. How can we get this, we would found out the force that we require to get a unit displacement.

So, whatever the force to get a unit displacement that is, the stiffness so we can actually measure m, we can measure K in a real structure. We can of course, mathematically model it also, m and k, so either way like for example, remember I got K as 24 EI upon L cubed assuming that, this is L and these are both EI and EI, I completed this. So, both mathematically and experimentally, I can get m and K but I cannot mathematically define it at all and experimentally, I have no clue what to measure, I have absolutely no idea.

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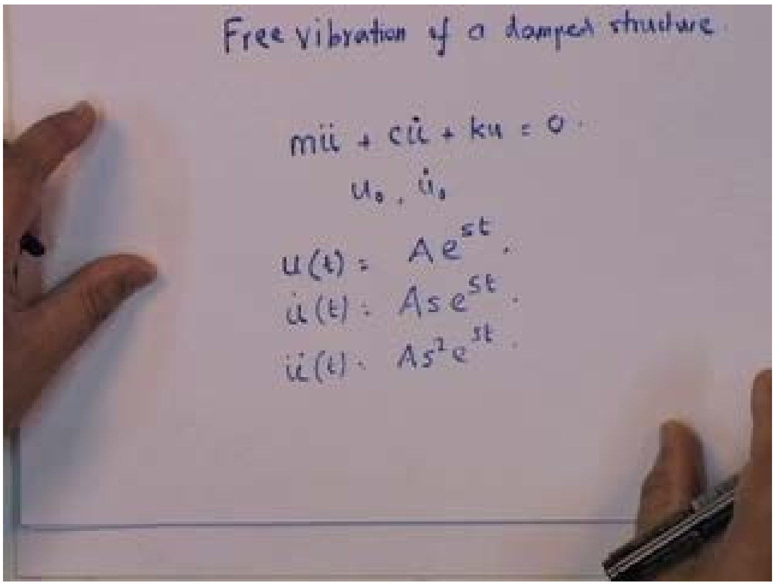

So, how does this help, let us go back, let us look at what I will call it as the free vibration of a damped structure. We are going to look at the free vibration of a damped structure so what is that then the equation becomes m u double dot plus C u dot plus K u dot is equal to z. And understand that, this is a viscously damped structure, so the model that we are using for the energy loss is the viscous damping. So now, again this is the solution, this is to be solved for initial displacement and initial velocity.

How do we solve this, we go through the same process, u of t is equal to A e s t, amplitude into e to the power of s t. And so if we substitute this where, u dot is given as A s e to the power of s t, u double dot is given as A s squared e to the power of s t. If we substitute all of these into the equation and then simplifying it what we get is m s squared plus C s plus K into A e s t is equal to 0.

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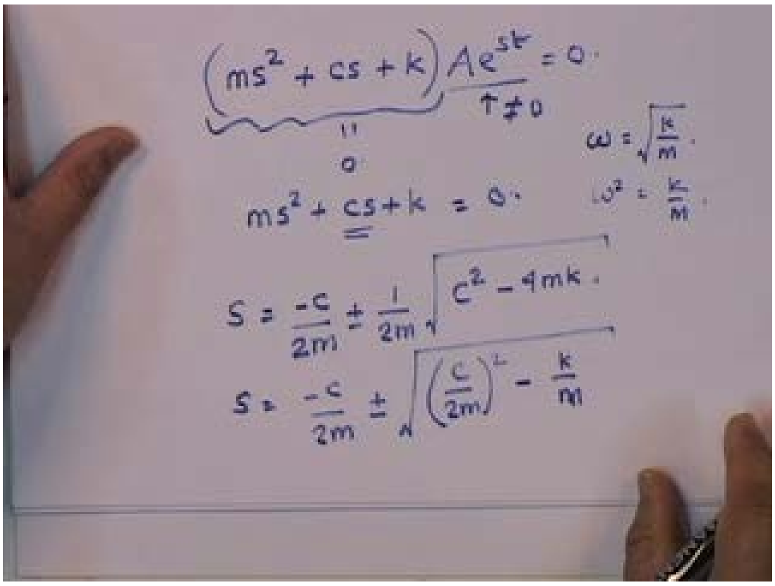

Now, we have already seen earlier that, this is not equal to 0 because if this is equal to 0, we get back the trivial solution. So, this is not equal to 0, if this is not equal to 0 obviously, for this equation to be always 0, it is implied that this has to be equal to 0. So, you see, this equation is identical to what we are obtained earlier, excepting for this additional term and this additional term is going to make it a very interesting solution. Now, let us see S, it is a quadratic equation and we can solve for this by minus b upon 2 a plus minus 1 upon 2 a square root b squared minus 4 a c.

