Structural Dynamics
Prof. P. Banerji
Department of Civil Engineering
Indian Institute of Technology, Bombay

Lecture - 17
Generalized Single Degree of Freedom Systems Equations of Motion

Hello, today we are going to be talking about development of equations of motions again and these will be for...

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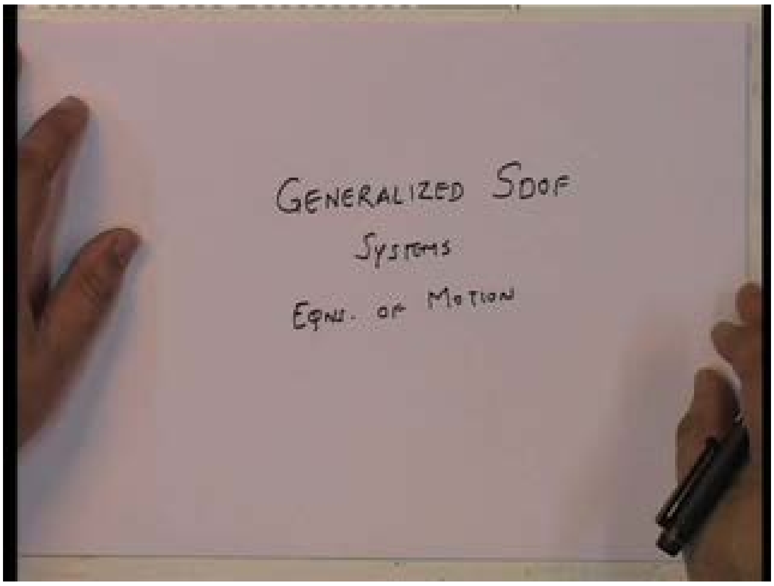

So, generalized single degree of freedom systems equations of motion, that is what we are going to be talking about in today. Now, what is the generalized single degree of freedom system, up till now I have shown a single degree of freedom system as one, which has this form.

I am showing all of these, actually in reality you do not have this dash pot, but I am just showing this and then we define this as power u of t and a load p of t and for this, we got the equation. So here, this is our degree of freedom and since there is only one degree of freedom, this is known as a single degree of freedom system and this is what, we been looking at in this particular situation. Now, the question then becomes is the following that, is this the only kind of a situation that we can have.

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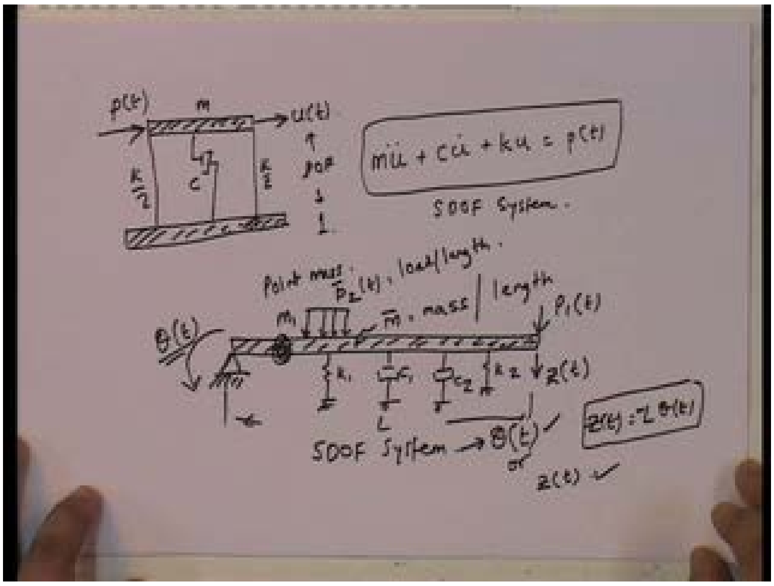

Let me look at another situation, let me take a rigid bar, a completely rigid bar supported on a pin end and having, I am just randomly putting some, this bar has m bar has the mass per unit length. And then let me put some point mass, so this now let us look at this particular system, if you look at this, is this a single degree of freedom system. If you look at it, this is a rigid bar, so what can it do, it can rotate about this point. Now, note that, one of the fundamental point is that, it can rotate, so I can define theta t as my degree of freedom.

