Structural Dynamics
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Lecture - 19
Generalized Single Degree of Freedom Systems Equations of Motion

Hi last time, we looked at how to look at the finding the equations of motion for single degree of freedom, which is a generalize single degree of freedom. We saw that essentially the kind of systems that we look at where assemblages of rigid bars and you know the entire system had 1 degree of freedom. Now, one single degree of freedom system and the overall point was that you know what that degree of freedom well, you know we saw many problems in which they were at least more than 1 degree of freedom that could be defined as a single degree of freedom. And of course, if you use that as a generalize displacement quantity, then the equation of motion corresponding to that generalizes displacement quantity was different from another one, which was using another degree of freedom.

But essentially, they were relatable to each other based on the relationship between the two generalize single degree of freedom, generalize displacements, because if it is a single degree of freedom system, you always have one unknown and obviously, any other thing is related to that one another; so we saw that.

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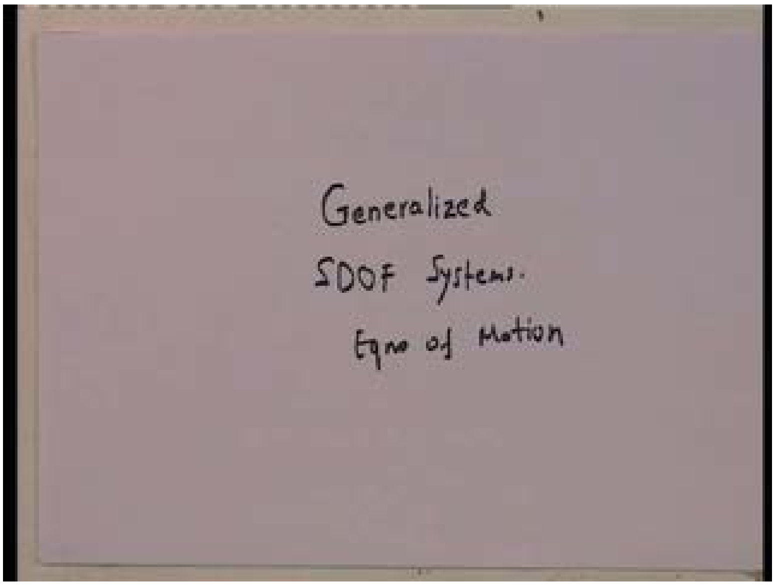

Today we are going to be continuing to look at generalized and equations of motion, the only difference is that, we shall be looking at essentially systems, which are not really single degree of freedom systems, but are rather they are actually multi degree of freedom systems. And in fact, we will see that there actually continues systems, which we can take a consider as single degree of freedom problem, let us see how this is?

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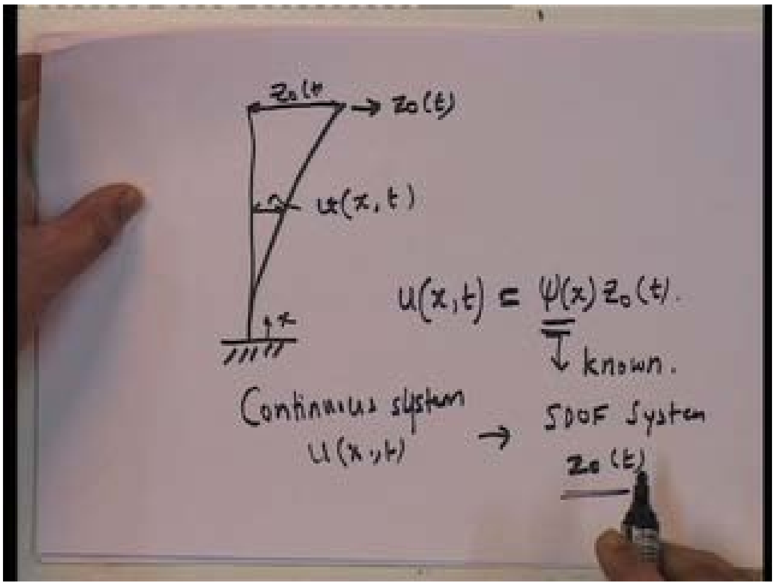

Let me draw a cantilever and this cantilever I am going to say they were when it deflects, what happens, it is going to reflect like this and if I look at, right if you look at this is with x. So, this is u x of t, now what we say is look, this u of x of t is given by a psi x, times z 0 of t and z 0 of t in this particular case, all define as this displacement right. So, that is z 0 of t the degree of freedom is z 0 of t, so if we write it in this form where, we say that this is known, if it say that then this continuous system given by u x of t transforms to single degree of freedom system with z 0 of t as the only unknown. Now, if that becomes then the question here is that I should know x I of x. So, let us look at a specific problem, let us look at the situation where let me take the cantilever itself.

