Structural Dynamics

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Lecture - 18
Generalized Single Degree of Freedom Systems Equations of Motion

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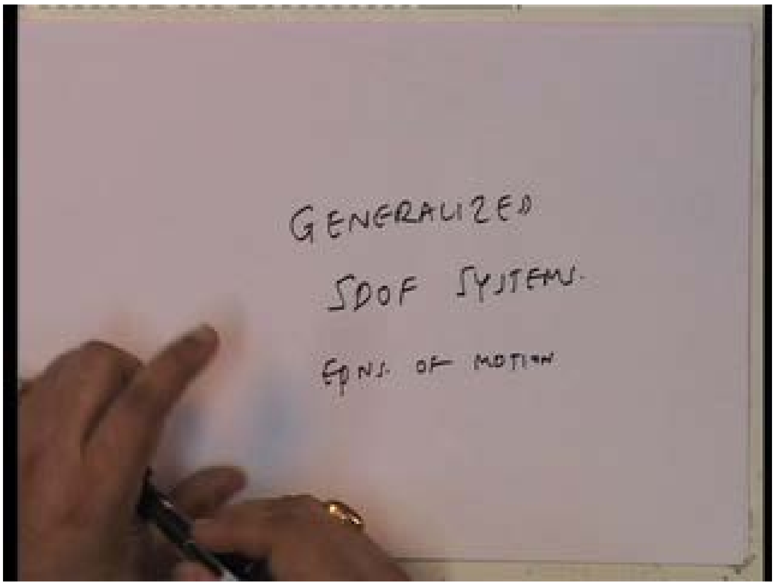

Hello there, so we are continuing our discussion on the equations of motion for Generalized Single Degree of Freedom Systems, and we are continuing with Equations of Motion. And today what I am going to be doing is, I am going to be looking at the possibility of developing the same equations of equilibrium. But you know last time I looked at trying to take see I due this particular problem right, I took this particular example problem. And try to solve it by writing down moment has equation of equilibrium. Now, you know in general it is already fine for us to say that look, this is you know your, it is a determinate system. So, we can find out we can actually write down equation of equilibrium, but you know it is always a good idea to use some alternate approach to the equation of equilibrium.

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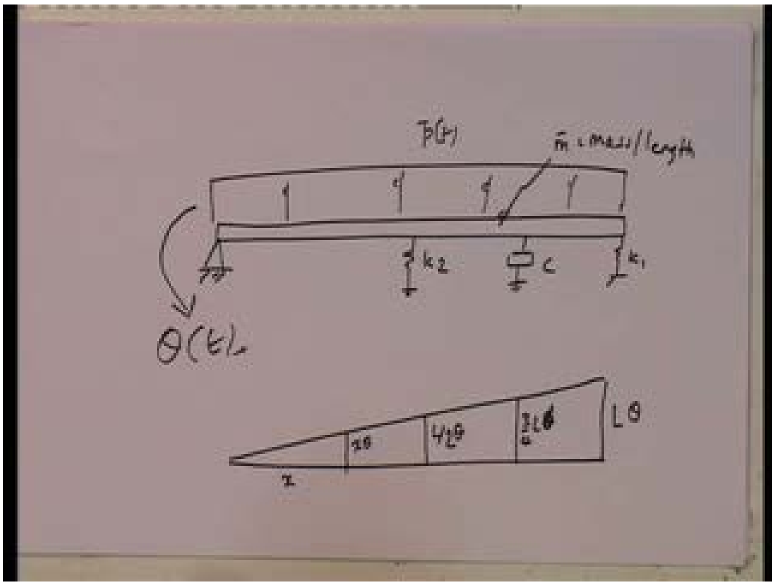

And the most common approach that we know today is the principle using the principle of virtual work to replace the equation of equilibrium, and the specific method that of the virtual work principle that replaces equations of equilibrium is the principle of virtual displacement. So, what I am going to do is I am going to solve the same problem, so the same problem that I solved in the last lecture, excepting that in this one we are going to look at, it from the prospective of what?

We do not look at from the prospective of using the principle of virtual displacement. So, now let me go ahead and look at it, and let me define theta as my degree of freedom. So, the first and for most is what is the first thing is to look at, what is going to be the displacements at various points. And in fact at any point x is going to be x theta, so that is the displacement pattern that we get and that is only going to be vertical right.

So, now let us write down the first thing is writing down the forces that the system has and you know without be labeling the point we have done this often enough by now we should be able to write this down what do we have, we have the u d L which is given by p bar of t. Then what do we have, we have force here, this is equal to k 2 into L by 2 theta, then we have k 1 into L theta, then we have c into 3 by 4 l theta dot. And we have m bar d x into x theta double dot.

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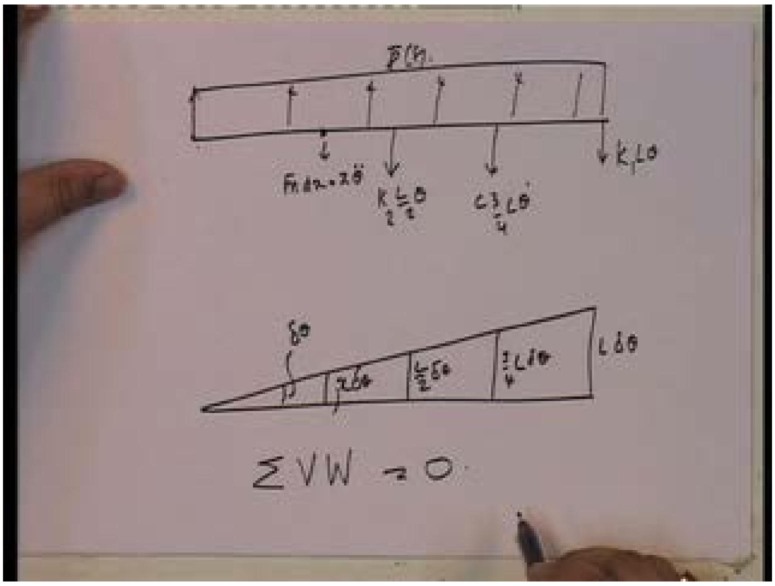

So, these are and this is of course, only an infinitesimal this has to be there's across the board right, this is everywhere its m bar d x into x theta, so these are my forces, these are the forces under which the system is in equilibrium. So, how do we use the principle of virtual displacement well, these forces which are an equilibrium now, there is actually you know these are the forces. And if we look at, this I give this system a virtual displacement and I will call this virtual displacement delta theta.