So, this is still a single degree of freedom system, but the point here becomes that, it is not obvious what my m, c and k are and p are in this particular case. So, let me have a situation, where I will put a mass, I will put a load also, because unless there is a load. So, I will put a load over here, I will call this p 1 of t and I will put some u d l, which I will call as p bar 2, this p bar 2 is load per unit length. So, now the question becomes that, look this is going to be an equation, it is still single degree of freedom system, because I can define this.

But, the question then becomes that, is this the only degree of freedom that I can define, no I could say that, instead of this, I could also define this and call it z of t. So, I could have a single degree of freedom system with theta t as a degree of freedom or z t as a degree of freedom. Note that, they are not independent degrees of freedom, because if you look at it, if I call this length as L then we know that, z of t is equal to L theta t automatically, where minus L theta t, where theta t is positive in this direction, it will go up in that direction.

So, it might but so they relate to each other, however I can choose this or this and it does not matter which one I choose, I can choose either one of them. So now, how do I write down the equation of motion for such a structure. So, let us now look at this particular problem, I will redraw the problem, I will define some length, because if we do not define length, it does not make sense.

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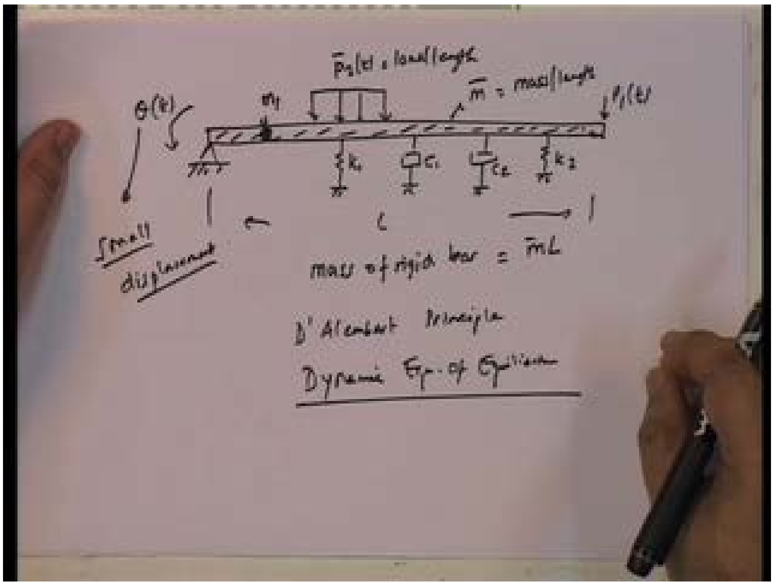

So, let me just redraw and draw it a slightly larger scale and I will call this as k 1, this will be c 1, call this c 2, call this k 2. And so I have a load here, which is p 1 of t then I have this as mass m bar, which is defined by mass per unit length. And since I will call this length as L, it is obvious that the mass of the rigid bar is equal to m bar L, obviously. Then I am going to put in another point mass over here, which is going to be m 1 and then I am going to put a load, which is going to be p bar 2 t, which is load per length and I will define my degree of freedom in the first instance of time by theta t.

So now, how do I solve this problem and the way I solve this problem is by using the D' Alembert principle, which is essentially dynamic equation of equilibrium. Note that, this by definition is small displacement, so now, what do I do, to begin with, in the D' Alembert principle what does it say, deform the body, displace the body equal to theta and the dynamic equation equilibrium is at any instance of time take theta. So, then theta becomes a specific value theta time t, take the displacement pattern and see, what are the forces that are acting on the body. And then they can equilibrium, write down an equation of equilibrium, so these are the steps that we are going to follow.