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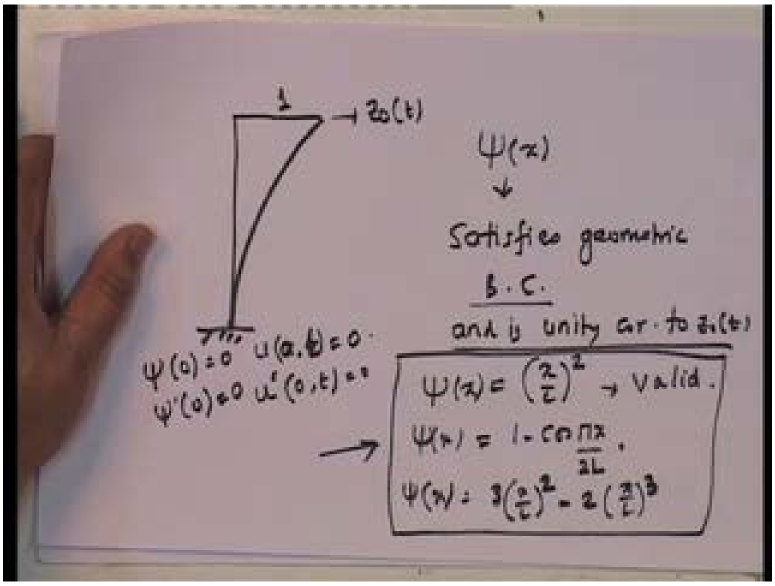

And let us say that how do I write it in terms of this, so let us look at what happens by enlarge what does x I x have to satisfy psi x is anything, which satisfies geometric boundary conditions and is unit corresponding to z 0 of t. So, now, if I look at it in this particular case, how what kind of x I x would I have what are the bound geometric boundary conditions, the geometric boundary conditions are that u at x 0 sorry, u at 0 t is equal to 0 and u dash 0 of t a 0, displacement in this direction is 0 and the slope is 0.

So, x I x is unit here and has to satisfy the fact that, so in other words this becomes psi 0 is equal to 0 and psi prime is equal to 0. These are the only 2 that, they have to satisfy and in that is the case then all, we have is well look at it, psi x equal to x by l squared, that satisfies the boundary conditions, we can look at it, if I put x equal to 0, I get 0, if I differentiated I get 2 x upon l squared put x equal to 0, you get 0.

So, this is valid right, psi x 1 minus cosine pi x by 2 l, well let us look at this, if I put x equal to 0 here, it become 1 minus 1, so it becomes 0 and 1 differentiate it. So, then the what this is become sin pi upon 2 l sin pi x upon 2 l, that is what happens put x equal to 0 this becomes 0. So, this also an equally valid system, then psi x is equal to 3 x upon l cubed minus 2 x upon l squared, if we take consider this as my you know equation, this also is valid, because note that, you know this is a higher or it is in this obviously, it will satisfy sorry, this is 3 x squared minus 2 x l cube.

Now obviously, we put x equal to 0, this disappears then you differentiate it, what do you get you get 6 x upon l minus 6 x up x squared upon l cubed. So, you put x equal to this, it becomes 0 minus 0. So, this is becomes 0. So, all of these are all valid and note that all of them at x equal to l, let us look at x what it has to be 1, this obviously, x equal to l is 1 put this x equal to l, this because l upon l 1 squared minus 2 1, look let us look at this put x equal to l what happens cosine pi over 2, what is cosine pi over 2 0, 1 minus 0 is 1 all of them satisfy the geometric boundary conditions and its unit. So, all of these are valid, which 1 is exact well, to find out which 1 is exact, we have to actually solve this continues problem.

Later on in the course, we shall be looking at that problem also and we will see what would happen as the exact answer to this, none of these by the way are the exact solution, because note that under any kind of load is psi x actually becomes as completely different. So, it does not matter and note that under load whatever you get and under the inertia force you get different. So, you know all of these are equally valid and all of them are equally approximate. So, this then becomes the fact that, this is how by defining any arbitrary shape function, I change the problem into just z 0 t, now the question is that, so if z 0 t is the degree of freedom then obviously, my equation has to be out of this form.

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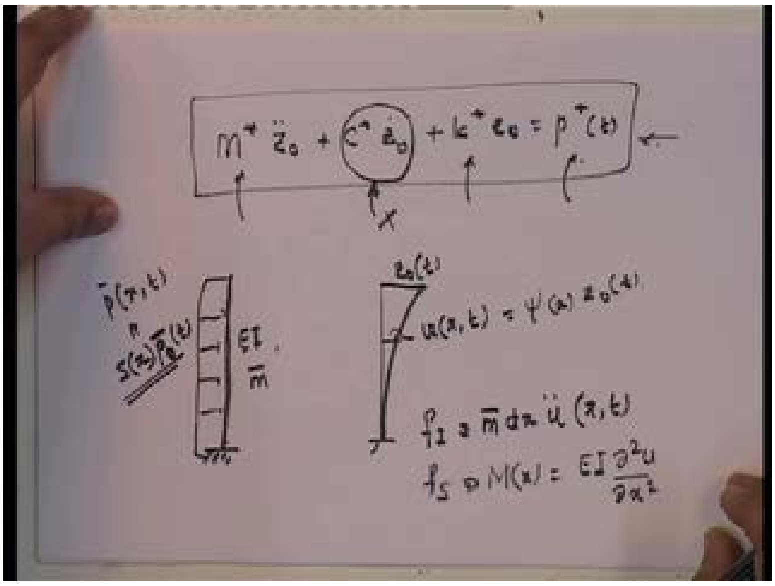

M star z 0 double t C star z 0 t plus k star z 0 is equal to p star t and this is the equation that, we have to get now note that, in this particular case, I am going to drop this term, because remember that, we had said that you know when I looked at rigid body assemblages, I put in dash parts etcetera. Now you know in a cantilever, you do not have dash parts, so for now, I will just drop this term and what we will see is we will, if we can derive these this and this for the specific kind of a deformable bodies.

So, let us look at this situations, I am saying that, I have this and let us say that, this is E I is its flexural rigidity and m bar is the mass and let us say that its subjected to a load, which is given in the following format, I will say that the load is given as zeta x into p naught of t. In other words what we are saying is remember, if it is a uniform load what is zeta x well its zeta x is nothing but 1.