So, what is this L into delta theta this point L by 2 delta theta this point 3 by 4 L delta theta and any point is x into delta theta any point. So, therefore, what we now do is write down what, the virtual work equation and virtual work is summation of virtual work is equal to 0. So, each of these forces under go these virtual displacements and we see what is the a work done. So, note that this force downwards undergoing upwards displacement does negative work. So, remembering that I can now write down, all the terms and that is the virtual work.

The virtual work is equal to what, let see, let me take the first one is going to be k 1 L theta into L delta theta. So, this is the real force undergoing the virtual displacement and this is negative because it is negative work. So, that is the first one that is the next one is, k 2 L by 2 theta is the force undergoing the displacement and this is also does undergoes negative work, then we have c 3 by 4 L theta double dot, undergoing 3 by 4 L delta theta that real force undergoing.

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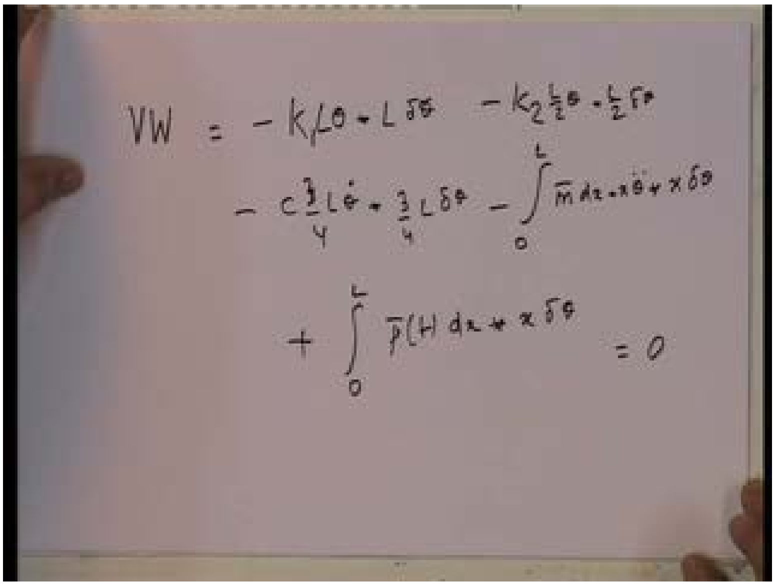

Remember that this is the real force and force into, the virtual displacement of that particular point. Then we have the following which is m bar d x into x theta double dot, this is the inertial force, this into x delta theta that is the work done by the infinitesimally length and so; obviously, that integrated over the whole length. And then so these are the work done by all these and finally, the work done by that, that is going to be equal to that is going to be plus because note that this goes undergoes positive work.

So, this is going to be plus now let see, p bar t d x is the load and that is multiplied by, the virtual displacement and that integrated over the whole length, is equal to that is all the work done that is equal to 0. So, now, this particular case I am going to just put together all the terms together and then if you look at it, this becomes the following. Integral 0 to L m bar x squared d x theta double dot this is minus, then I have minus c into 3 by 4 L squared the whole squared into theta dot minus and I am going to put inside this, k 1 L squared plus k 2 L by 2 the whole squared theta plus 0 to L p bar into x d x and the entire thing into del theta is equal to 0. Note, del theta exists and all of them, so I am taking the del theta all side is equal to 0, this is the virtual work equation that I have.

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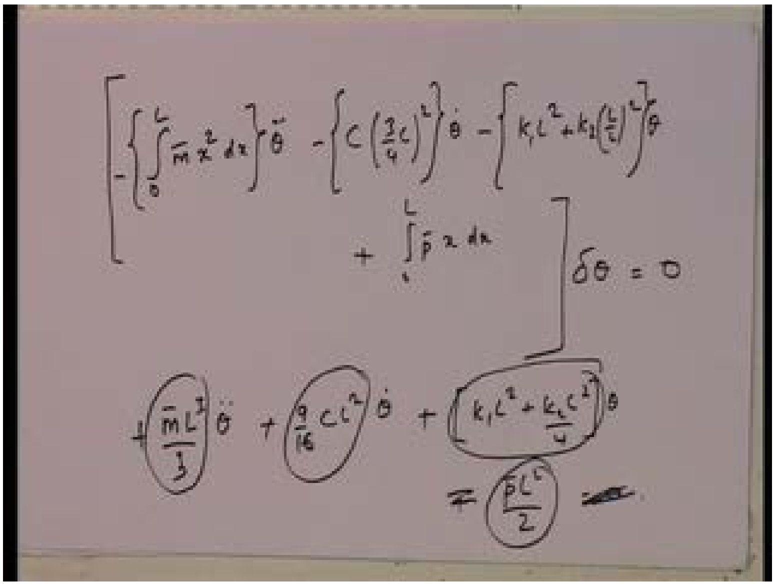

Now, the virtual displacement is an arbitrary displacement since, it is an arbitrary displacement for this to be equal to 0 it implies that the inside has to be equal to 0. If I put the inside equal to 0 what do I get, I get the following I get m bar L cubed by 3 minus theta double dot minus 9 by 16 c L squared theta dot minus k 1 L squared plus k 2 L squared by 4 theta dot plus p bar L squared by 2 is equal to 0. And if I rewrite this what I get is that you know I am going to just put it, in this only this plus this plus this is equal to this that is all I get, this is my m star, this is my c star, this is my k star and this is my p star.