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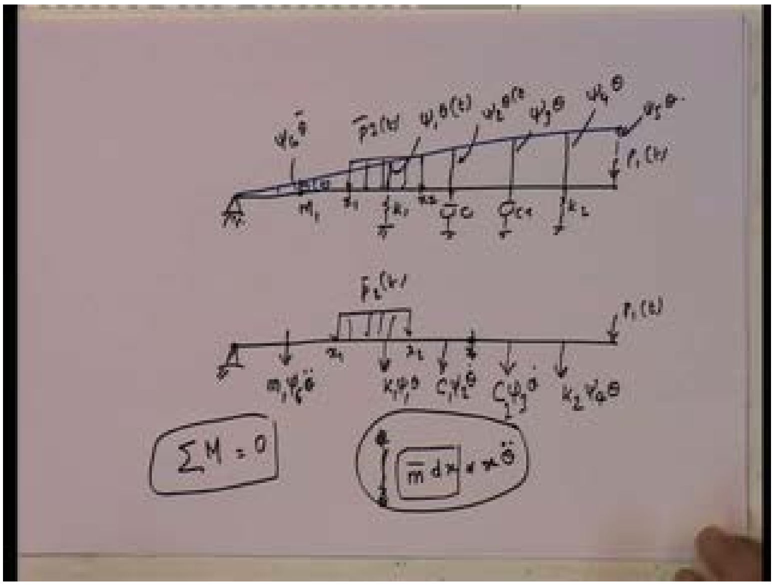

And if you follow those, so let us look at this, so then this becomes the following that, I have this bar, normally represent the bar by a straight line and if I displace it and it is note that, this is small displacement. So, this is my theta t, this is the small displacement, in other words this point where does it go, it goes here, every point over here goes like this. And so if I were to plot, this is where k 1 is, this is where c 1 is then this is where c 2 is, this is where k 2 is.

And therefore, and this is where the mass m 1 is, the load is here and what do we have, the other load is here and this is p bar 2 t. Now, let us look at it, so under this what I have is, at various points due to theta t are called these points as equal to this one, is equal to psi 1 into theta t. This point is going to be psi 2 by theta t, this point is going to be psi 3 theta t. I am not going to write down t anymore, because we know this going to be psi 4 by theta, this I will call as psi 5 by theta and this point I will call this psi 6 and here, I call this as point x 1 and x 2.

So, the point here becomes that, once I know theta t at any instance of time theta t, I can find out the displacements of all the points. Once I know the displacement of all the points, let us see what happens. If this bar, if this spring is being moved up by psi 1 theta, now I am going to write down all the forces that this structure is being subjected to, structure here p 1 of t. Let us look at this one, this is being extended what is going to happen, this is going to have an opposing force, which is going to be equal to k 1 into psi 1 theta.

Let us look at this one, here if this is a dash pot and this is the displacement then the velocity will be given by what, psi 2 into theta dot, because psi 2 is a constant, psi 2 depends on where this positions is. So, this is going to be psi 2 theta dot is going to be the velocity, so the force, since this is going in this direction, this opposes the force and that force is going to be equal to c 1 psi 2 theta dot. Similarly, we are going in the details, this is going to be psi 2 theta 3 theta dot, this is going to be k 2 psi 4 theta and what else, this force going from x 1 to x 2 and this is p bar.

And note, what about this one, this mass, now let us look at this, if we look at this mass, this entire mass at any instant of place, what is it being subjected to. Note that, every point is being, let us take a mass at a particular point and let us call it x, the mass at that particular point over an infinitesimal length d x is going to be equal to what, the mass is going to be m bar d x. Because, m bar is the mass per unit length and d x is the mass and that is going to be subjected to what, it is going to be subjected to an acceleration that is given by x theta dot.

Remember, x is the position, so x does not vary with time, however theta t varies with time, so this is going to be subjected to this acceleration. So therefore, what we have is that, all of this particular one is going to give rise to an inertial force that opposes the motion and that inertial force is equal to this. And so therefore, this one is to be integrated from 0 to L to get the entire mass, that the inertial force, however this is not to be integrated to get the total inertia force.

But, understand the inertia force by itself does not give anything, on top of that, we have a mass m 1 here. So, m 1 is being subjected to what acceleration, it is going to be subjected to acceleration psi 6 into theta double dot. So, this is going to give rise to a load m 1 psi 6 theta double dot and these are the inertial forces that happen here. So now, these are the only forces that acting on the structure, so once I have all the forces acting on the structure what do I do, I have to write down an equation of equilibrium.