So, this zeta x looks at the variation of the load and this defines, this is the load per unit length, which defines the intensity at a particular point. So, this then is typically, how this load, you know per unit length is define. So, this is typically how a dynamic load looks, because a t variation is independent of the other variation, so because its typically like, what sin omega t or its, you know it is some harmonic or its some you know pulse load all of those kind of things, but unit load. So, that is why I am saying that, this gives you the special variation and otherwise it is this thing.

So, this is how this entire thing is defined, so e I is the now how do, we solve this problem, now let us see let me give it a displacement and the displacement is where, this is z 0 t and u x of t is equal to psi x z 0 t. So, this is my displacement at a particular point, so if I have displaced particular point given by this the inertial force is automatically equal to m bar d x, which is infinity symbol length at a distance x.

So, that is m bar d x into u double dot x of t, so that is the inertial force that, you have what about due to the flexure, what kind of a load is set up over here, if you look at their the internal force what should I say or call that spring force and that is actually M x at a particular point, M x by definition is given by E I into del square u by del x squared. So, that is the flectional moment due to this displacement and you know well, that is it will just let go with these are the forces.

Now, the question then becomes that, we found out the forces on the another deformation, now what do we do, well the best way to do is write down an equation of equilibrium, because ultimately if you look at this looks likes an equation of equilibrium. And we already looked at last time, that you can use the principle of virtual displacement to solve this particular problem. So, then how does the how do we get that well.

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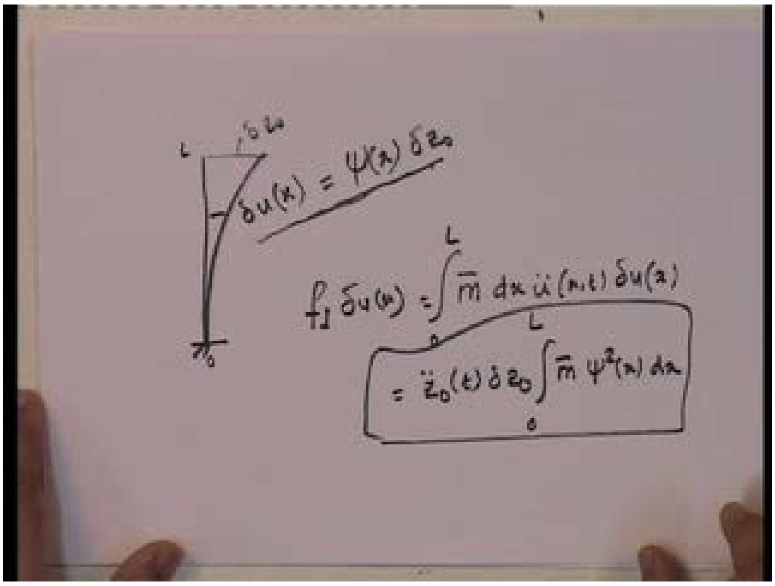

Let us see virtual displacement look like, I will give at virtual displacement where, virtual displacement is note that, this does not vary with x, because this is the virtual displacement right. So, those are the forces under going this virtual displacement an what kind of virtual displacement pattern do I give well, I will give psi x into del z 0 right, in other words over here, this is the l z 0, I will give the same I mean the this will be given by this, because even when I give it up virtual displacement, the pattern has to remain the same as the previous 1.

So, let us see what do I get well the work done by the inertial forces into del u x is going to be equal to m bar d x u double dot x of t into del u x. And this for interest able, so we have to go from 0 to this is 0 and this l, so it has to go from 0 to L. So, what does this become lets substitute all of these terms, see u x of t is given by psi x z 0. So, u double dot is going to be equal to x I x into z 0 naught, so this is going to be equal to z 0 naught t that goes outside and del 0 is del 0, so this goes outside and inside, what do we get 0 to l m bar.

Now one of them is psi x coming from z 0 and the other is del use I x, so I basically, get psi squared x d x and that is the work done by the real inertial forces, under going the virtual displacement. So, this is what we get over here, now let us look at the next step and which is the work done by the bending moment undergoing the virtual displacement and if we look at that then this is what happens, let us look at what happen in the next term.

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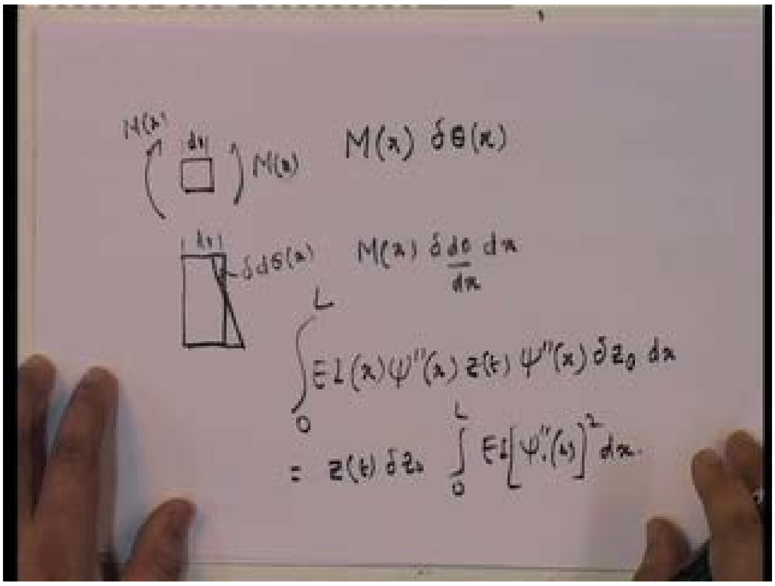

A moment M x, so the work done by the M x is going to be equal to del theta x, note that this is M x into del theta x, now this del theta x is actually equal not del theta x, but it is equal to in other words, if I look at this is my d x note that, I have M x here and M x here and note that, what we have is that I have M x and M x. So, if you look at it, this particular term, if you I am going to draw it larger, you will see that this is equal to if this is my del d theta x right. This is the so then the work done is actually, M x into del d theta x by d x into d x right.