So, you see this is the quadratic, this is b, this c is b, A s squared plus b plus c is equal to 0, big S is equal to minus b upon 2 a plus minus 1 upon 2 a square root of b square minus 4 a c. So, that is all I have done, I have just substituted all these terms in now, what am going to do is, I am going to actually put this inside. So, if I put this inside, what I get is minus C upon 2 m plus minus C upon 2 m square because 2 m when you put it inside, you get 4 m square so when you put 4 m square over here, you get K upon m. Now note that, earlier we have defined the natural frequency of the structure omega as square root upon K upon m earlier so which basically means, omega squared is equal to k upon m.

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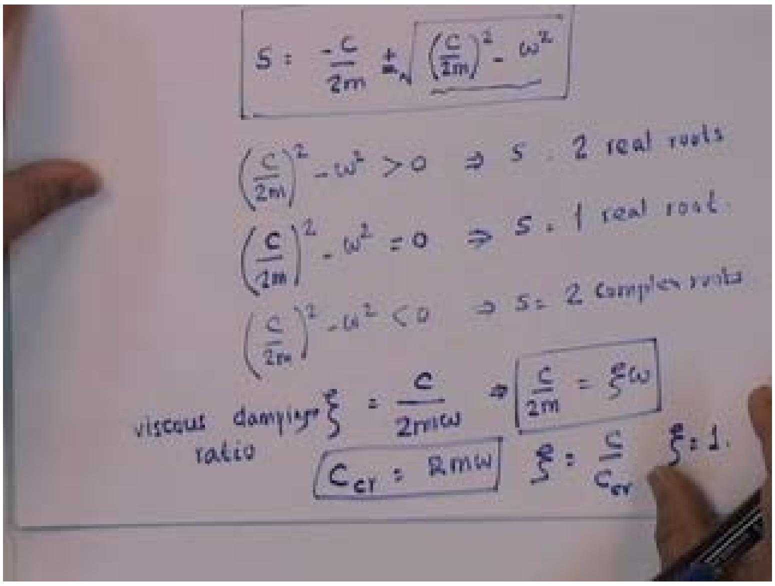

So, I can substitute this for here and what I get is this solution, the solution turns out to be S is equal to minus C upon 2 m plus minus C upon 2 m minus omega squared. Now, you see how the solution is depends on this term, if C upon 2 m squared minus omega squared is greater than 0 then S you have two real roots of this equation. Because, if this is greater than 0 then what you get is plus minus square root of a real number, which is greater than 0 and so this entire S has two real roots.

If C upon 2 m minus omega squared is equal to 0, think of it this is 0, so what you have is a S has one real root and if then S has two complex roots why because if this is less than 0 then this becomes a negative number and root square root of a negative number will become i times the omega squared minus C m upon 2 whole square. So therefore, we see that, the value of C actually determines, whether what kind of form the solution takes.

So therefore, this C is a very critical parameter and what we say is, we define a parameter zeta, which we call as C upon 2 m omega. This is a parameter, I am defining a parameter this implies then that C upon 2 m essentially, then becomes zeta omega. Now, this term is known, is actually called as the viscous damping ratio, so this is known as a viscous damping ratio and what is it, is a ratio of the dash pot constant to 2 m omega. So, in other words, we call C critical as 2 m omega think about it, C critical is 2 m omega and essentially, zeta becomes ratio of C to C critical.

Why C critical 2 m omega let us come back to this, in other words if C to C critical xi is equal to 1. So, xi is equal to 1 so that means, if you look at this particular term, if xi is equal to 1, what does this become, this becomes xi omega minus omega squared. So, I can actually rewrite this entire thing in this following format, let us just go back to that solution.