And what is the easiest equation of equilibrium in this particular case, easiest equilibrium equation is that, since this is the pin here, the moment about this point has to be equal to 0. So, at an instant of time t, the moment has to be equal to 0 instantaneously, so we are taking an instant of time t, remember that. So, this is not a static equilibrium, it is a dynamic equation equilibrium and this I remember, if we go back to the early lectures that I have talked about, the D' Alembert principle essentially boils down to a dynamic equation of equilibrium, it is an equation of equilibrium taken at instant of time t.

And since t itself is variable, instance of time t all that it gives me, is it gives me an equation that is time dependent, that is all. So, I am going to take moments about this particular point and if I take moments about this particular point and I am going to keep this on the top and show it at the bottom here, before we do this. And this is going to be if you look at this, these are the forces, I am going to take moment about the axis and so this is going to be equal to, take moments about this, this one is, all of them are going to give you clockwise moments.

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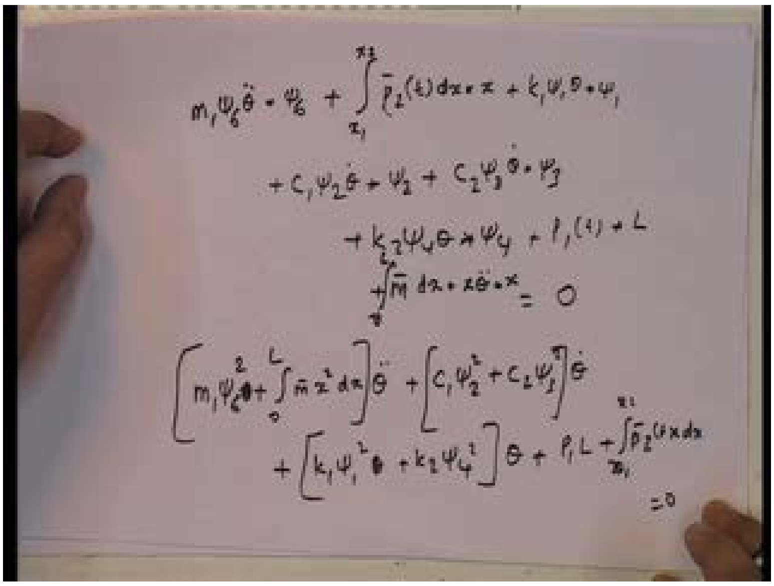

So, the first one is m 1 psi 6, this is the force and what is the kind of this thing, if this is psi 6 it implies that, the distance from this point is psi 6. So, this is going to be equal to psi 6, that is the only reason why this is psi 6 theta, because theta is this one, all of them are L theta, so this one is going to be psi 6. Since all of them are clockwise moments, I am going to take clockwise is positive in this particular case. And if we take anticlockwise as negative, all that will happen is that, you will get them all as negative, that is all, does not matter.

Then, I have this load, now this load is, if you look at this particular load, I could do one of two things, I could look at it in this fashion, p bar t d x, that is the load. And that at point x is going to have x as the distance and that I integrate, if I integrate from x 1 to x 2, it gives me, this is the load into the thing, that gives me the moment due to this load, there I have this one, this is equal to k 1 psi 1 theta. And since psi 1 is the distance from this point, so I am taking psi 1 as the distance, so that is gone.

Next is this one, so this is c psi 2 theta dot into psi 2, because that is the distance again, plus c 2 psi 3 into theta dot into psi 3 distance. Remember that, all of these implies that the distance from here to here is psi 1, only then will you get psi 1 theta, because theta is the distance. So, that is the point that we have and so therefore, this is psi 2 equal to psi 3, the last one is plus k 2 psi 4 theta into psi 4 as the distance and plus p 1 into, psi 5 is L actually, so this is L, is the distance.