That is what we have, because if you look at it this del theta x is going to equal to over length d x whatever the rate of change of the rotation, that is given over here, now m if I substitute, this is the work done by the internal forces. So, if I look at M I, M I is equal to E I into psi prime z of t, because again, if you look back. If u x of t is given in this form u double prime, that is del u x of t by del x square is going to be just the di phi prime by d x squared into z naught, d I phi is given by psi prime. So, this into z t is my m x and this one, if you look at it is going to be equal to what is d theta by d x by definition is del squared d by del x square. So, this entire thing becomes then nothing but psi prime into del psi z 0 into d x and this for in the, so this entire thing has to be from 0 to l. So, if we look at this, this is going to be equal to z of t del 0 to l, E I double prime x, the whole squared d x, this is the work done by the internal forces and now let us look at finally, the work done by the load.

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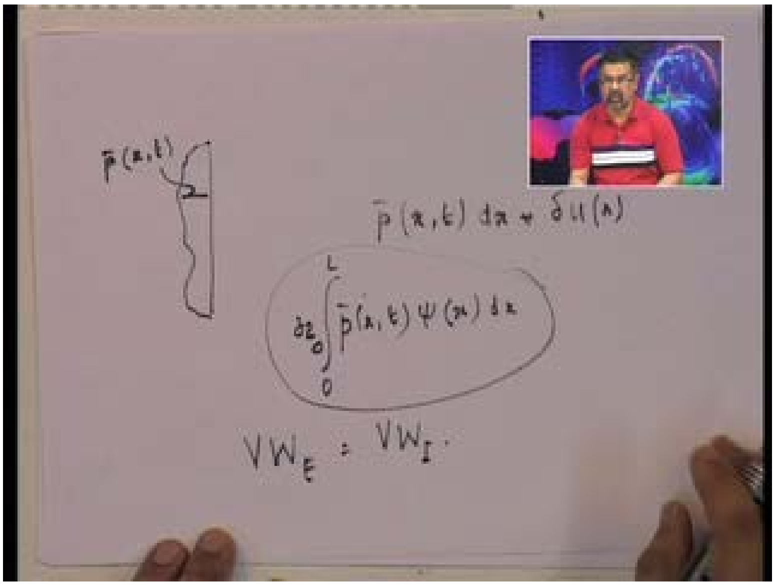

Now, the load is what at any point the load is given by p bar x of t, so now, if you look at this p bar x of t, so then the load is given as p bar x of t into d x, that is the load per infinitesimal length, that is the load and what is the work done its equal to del u x. So, if you look at this, this is equal to p bar x of t into psi x d x 0 to L, because this is obviously, integrated and I have del z 0 here. So, now, the question becomes that look the work done by your external a forces is always positive and the work done by the internal forces is always negative, because so that is why this one, actually becomes something like this.

So, the external forces are these and that internal forces is the work done by elastic forces, which is the bending moment and the work done by the inertial forces. So, putting all of those down I get this, integrated from 0 to L m bar x psi squared x d x into z 0 does 0 plus integral from 0 to L, E I x d x into z 0 does 0, these are the internal forces and the work done by the external forces is 0 to l p bar x of t psi x d x. So, in other words if you really look at it, this is all of them have del z 0, so that you know, I am just l made that, because del psi 0 is an arbitrary displacement.

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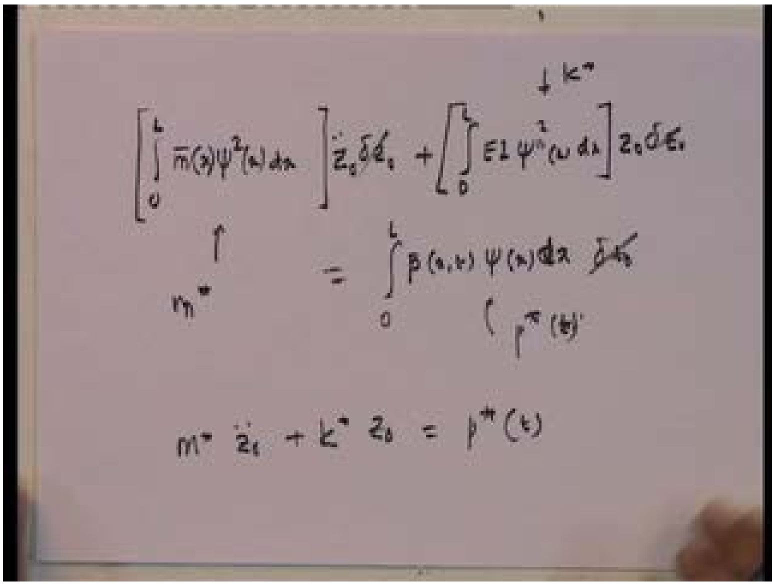

So, this term is my m star, this term is my k star and this term is my p star, so the final equation then becomes m star z, so that is what happens over here. So, now, the question then becomes that, this is this is the equation of motion and my m star, I will just rewrite these terms that, we have looked at we will just describe these.