Note, that I got exactly the same equation that I got by the equation of equilibrium, where I took moments about the left point to be equal to 0 because that that is; obviously, in order to hinge it has to be equal to 0. I have use the principle of virtual displacement, where I all I have done is a given a displacement that satisfies all boundary conditions and I have given a virtual displacement, that also satisfies all geometric boundary conditions and I have just taken the virtual work done and put it equal to 0. And, so for theta equal to 0 I get exactly the same equation. Now, let us try to see if I use z of t whether I get exactly the same.

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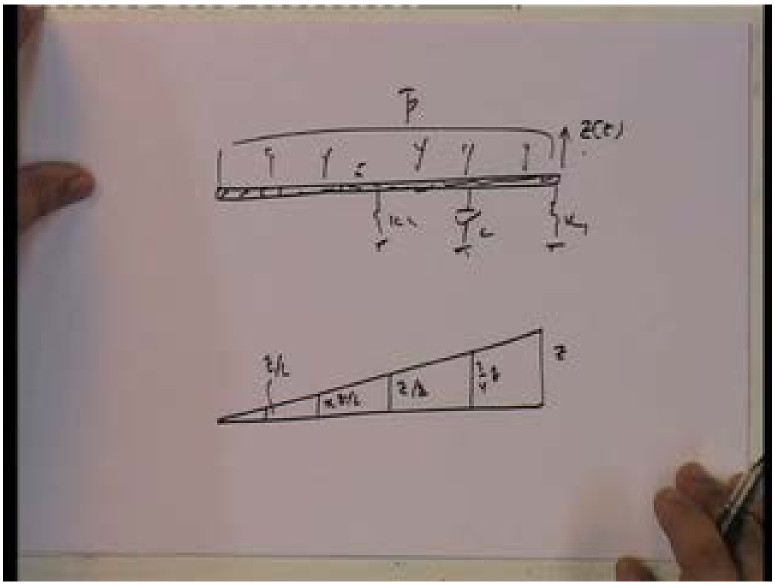

So, I am going to redraw the problem, excepting that now I am taking z of t as my degree of freedom. So, I have this, I have this I am not even writing down the values we already know and this is m bar and I have load, all of this is a exactly the same problem that I am solving. So, I am not writing down all other things, we know this k 2 k 1 c we know, this excepting that now I am using the z of t as my degree of freedom. So, if I look at the displacement again, you know this is the same as what we have done last time z this is z, this is z by 2, this is z by L and so at any point it is x z by L z by 2 this is 3 4'th z and having done that the force is, in the various you know things are going to be the following.

It is going to be here k 1 z, this is going to be c into 3 4'th z dot this is going to be k 2 into z by 2 and the mass, infinitesimal mass is going to be m bar d x that is the mass, it undergoes x by L into theta double dot. And of course, I have my load p star these are all my forces that I have. Now, I am going to use the principle virtual work, so principle of virtual work, I am going to give a virtual displacement corresponding to the degree of freedom, and find out the work done by all the forces.

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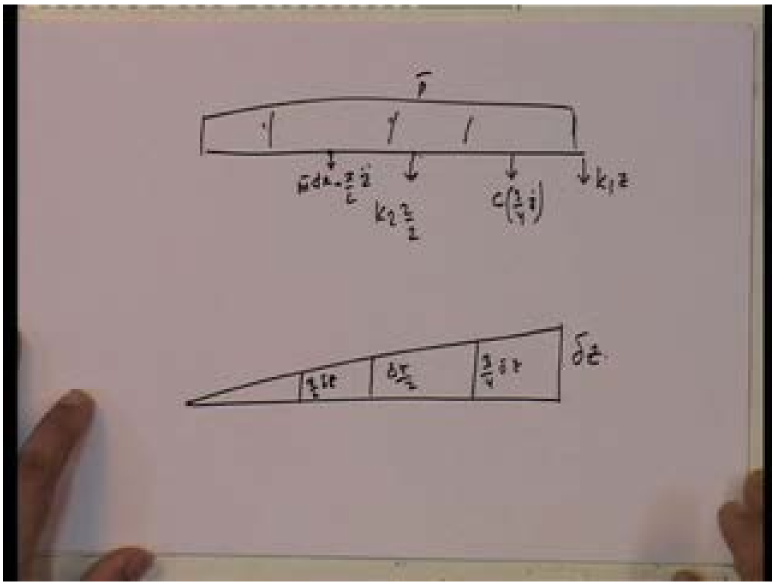

So, if this is delta z, this is going to be 3 4’th delta, z this is going to be delta z by 2, this is going to be equal to x by L delta z. And so if I take the work done by all the forces what do I get, I get it equal to let me star of k 1 z into delta z by the work done is negative because if we go back this is this direction this is upwards. So, always the work done is negative.

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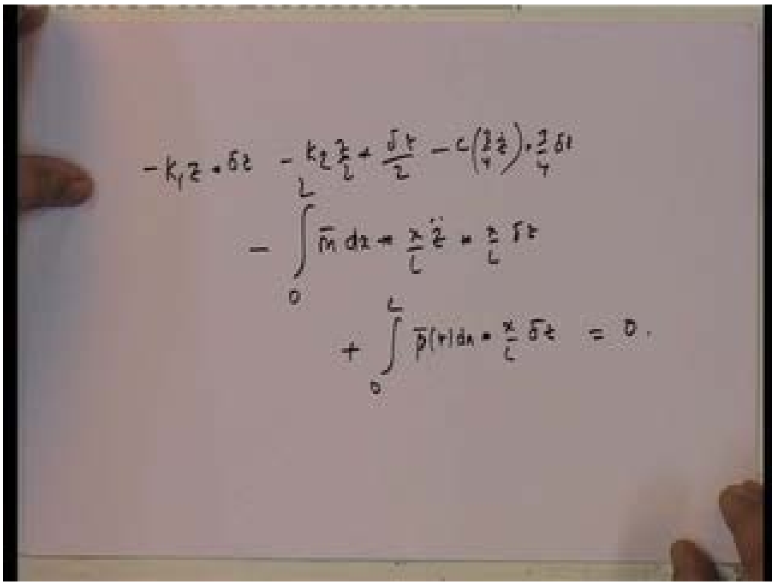