Now, that is all the forces and that has to be equal to 0, so this becomes then my equilibrium equation and if I look at this, this is my equation of equilibrium. And if I rewrite this, what I get is m 1, I forgot one very fundamental point and that is, I forgot this load. So, at infinitesimal length I have forgot that one and so that one is going to be equal to plus m bar d x into x theta double dot. So, that was the force at point x and so that one multiplied by x and integrated from 0 to L would take all the masses.

Look, I have taken an infinitesimal mass, if you look a back at this, I have taken an infinitesimal mass at length x and found out that, this was the inertial force. So, this inertial force if I take this infinitesimal inertial force, this is length x, so what is the moment about this point, moment about this point is going to be m bar d x into x theta multiplied by x so that is what you have, m bar x. But note, that is the infinitesimal length, that is the mass at a particular infinitesimal length.

Now, this mass exists all across, so this would have been 0 to L and this is equal to 0, so what I have then is, putting down all the respective ones together, this becomes 0 to L m bar x square. Now, this is, let me take this out, m bar theta, this into theta double dot plus c 1 y 2 plus c 2 y 3 theta dot plus k 1, this is squared, psi 2 psi 2 psi 3 psi 3 this is squared then k 1 into y 1 into y 1. So, that is psi 1 squared, forget the theta here and then this one also has k 2 psi 4. So, this is plus k 2 psi 4 square into theta plus p 1 L plus x 1 to x 2 p bar 2 to x d x, this entire thing is equal 0.

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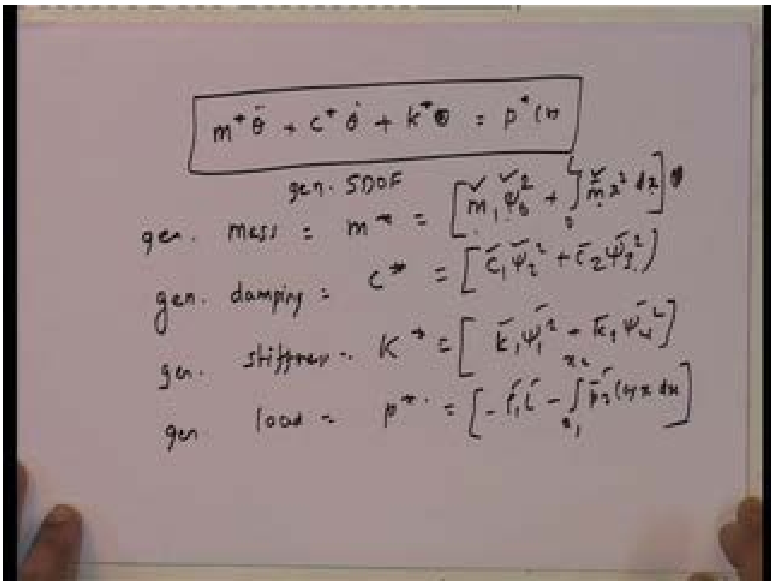

So, if I rewrite, if I write this in this form that, I am going to write this in the form, m star theta double dot plus c star theta dot plus k star theta is equal to p star of t. Note that, this looks like what, it looks like a single degree of freedom, where mass and this I will call, this is a generalized single degree of freedom. The generalized mass is equal to m star, the generalized damping is equal to c star, generalized stiffness is equal to k star and generalized load is equal to p star.

If we do that and if we take mass, dash pot, constant stiffness and load, look this is nothing but a single degree of freedom, excepting that now, what is generalized mass equal to. If we look at this, the generalized mass is equal to m 1 psi 6 square plus 0 to L m bar x square t x, that is my m star. My c star is c 1 y 2 plus c 2 y 3 square, k 3 is k 1 psi 1 square plus k 2 psi 4 square and p star in this particular case is equal to p 1 L minus x 1 to x 2 p star t x d x.

So, now here, all of these are given, this points where they are occurring also are given, so I know this, I know this, this one if m star is this thing, this goes out and this becomes essentially m bar L cube by 3. So, this one I know this this, I know this ((Refer Time: 28:53)) this, I know this this this, I know this and this, I know this. And if p bar is a constant if we take it out, this essentially becomes equal to x 2 square minus x 1 square upon 2.