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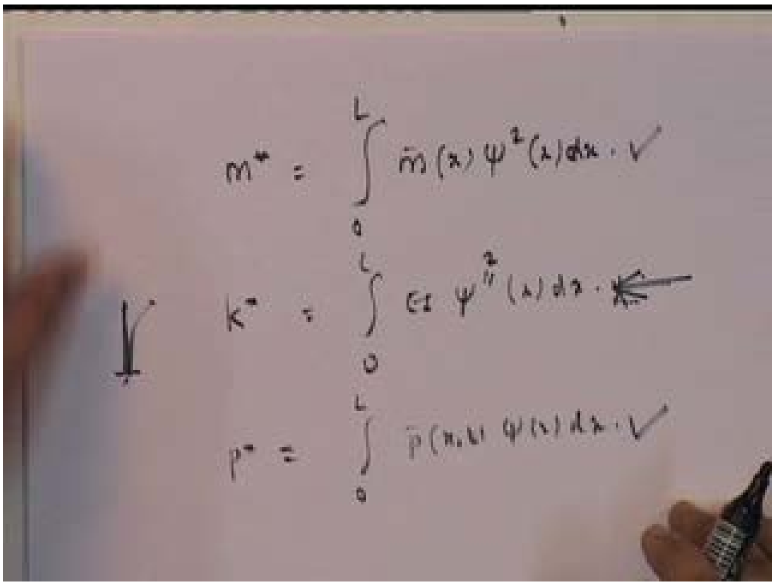

If you look at m star m star is equal to 0 to L, m bar x psi squared x d x k star is equal to 0 to L, E I psi prime square x d x and p star is 0 to L. Now, if you look at this particular thing, you will see excepting for this and this are things that, we have developed already for the generalize single degree of freedom system, which was for the rigid body assemblages right. These are exactly the same as that, now the question then becomes is that this is only term, that comes in and this comes in due to the deformation in other words, when you deform a body in this form, it tries to go back to this. So, in a way this is like a spring constant equivalent spring constant that represents the inertial the elastic force, that comes from the that comes in to the equation, because k star in to z. So, k star is like a equivalent spring force, now having develop this, let us now try to solve this problem for a specific case.

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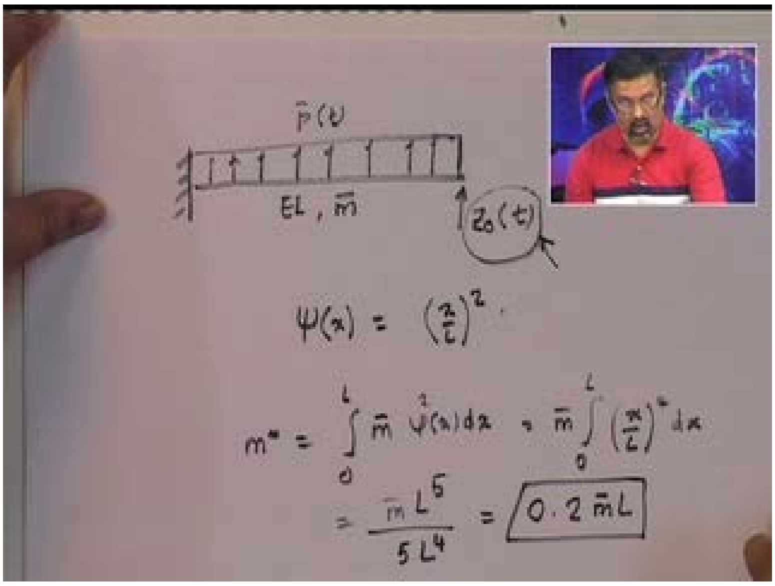

Let me take this specific case, I am going to put u d L on it and then I am going to put this as my degree of freedom. So, this is p bar t, in other words it is a constant, it the its not function of x. And I have E I and m bar also as constants, in other words what we have is that this is uniform bar with uniform distribution of mass subjected to uniform distributed load and I want to write down the equation of motion in terms of this. And I shall use psi x to be equal to x upon L squared, I am assuming psi x to be equal to x upon L squared. So, if I do that, well, let us see what does the equation happen in terms of this, now you know without going in, let us go and look at the way we have developed it.

So, if you look at it, this was m star is equal to 0 to L m star psi x squared d x right. So, if we look at this what does it become m star is a m bar is a constant. So, m bar goes outside and this becomes 0 to x and this is psi squared, so this becomes psi L forth the x.

So, this becomes m bar L 5th upon 5 L 4th, so this is equal to 0.2 m bar L, so my m star is equal to 0.2 m bar L.