And therefore, what we have is minus and then we have, minus k 2 z by 2 into del z by 2 minus c 3 4’th z dot into 3 4’th del z then what do we have, then we have minus m bar d x into x by L z double dot that is the force multiplied by x by L delta z that is the integrated from 0 to L that gives me the total work done by the inertial forces. And finally, the work done by p bar of t d x and that is equal to what that is to be multiplied by x by l del z. So, that is what we have this is the work done by the infinitesimal load. So, that has to be integrate from 0 to L and that is equal to 0, if I rewrite this particular a function in another a format what do I get I get the following.

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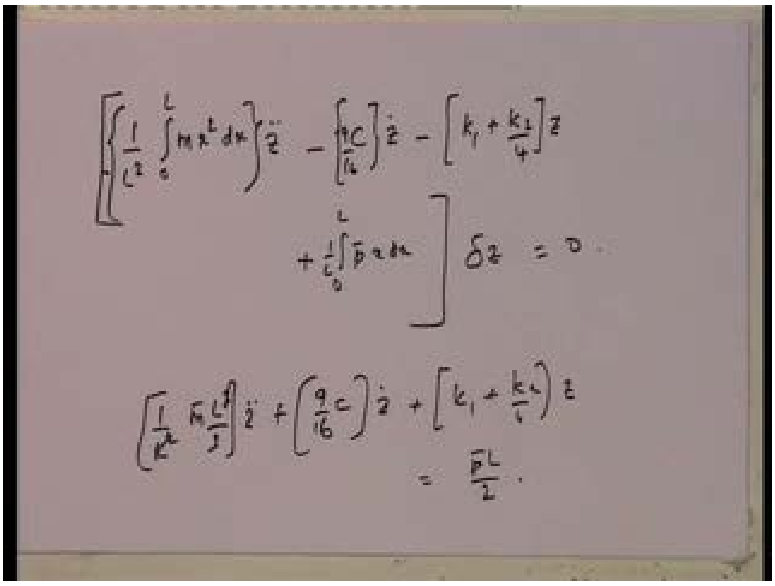

I am going to put them all in this specific part 1 upon L squared 0 to L m x squared d x and this is negative of course, all of this negative d x, x squared d x into z double dot minus this is going to be equal to c into 3 4'th by 3 4'th. So, it is going to be 9 by 16 c z dot minus k 1 plus k 2 by 4 z plus 0 to L 1 upon L p bar x d x and this entire thing into del z equal to 0, this is the equation there is all at is done is put together, the terms in the previous equation and to put them together.

Now, of course, this is an arbitrary virtual displacement. So, obviously; that means, this has to be equal to 0 and all I am doing is this is equal to 0, I am just putting these on the other side and so what I get is the following, I get 1 upon L squared m bar L cubed by 3 minus z double dot 9 by 16 c z dot plus k 1 by k 2 by 4 z is equal to x squared. So, this becomes p bar l upon 2 and so this one essentially this goes it becomes m bar l upon 3. Now, note something very, very interesting. And that is now, if I look at this equation, let me look at this equation very, very carefully.

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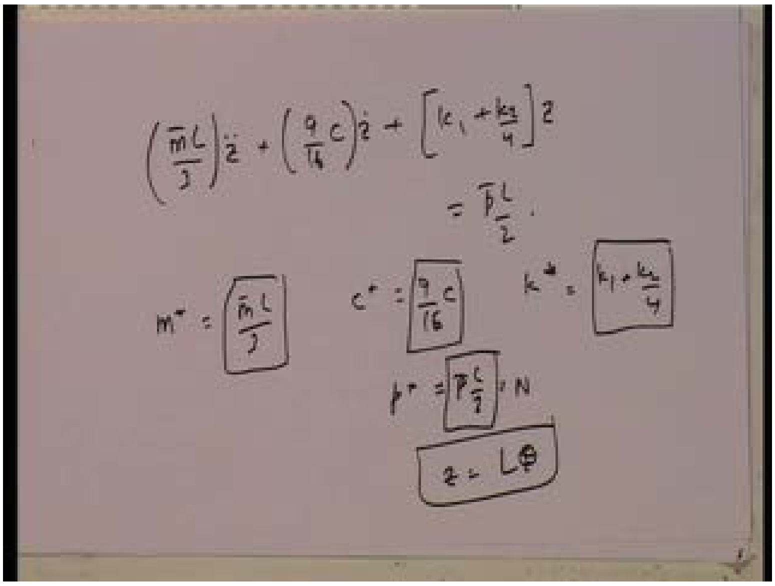

This equation tells me that, it is m bar L by 3 into z double dot plus 9 by 16 c z dot plus k 1 plus k 2 by 4 z is equal to p bar L by 2. So, what is m star m bar L by 3 c star 9 by 16 c k star is equal to k 1 plus k 2 by 4 and p star is equal to p bar L over 2. Let us look at this, this is the beauty of the principle of virtual displacement, the beauty of the principle of virtual displacement is, when we use theta has the degree of freedom the m stars that you get or all given you know the m star is mass momentum inertia, k star is rotational spring constant and p star is a moment.

Now, if z which is the displacement degree of freedom, if that is my degree of freedom then what is my m star my, if you look at its m bar is k g per meter. So, k g per meter by k g it is beings this is k g, this is Newton second per meter, this is Newton per meter, this is Newton per this is Newton. So, if you look at it the entire equation essentially becomes very, very clean this is mass because if it is z this is mass.