So, if I write this in a sense, what I am doing is, I got an equation that gives a generalized single degree of freedom. Now, what I will do is, this looks very very complex and you know, but the point then I am trying to make is that, I will take an example and then I go ahead and try to put this in perspective.

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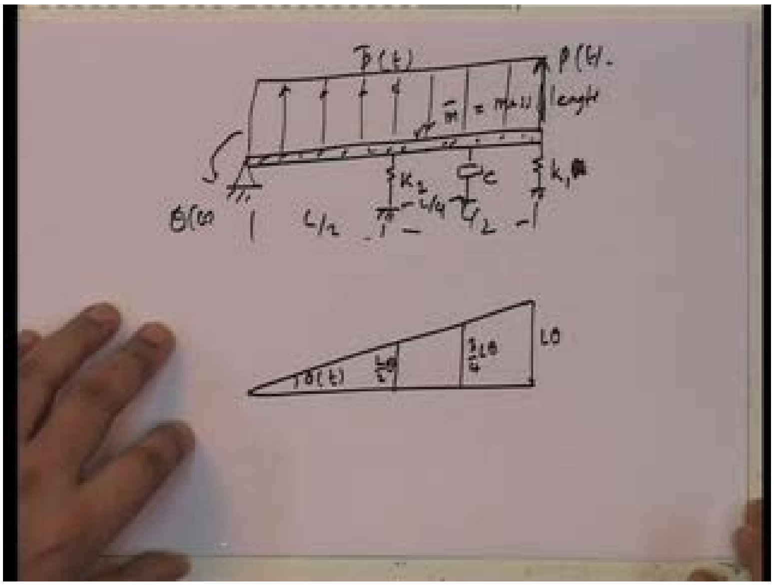

So, let me take an example, for the example I will take, so this is my structure and I am going to derive it with theta t as my degree of freedom. And note that, k 1 is the load here I mean, spring here, k 2 is the spring at the midpoint, put a dash pot here, which is at L by 4. And I have m bar as the mass per unit length of this rigid bar and then I put a concentrated load over here. Let me not put a concentrated load, let me just put a UDL, let us put a UDL here, which is p bar by t, which is a load per unit length.

Let us do that, let me do that, that is the problem that I have to solve, now one idea is to use these formulations that we have. But, I will not use them, I will go back to first principles and show you that, what I have derived for a generalized kind of a structure can actually be used for any system. And in fact, I will take this example that I have, by the way this example is also not, because I am not putting k 1 and k 2 values, I am not put c value, I am not put m bar value, so it is still a generalized.

But, let us look at this particular problem and actually get some very specific numbers for those in terms of k 1, k 2, m bar and p bar of t. So, the first thing to do is, what displacement, note that although I draw it very large, it still every point only moves along the tangent, because remember small displacement means, the arc and the tangent are the same, that is the fundamental point.

And so therefore, I need to find out all these values of where they are, at this point, this is going to be equal to L theta, this is going to be equal to L by 2 theta, this point where the dash pot is going to be equal to three fourth L theta. And I think that is all, these are the only three that are point, everything else is UDL, so let us look at this, so these are the displacements then what are my forces, let me write down the forces.

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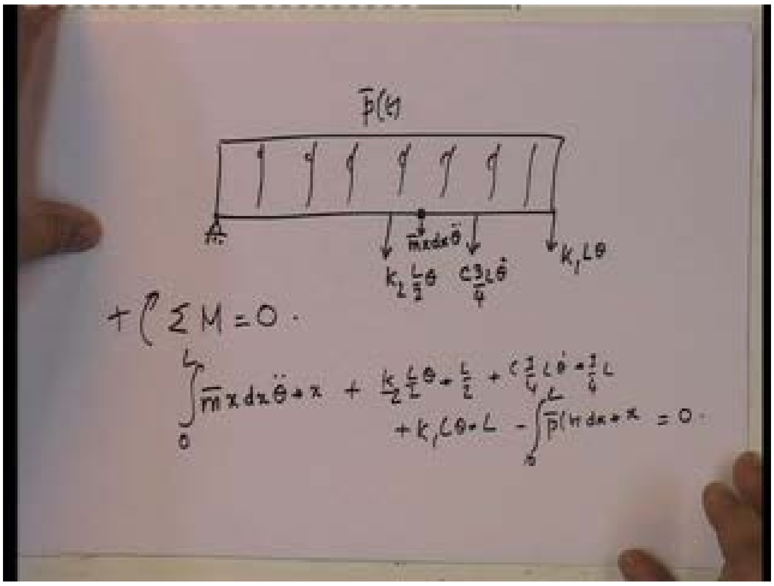