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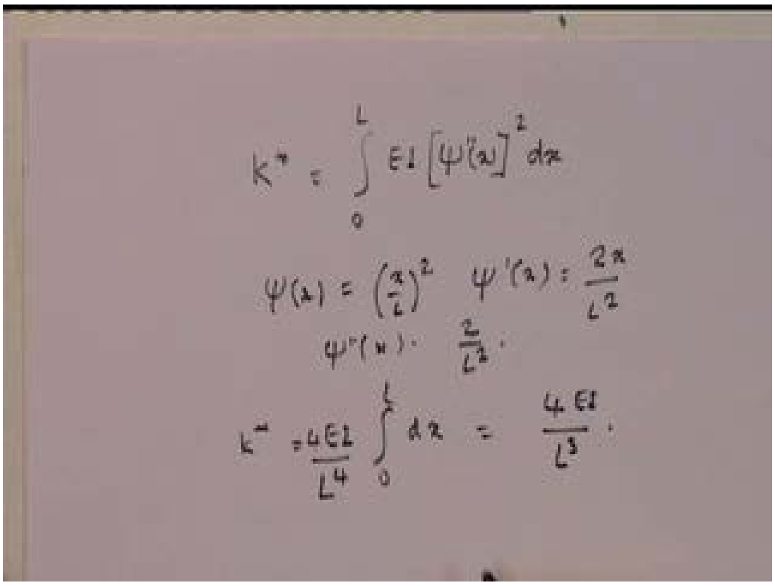

Now, what is my k star, my k star is equal to 0 to L E I, so if your psi x is equal to x by L squared, then psi prime x becomes 2 x upon L squared and psi double prime becomes 2 upon L squared. So, if I just do this, so 2 upon L, the whole squared this basically becomes k star is equal to E I 4 E I upon L 4, 0 to L d x, which is equal to 4 E I upon L cubed.

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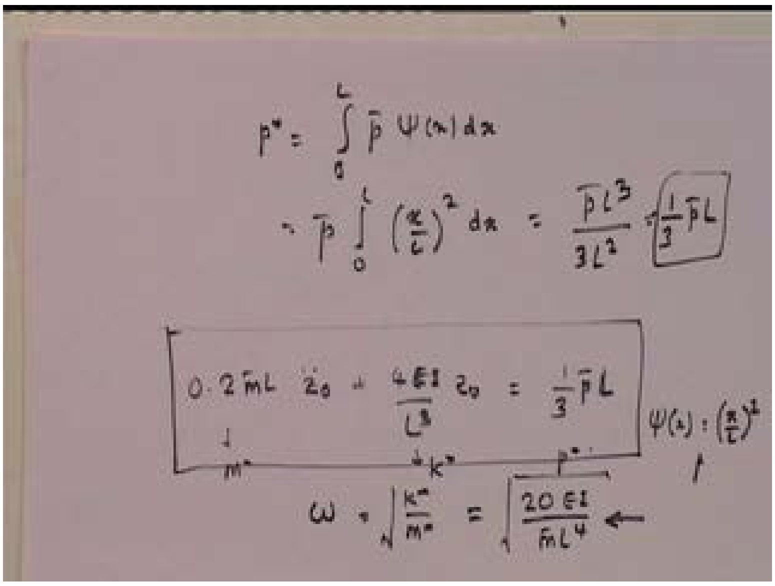

So, that is my k star and what is my p star, my p star is going to be equal to 0 to L p bar psi x d x. So, p bar goes out 0 to L x upon L squared d x, so this becomes p bar L cubed upon 3 L squared, so this becomes 1 3rd p bar L, so this becomes this. So, automatically what is my equation look like for that particular for this particular problem, it become its look like this 0.2 m bar L z naught plus 4 E I upon L cubed z naught is equal to 1 3rd p bar L. Now, let us look at say units, that we have m bar L is mass units, because m bar is mass per unit length m bar L is mass per unit.

Now, E is Newton per meter squared I is meter 4, so this become Newton meter squared by meter cubed become Newton per meter. So, this is units of a linear spring constant Newton per meter and p bar L is load p bar load per unit length into l is load. So, you see this entire thing this is my k star, this is my m star and this is p star, let us look at something interesting, what is the free vibration frequency going to be equal to, it is by definition k star upon m star and this, if you look at it is equal to 4 E upon L square divided by this basically becomes 0.2 goes up.

So, this becomes 20 E I m bar L 4, so now let us look at this Newton meter squared, this is k g meter force. So, if you look at this basically becomes radiance per second square root, this basically becomes per second squared and if you look at this is, so this is my if I use psi x equal to x by l squared by omega turns out to be this. Now, is this the true frequency, first frequency of vibration of this particular cantilever, well no why well this is approximation, the approximation, it is in approximation to the true value.

Now, now lets us look at put another situation, let us look at using the 1 upon x by L as my value. So, if you look at that, then what we get is, so I am going, you know, I just used psi x equal to x upon L the whole squared, note that all 3 that, we have define before or equally valid, in terms of equation, I mean as a ship function. So, psi x is known as the ship function by the way, because it is it defines the shape with unit amplitude corresponding to the degree of freedom. So, that is why it is call the ship function psi x is the ship function.

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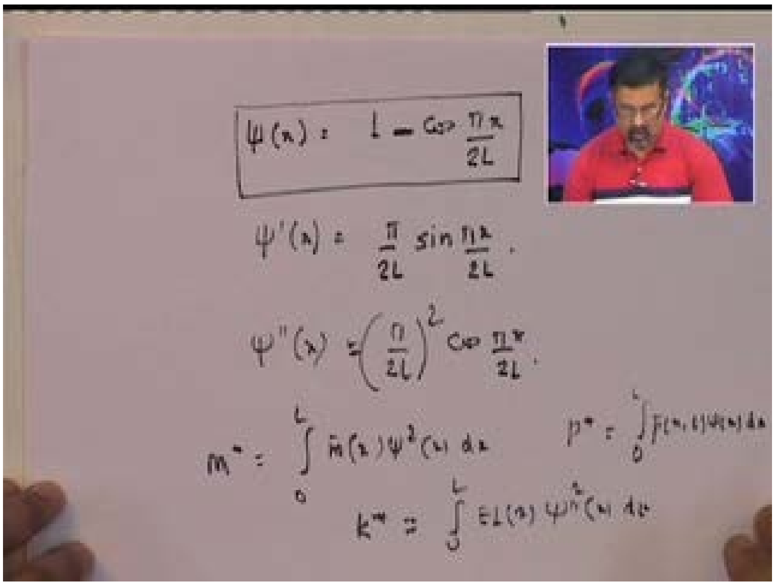

So, now let us look at the same problem excepting, that I am going to define psi x to be given by 1 minus cosine pi x upon 2 L, I am going to define this as my ship function what why am I doing to because I want to show you, that whatever this equivalent single degree of freedom, that I get is actually an approximation. Note that earlier when we looked at for the specific things about the generalized single degree of freedom with rigid bar assemblages.