So, this has units of mass, if this is displacement, this has unit of dash pot constant, if this is displacement to this has units of linear spring constant and if it is displacement the p star is a force and this is the beauty of this. Now, if you look at it and you know, I am I do not want to going where we will see that if I look at you know, the equations they look identical, if we were to use you know theta or z just the fact that z is equal to L theta, if we substitute that in you will get that equation back.

So, essential point that I want to make is that the, if for a generalized single degree of freedom you are this thing you are specific equation is consistent. If we, use the principle of virtual work which is principle of virtual displacement this for you get consistent, it does not matter where the degree of freedom is define, it does not matter as long. As the system is a single degree of freedom the equations of motion that you get are really identical, is just that they are you know in somewhere related to each other they are really the same equation of motion, written in different formats. So, that is all there is to it, now what I would like to do is once you have this I can now generalized the equations for m star and k star, etcetera.

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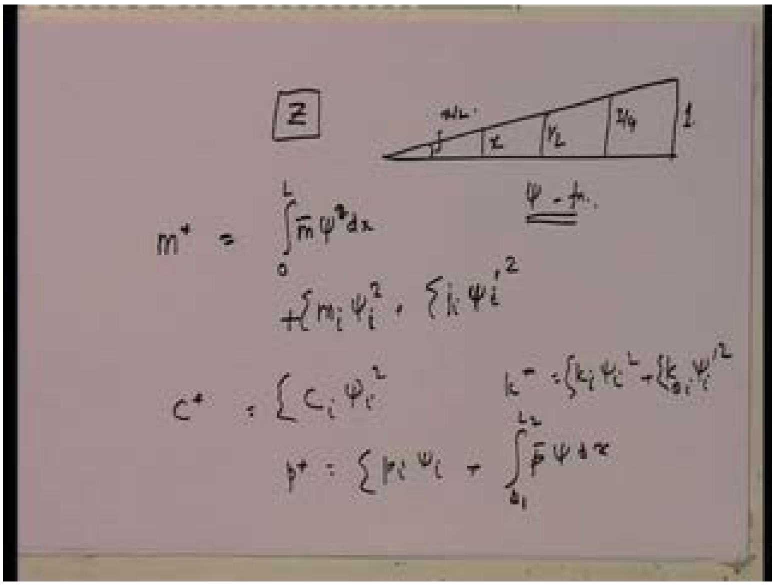

Now, this is where the beauty comes because now if we really look at it this is nothing, but if you have a displacement as a degree of freedom, then your m star if you have a linear mass at a particular point, a linear mass. Now, if we look back at it, now let me go back a little bit and define a something ((Refer Time: 25:03)), if you really look at this I could define this thing in terms of a xi times g at any point. Because, that xi times g if I look at it, I could draw it in this fashion this g and I take xi, xi I take it as 1 r 3 4'th any x..

So, this is my xi function, if we look at this the xi function then what is m star in this particular case, the m star becomes if we look at it m bar xi squared d x integral from 0. Let see, what is the xi function is going be xi in this particular case x squared, you put it a x and so m star then becomes essentially the following right. If you look at this, this is m star is going from 0 to m bar xi squared and you look at xi squared it basically becomes m bar xi squared, if you look at it become xi cubed it becomes m bar L cubed I am sorry this is not true this is x by L, so this is x.

So, now, m star can be written if you have m r it is in this fashion, if we have a linear m any moment m i is going to be equal to, what where is the xi I squared at that particular time. If you have mass moments of inertia, you just need to do the slope at that particular point of time squared and in a sense, what happens is you know you essentially land up having that m star, all you need to do is find out the shift function find out the values.

Similarly, c star is going to be equal to c i into xi i squared k star is going to be equal to k i into xi squared, if I have theta if I have a rotational spring this is going to be equal to xi I, xi prime i squared a slope. And p star is equal to p i into xi i, if there is a specific load multiplied by this, these are all summations, summation, summed of over all the things this is summation.

And then if I have loads they basically become p star xi i over the length, I mean over the length of which not 0 to L, let me also say it can be L 1 to L 2 depending on if you have a particular length over which the load acts. So, this is the beauty of this particular method now you know, so I have actually got expressions for specific problem. Now, given that I am actually now going to solve a specific problem that I am going to give you, and that is that let us put it down another problem and I did that example problem then I am going to solve.

Let me take a situation, when I have k, k. So, I am defining these as k, k and I am not putting a dash pot in this particular case it does not matter, this is a u d L and I have a loading, which I will say is of the following form, the loading is only here, where this is at this point the load is given as p bar of t. So, and this has m bar constant mass per length and what I have to do is and what is my degree of freedom, I will define my degree of freedom as the rotation here, for define that degree of freedom rotation here.

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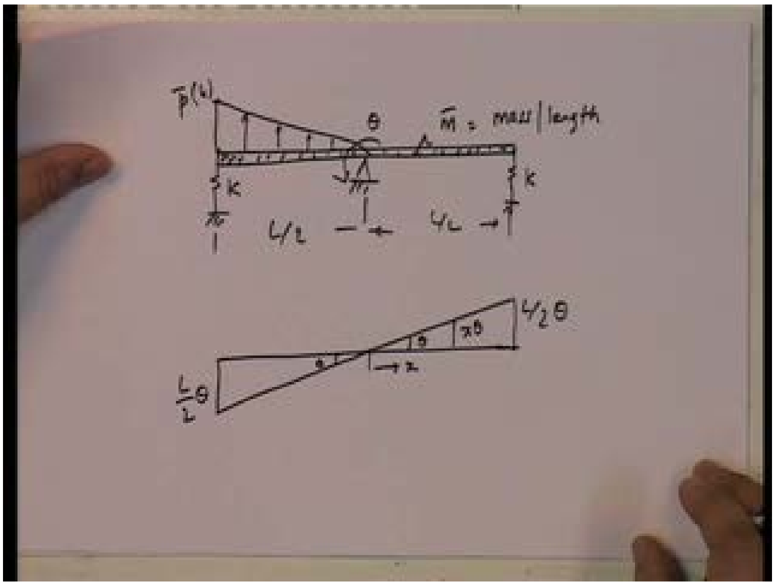

Now, I am going to solve this problem in two ways, what I am going to first 2 is I am going to solve this problem from, first principle and then I am going to use these expressions that I have I am going to use these expressions in it. So, now let us go step by step, in that is first step what well let us give it the displacement and if this is unit, if this is theta then at this point I have I forgot to put down the values and so I am going to put them as L by 2 and this is L by 2. So, the total length of the bar is L, so there over here I have L by 2 theta, this is also L by 2 theta and at any point, so my x I am going to take as positive in this direction. So, this is x theta, this is my displacement.