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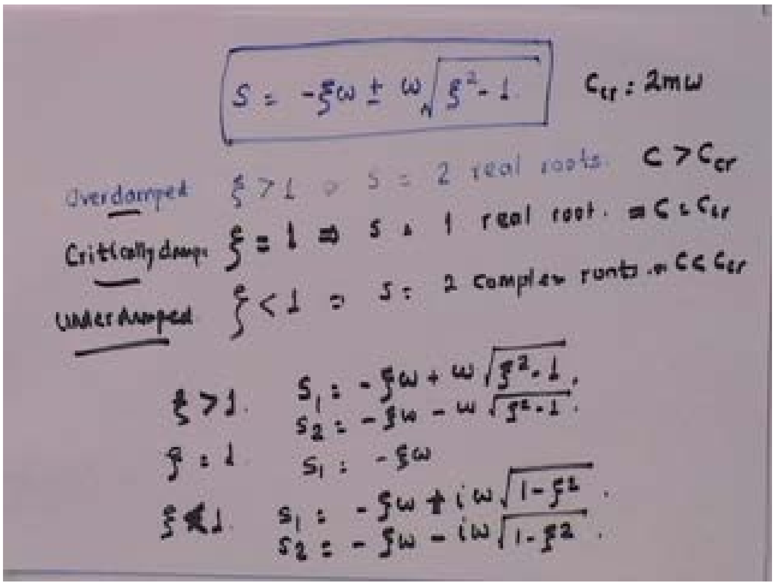

And let us look at S is equal to now, minus C upon 2 m is minus xi omega plus minus omega is going to come outside, square root of xi square minus 1, this is what that solution turns out to be. So, if you look at it, this equation here by substituting the fact that, C upon 2 m is xi omega, I have substituted that into this equation and I have got this equation. And therefore, we basically have this situation that, if xi is greater than 1 then we have two real roots.

And what does xi is being greater than 1 mean that means, C is more than C critical which basically means, we call it as over damped. Then if we have xi equal to 1, we have that situation that S is one real root, this is known as critically damped because xi equal to 1 implies that; Here, over damped means, xi is greater than C critical note that, C critical is equal to 2 m omega. If C is equal to C critical, xi is equal to 1 and that is known as critically damped and if xi is less than 1 then S is two complex roots.

C is less than C critical that is why, xi is less this is known under damped so this is a essentially so therefore, a new critical parameter called xi, xi defines the damping in the system. Because, xi is equal to C, which is a dash pot constant upon 2 m omega and this particular parameter xi, it is very important because if xi is greater than 1, the roots of the equation S are both real. If you have xi equal to 1, which implies that you have a critically damped system, you have S only one root and that root is minus zeta omega.

So, there is only a single root, S is equal to minus xi omega and you have under damped where, xi is less than 1, in which case both the roots are complex roots so let me write down those roots. So, if I have xi is greater than 1, the roots S 1 is equal to minus zeta omega plus omega into xi square minus 1. Then you have S 2, S 2 is equal to minus zeta omega minus omega square root of xi square minus 1. If xi is equal to 1, you only have one root, which is minus xi omega. If xi is greater than 1, you have two roots, one is minus zeta omega plus minus i omega 1 minus xi square. Note that, 1 minus xi squared is a positive quantity because xi is is less than 1 then s 2 is equal to minus xi square, this is plus and then minus.

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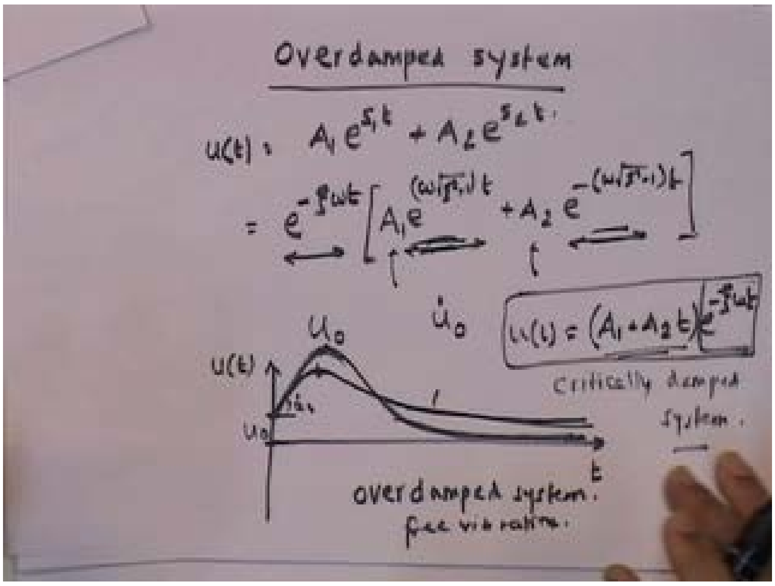