Those where exact single degree of freedom, the psi x was unique here, you know psi x all that is has to satisfy is it has to satisfy the boundary conditions and it has to satisfy that corresponding to the degree of freedom its 1. So, that all the shape function also satisfy, so let us see plugging psi x equal to and lets us see psi prime is equal to pi upon 2 L sin pi x by 2 L and psi prime is equal to pi upon 2 L, the whole square cosine pi x upon 2 L. So, this are I am deriving these up from an also the fact, that m star is equal to remember m bar psi squared x d x k star is equal to 0 to L E I x psi prime squared d x and p star is equal to 0 to L p bar x t into psi x into d x given that, let us now try to find out what values do I get for with this psi x.

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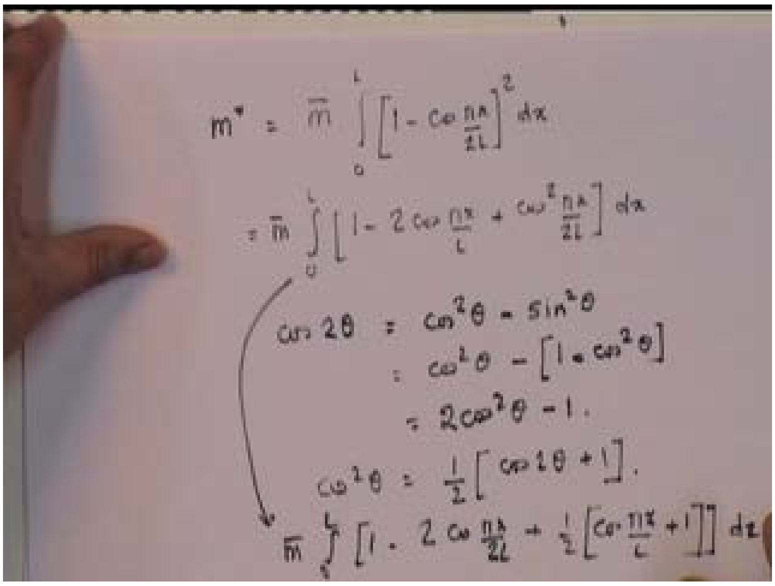

So, let us try to find out m star, so now, m bar comes outside, because it is a constant 0 to L 1 minus cosine pi x whole squared d x. So, therefore, what we get is equal to now this becomes what this becomes equal to m bar 1 minus 2 cosine by x by L plus cosine squared by x by 2 L. Now, note that cosine essentially, if you look at it becomes nothing but when I am going to use another this thing and that is that, I know that cosine 2 theta, I am going to define, this is equal to cosine squared minus sin squared.

And note that since this is equal to see cosine theta if I subtract 1 from this if anyway let me rewrite this becomes 1 minus cosine pi x remember that, this is 1 minus. So, what you get is you get it equal to 2 cosine theta minus 1 and so if you look at this cosine squared theta becomes half of cosine 2 theta plus 1. So, I am going to plug that in, so what is that going to be equal to this implies that, this is equal to m star 0 to L 1 minus 2 cosine pi x upon 2 L plus half of cosine pi x upon L, note that 2 of this is equal to this plus 1 d x.

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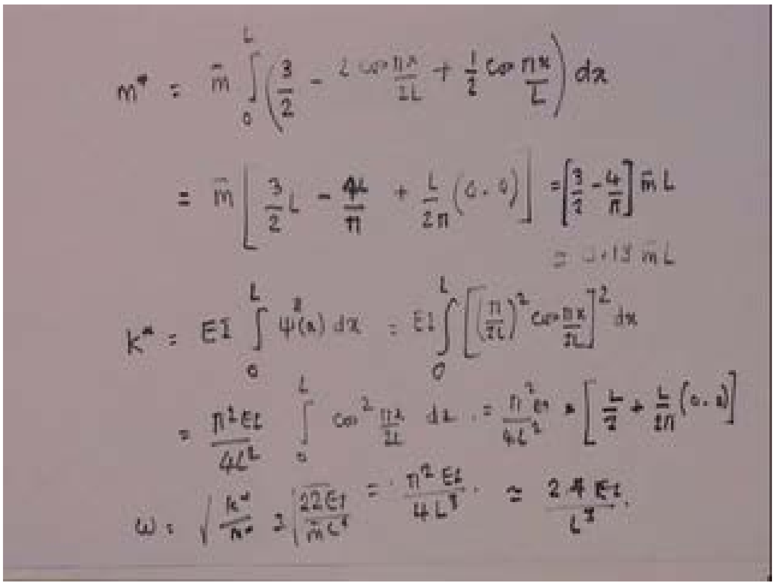

So, if you look at that what we get is this is equal to so m star becomes equal to m bar 0 to L, this becomes 3 by 2 minus 2 cosine pi x upon 2 L plus half cosine pi x upon L. So, this is what we get, this into d x, and if you look at this becomes m star and then inside, I am going to put 3 by 2 L then minus now let us see when I differentiate this becomes 2 L.