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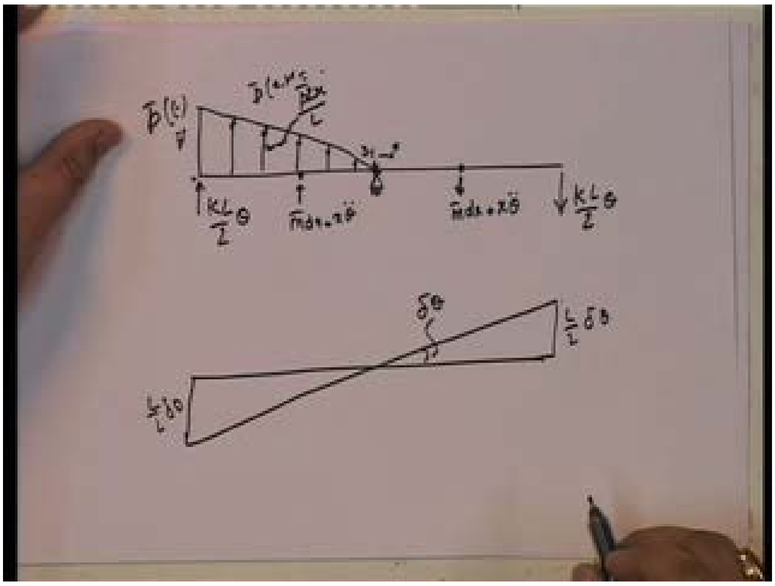

And if I look at it, so therefore, so now, I need to find out the forces I am going to write down the forces over here and the forces are, the following I have since this is undergoing displacement in this direction, the force is going to opposite. So, the force is going to be here and this force is going to be equal to k L by 2 theta. Note that I have put both of them as identical springs k, k.

Now, here this spring is being subjected to this direction. So, it is going to opposite in this, so this force is going to be equal to k L by 2 theta, then I have p bar of t, so this is my upwards and so where this is p bar of t. So, this basically becomes then given by p bar it is p bar at any point, the intensity is given by p bar x 2 x upon L look at it because you put x equal to L by 2 this becomes p bar that is p bar, if you put x equal to 0, which is 0 x starts from here, right x starts from here.

So, if we go to x equal to 0 you get 0 and so that is the intensity, intensity at any point p bar x of t is equal to this. So, that is the point, so we have that, so these are the forces and on top of that what do we have, well if I look at it if I look at this particular direction, this is rotating in this direction right. So, if I take in this direction, so this is going this is going up, so this is going to give rise to m bar d x into x theta dot and if I look at in this direction.

Since, this is going down this is going to give rise to m bar d x into x theta double dot in the opposite direction. So, understand from here to here the forces are, inertial forces are upwards and from here to here the inertial forces are downwards. So, this is the very important point to note that because it is this is rotating about this point, so I am got all the forces. So, now, I need to just put down the displacement, virtual displacement and the virtual displacement is del theta. So, I know that this is L by 2 delta theta, this is L by 2 delta theta, so now, so this is the work all the work done by all the forces right. So, note that these forces will do negative work, these forces will also do negative work, so every term is negative right. So, I am going to write that down.

And if I write it down what do I get I will take the first one, this is k all of them are negative. So, all of them are negative k L over 2 theta into L by 2 delta theta minus k L by 2 theta into L by 2, these are the 2, these are the this one into this negative, this one into this negative. So, both of them are negative and in this form and then I have the mass, so the mass infinitesimal mass in this from this 0 to L by 2 I have the work done has negative and on this also, the work done is negative, this is equal to x delta theta and this is x delta theta, this goes under, this goes opposites all I need to do is it is 2 times.

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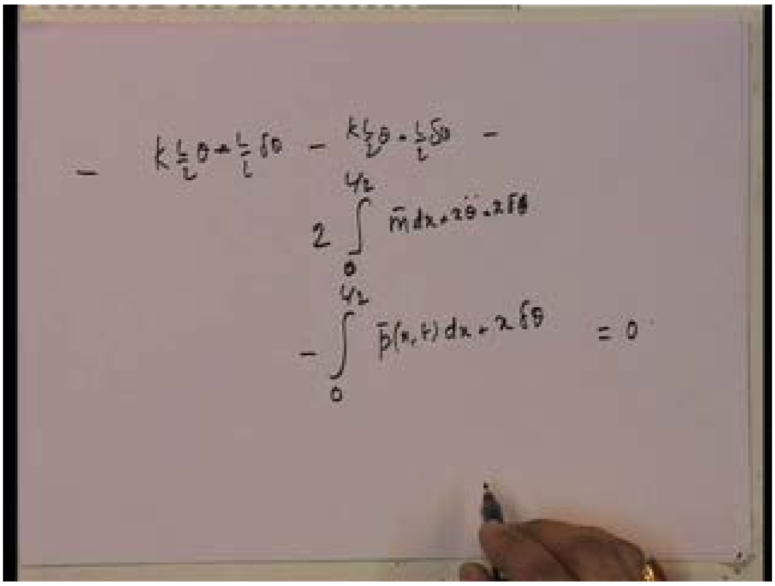

So, this is going to be equal to 2 times 0 to L by 2 and this is going to be the force is, m bar d x into x by theta this is the force, into x delta theta this is the work done. So, integrated from 0 to L by 2 and twice because the both sides are doing identical work and then finally, I have minus and this is the, work done and this note is again negative work done. So, this is going to be 0 to L by 2 now, note that p star x by t that is at any point, this is going to be equal to d x that is the load and that into x by x delta theta is the work done and so this is equal to 0.