So, let us look at the over damped system, what is the solution, the solution is is u of t is equal to A 1 e to the power of s 1 t plus A 2 e to the power of s 2 t. And so if I can substitute these two values and note that, if S 1 this is exponent and you have minus and plus. What that becomes is that, we can rewrite this equation in this form, e to the power of minus xi omega t into A 1 e to the power of omega xi square minus 1 t plus A 2 e to the power of minus omega into xi square minus 1 t.

So, in this particular equation, if you look at it, what does it represent and of course, how do I get A 1 and A 2. I can get A 1 and A 2 by substituting u 0 and u dot 0 into these equation, how do you do it, you find out u of t, plug in, put equal to 0, this will be e the power of 0, 1 so this will be 1. So, it will be A 1 plus A 2 is equal to u 0 and then similarly, you can differentiate u of t of time, I am not going to go into this. I am not evaluating the over damped system response, I am just writing it down to show you, what actually is the form.

The form if you look at it, is consists of two terms, one an exponentially exploding function, one an exponentially decaying function superposed with an enveloping exponentially decaying function. And if I look at this, I am going to only draw it qualitatively, if you are interested in, look at any of the two books that I have talked to you about and you can see, what the solution is. By substituting, you can get A 1 and A 2 but I just want to show you qualitatively, what this looks like.

Initially, the exploding function takes over and then this function essentially starts kicking in and what we have is, we have an exponentially decaying function. So, this is how, an over damped system free vibration looks like, note that you see any vibration here, no. You know, the free vibration term is actually a misnormal for an over damped system because an over damped system, if you give it an initial displacement and initial velocity and it is over damped.

What it will do is, it will go like this and slowly come back to this, a classical example is an automatic closing door, have you seen it, you open it and then it slowly closes, that is what happens, there is no vibration. So, an over damped system, an over critically damped system does not vibrate at all, it does not vibrate at all, it just exponentially goes up and then exponentially decays, that is what an over damped system is. Now, I just want to here itself show a critically damped system, without going a about it.

Since a critically damped system only has one root, the solution u of t is actually of the form A 1 plus A 2 t e to the power of minus zeta omega t, this is how it looks. Now, if you look at this, this is a linearly exploding function superposed with an exponentially decaying function. So, what this looks like, if you give the same initial displacement and same initial velocity, what this will do is, this will go up like this and come back. Look at it, a critically damped system has a linear exploding function and then slowly the exponential decay takes over after period of time.

And it comes down very, very quickly because note that here, you have an exponentially exploding function. Here, you only have a linear exploding function, so once this function takes over, this comes down really quickly and goes to 0 really, really fast. So, the only difference between an over damped system and a critically damped system is that, both do not vibrate. Both, when they are subjected to initial displacement and an initial velocity, go up and then come down.

The reason, why this goes up more than this is this, even though you have an exponentially decaying exploding, it is added with an exponentially decaying function. So, actually a over damped system response is lower than a critically damped system response. But, once the exponential takes over here because you have an exponentially exploding function, this goes down much, much lower. This one, this one takes over and this comes down now, you can actually get A 1 and A 2 and you can actually plot these yourselves and you will see that, what I have shown qualitatively, is what you get.

So, I am done with over damped and critically damped systems and note that, they do not vibrate at all. The thing is, an over damped system goes like this, takes a long time to come back, a critically damped system is less because zeta is greater than 1 for an over damped system. A critically damped system goes further comes back quicker, but still does not go in the other direction. Note that, all of them are positive that means, if I let them go in this direction, they will only come back to 0, they will not go back this way, so there is no vibration.

So, an over damped system and a critically damped system do not vibrate now, why am I not interested in the solution because in a structure, we do not get fortunately an over damped or critically damped system. A structure always is under damped system and so in the next lecture, we are going to be looking at, how to look at a solution of an under damped system and see how we can solve that.

Thank you, bye.