So, this 2 and 2 this becomes 4 L upon pi and this integration becomes sin pi by x, so if you look at x this going to be 0 and when you do this, you get sin pi by 2. So, that becomes 1, so this is minus 4 1 by 0 and plus on this side again, you have L upon 2 pi and look this is cosine. So, when I integrate this I get sin pi x upon L, so when I put x equal to L, I get sin pi 0 and so both are 0 minus 0. So, what I get is equal to 3 by 2 minus 4 by pi, so this becomes 3 by 2 minus 4 by pi m bar L and if you look at this one, this one turns out to be equal to 1.5.

And this particular 1, this anyway let just leave, it in the this fashion, we will derive this later, but you will see that 3 by 2 and 4 upon pi is approximately about 1.32. So, this is going to be a 0.18 about 178 L, this is approximately about 0.18 m bar L, that is what you get and if you look at k star, k star turns out to be equal to E I 0 to L psi square double prime square d x. So, this becomes equal to E I into 0 to L, hence I double prime, we already done psi double was prime is equal to pi upon 2 L cosine pi x upon 2 L, the whole squared d x.

So, this is going to be equal to pi squared up on 4 L square, so I am going to take that outside, so this is going to be equal to pi squared E I upon 4 L squared into 0 to L cosine squared pi x upon 2 L into d x. And this we have already seen what this equal to pi over 2 sorry, E I 4 upon L squared is equal to this one become half and half the other one disappears. So, this going to be equal to this into inside is going to be equal to half L by 2 plus L by 2 pi 0 minus 0. So, what we ultimately get is equal to pi squared E I upon 4 L cubed, which is equal to approximately about 2.4 are there about E I upon L cubed.

And if you look at this, if I look at omega, it is going to be 2.4 divide by point m bar, so it is going to be E I m bar, so if you look at omega, it is going to be equal to k star up on m star and this is going to be equal to approximately just over 20, about 22 E I upon m bar L forth are pointed. Different you see that, this value is different from what we have got earlier and if you look at it. So, in other words become why well why should not be different they both approximation and they are not the same shape function, no way in the in the case of this thing, you saw that the psi prime was a constant, here the psi prime varies with cosine.

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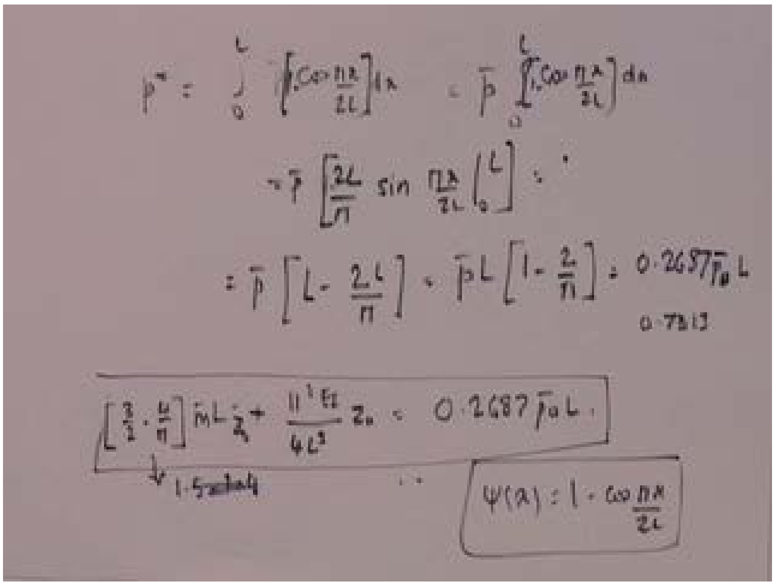

So obviously, there are the same function, let me finally, get the p star, since we have been complete lets also do the p star, p star is 0 by L p bar into cosine pi x upon 2 L d. So, this becomes p star 0 to L cosine pi x upon 2 L d x, this is equal to p star into 2 l upon pi sin pi x upon 2 L 0 to L. So, what we get is its equal to I am so sorry, this is 1 minus cosine, so there is a 1 minus cosine, so there is a 1 minus.

So, this basically becomes equal to p star into 1 minus the all becomes L, actually L upon 2 pi into L. So, this is equal to pr prime 1 minus 2 upon pi, which is equal to 0.2687 p bar L. So, the equation then becomes the following, it becomes 4 minus pi m star plus pi squared E I upon sorry, z double naught 0 plus pi squared upon 4 L squared z 0 is equal to 0.2687 p bar p naught upon L.

So, this is what we get as our definition of this thing, so this becomes essentially, this point exactly becomes equal to point this is 0.7313, so this becomes 1.5 minus 2 times. So, this becomes 4 upon pi becomes 4 upon, so this becomes 1.4, we will do this later, I will get you exact values for this, the question then becomes at this equation that, you get for, which psi x equal to 1 minus cosine pi x upon 2 L, this is the equation that you get, looks completely different. This is in other words, what we are looking at, I shall leave it up to you to do the 3 x upon the other psi x term, that I have given, that is 3 x upon L, the whole square minus 2 x upon L the whole cubed, that psi x, I leave that is an exercises for you to do. So, I shall stop over here.

Thank you very much, bye.