Now, so if I put this in properly what I get is the following, I get now m bar x squared 2, so into l upon 2. So, this is going to be, if you note that this is going to be x squared, so this is going to be x cubed by 3, x cubed by 3 going from 0 to L by 2 x cubed becomes upon, you know L cubed upon 3 and a you know x cubed upon 3, in and x cubed is l upon 2. So, this becomes what do we get l cubed upon 24 2 times.

So, this becomes equal to m bar L squared by 12 into theta double dot and I am taking all the negative I am going to making it positive. So, plus now k L by 2, so this becomes 2 into L squared, so this becomes k L squared upon 2 theta is equal to now the integral over here, basically becomes if you look at it becomes p bar x upon 12. So, this becomes p bar goes outside, p bar upon L goes outside and this goes from 0 to L by 3 and this is x squared d x.

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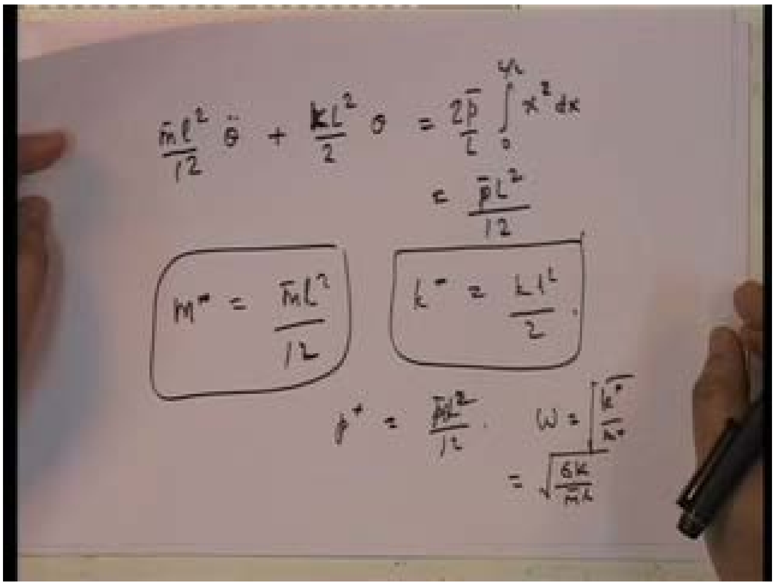

And, so this becomes what p bar L cubed by 3, this is become L by 8 x cubed by 3. So, this becomes L by 24, so this becomes p bar oh I am so sorry I made a mistake here, please forgive me that this goes from 0 to L by 2 right exactly it is 0 to L by 2. So, this becomes l by 24, so what I have is essentially p bar L squared by there is a please note here, I made a mistake this 1 is 2, so this there is a 2 coming here 2 p bar. So, what I have is 2 p bar, so this becomes 24, so this becomes p bar l squared by 12.

So, what I have is my m star is equal m bar L squared by 12 k star is k L squared by 2 k and p star becomes equal to p bar L squared by 12. So, this is my solution now, if you want to look at what omega is equal to k star upon m star square root and so this becomes equal to a k L squared upon 2. So, this becomes square root of 6 k upon m bar L that is your omega. And a note that understand that p bar is a function of t, so it is not as if we do not have a function of t, so now, let see if I can use the equations that I have used. So, let us look at it, I am only going to find out the k star because everything else is again this thing.

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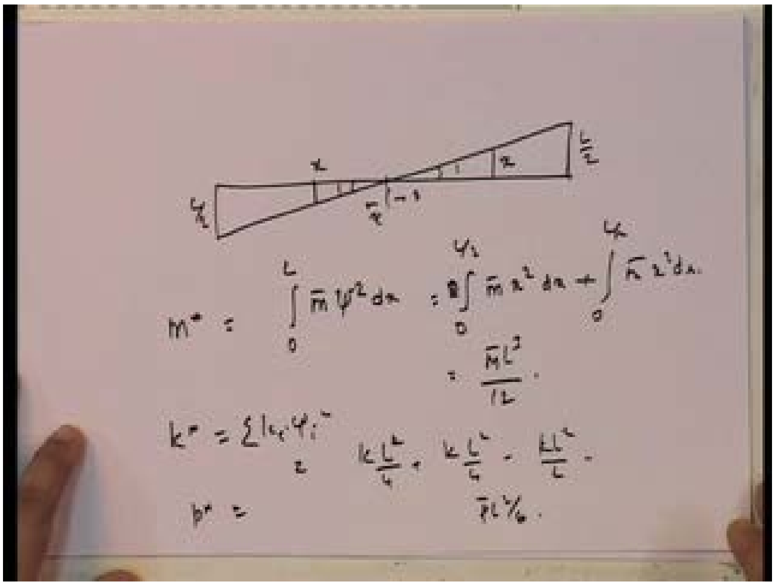

So, let us look at what this is, this is a situation where, what is xi, xi is the one where you give unit displacement corresponding to the degree of freedom and you find out what are the corresponding values. So, corresponding values are were here L by 2, L by 2 and here, this is going to be equal to x and this is going to be equal to x. So, if you look at this, if you look at m star, m star's is what it says 0 to L m bar xi squared d x.

Now, note that 0 to L in this particular because xi is actually a little bit different from here. So, why we get it is, it is going to be equal to 0 to L over 2 because x goes from 0 to L by 2 in both directions. So, m bar xi is what x squared d x and twice of that because I have m bar x 0 to L m bar x 0 to L by 2, so 2 of them, so that becomes essentially or I can pull it as 0 by L by 2 m bar h squared d x. So, note that you know x always remember that x is what, it goes from 0 to whatever is my thing.

So, if you look at this, this you get as m bar L cubed by 12, look at k star, k star is k i xi i squared summation. So, k i is k into xi squared L squared by 4 plus k into L squared by 4 is equal to k L squared by 2 which is what you get. And p star is you know the same thing you will get it, you know you will get p bar L squared by 6. So, therefore, xi by this is xi if you use this, so you see whether you used this formulation or whether you used the other formulation, from first principles you should get the same equations.

So, I am going to stop here in terms of the generalized single degree of freedom, I have spent enough time on this generalized single degree of freedom. And I shall give you some problems to do, which will help you in your own way get the I mean get you know understand, how to get m star, k star, c star, p star and all of those terms. So, that once you get m star, k star you see the point that goes back is the following and that is the point that understand to make.

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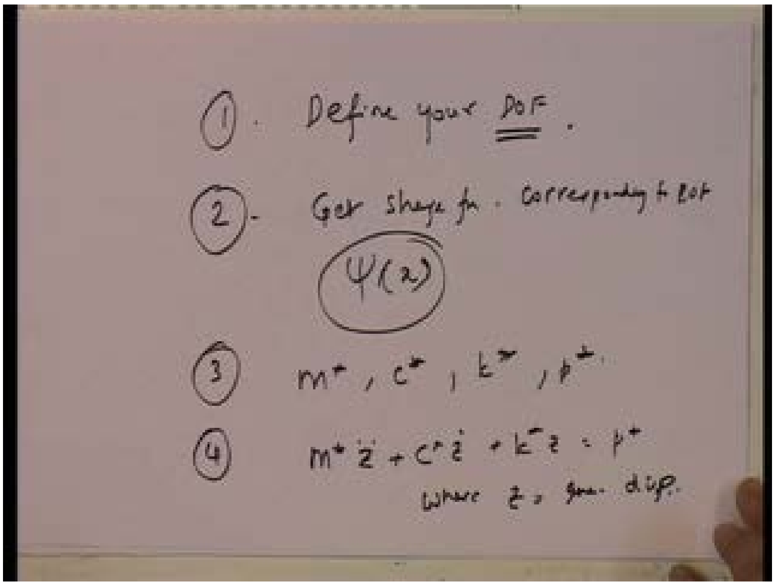

So, now I have shown you that given a generalized single degree of freedom problem first what we have to do, first is define your degree of freedom that is important because if you do not define your degree of freedom. So, therefore, your entire thing is based on thing, next get shape function corresponding to degree of freedom, in other words get xi x plot it, put degree of freedom equal to 1 and then put your system, you know put it equal to 1 and find out what are the values that you get a at different points along the length of the ball you know.

So, that is this first thing, third well get m star c star, k star and p star. Now, you can get it from first principles or you can get it from the formulation, the formula that, I have given you I would prefer you to solve it from first principles purely because it just mixed that much easier for you to understand, use the principle of virtual displacement and get it, so get your m star. So, what is there fourth well now you see this becomes, where z is the generalized displacement, note that z can also be theta in a particular case. So, in which case m star, c star, k star and p star will be appropriate. So, the point m becomes that once I have this what happens well.

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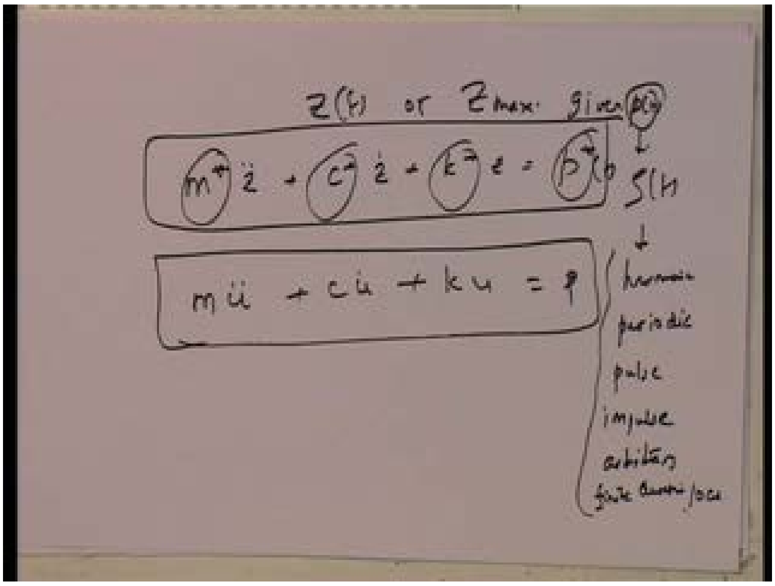

I have m star z double dot plus c star z dot plus k star z equal to p star, how is it any different from the single degree of freedom problem that I have solved. If I get this I should be able to solve for z t or z max given you know p star of t will have some form of xi t, which would be harmonic or periodic or pulse or impulse or arbitrary finite duration load, it does not matter, we have already spent 14 lectures looking at how to solve this for a variety of different kinds.

So, therefore, the solution once you got the this, the whole problem is actually in getting the equation of motion getting, these once you get these solving this is identical to what we have already done earlier. So, that in a sense brings you down to the solution that look, whether I have single degree of freedom given by that one you know, one story, one rigid beam a frame or whether I have a more general kind of problem, which can still be define by a single degree of freedom.

Of course, note that in such a situation you have to have it is an assemblage of rigid bars because you know the bar can only move in a particular manner and you know exactly how it moves. So, that is one, so it is an assemblage of rigid bars, so we have actually solved the even getting equations of equilibrium for general problems. Next time I am going to show you that this generalized single degree of freedom problem can be extended, for other kinds of problems.

Thank you very much bye, bye.