And so the forces are going to be equal to k 2 into L by 2 theta, this one is going to be k 1 L theta over here, we are going to have c into 3 by 4 L theta dot. Remember the dash pot is proportional to the velocity, so it is three fourth L theta dot, where displacement is L theta. If you differentiate with respect to time, that becomes L theta dot, apart from that we have the load, which is given by a constant p bar of t. In addition as I said, you have the situation, where an infinitesimal length and this is equal to m bar x d x theta double dot.

Because, m bar d x is the mass, x theta double dot is the linear acceleration of the mass is, so this is, that every instant, so these are the forces. So, if I take moments, again since this is the point about which I know, so moment equal to 0 then what do I get. Because, now I can actually put this down, I get one which is m bar x d x theta double dot, which is the infinitesimal area that multiplied by x and this integrated over the L length then that is what gives me the moment about this point of all the inertial forces of the bar plus k 2 L by 2 theta that is, the displacement into L by 2.

And so I am taking this as positive, so this is clockwise, this is ((Refer Time: 36:50)) clockwise, this is also clockwise, three fourth L theta double dot into three fourth L, that is the moment plus k 1 L theta into L. And then this one is anti clock wise, so I have minus, now p bar t d x is the load at any point x, so that has to be multiplied by x and integrated from 0 to L, because this load acts everywhere. So, that is the moment of the infinitesimal area of the infinitesimal load, so that integrated summed up over the whole length is integral, is equal to 0. So now, m bar is a constant, so I can take m bar outside and I am going to rewrite this on another sheet for cleanliness.

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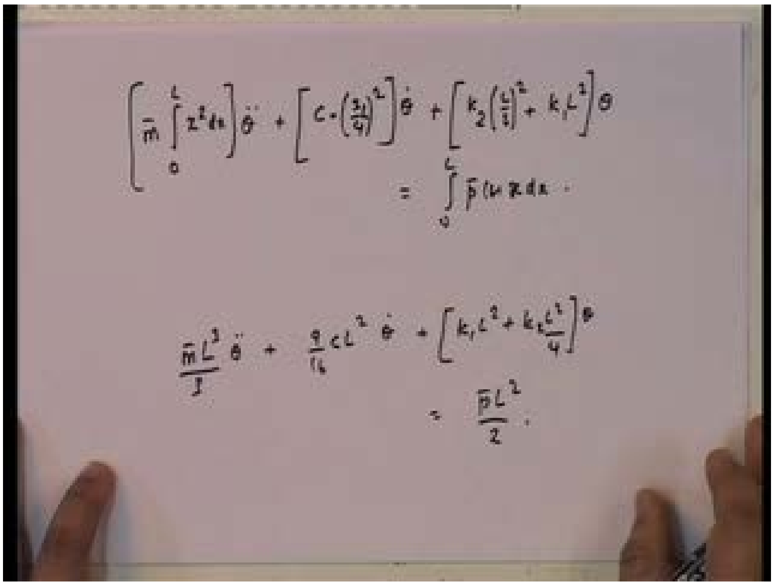

And so if I do this what do I get, I get m bar into 0 to L x squared d x theta double dot plus c into 3 by 4 L whole squared into theta double dot plus k 2 L by 2 squared plus k 1 L square theta. And if I take the, on the other side I get 0 to L p bar t x d x, so now if I do this, I know that p bar is a constant, because I have said it is uniformly distributed load, m bar is a constant. So, this becomes m bar L cubed by 3 theta double dot plus 9 by 16 c

L squared theta dot plus and this one still remains k 1 L squared plus k 2 L squared by 4 theta is equal to, since p bar goes out, this becomes p bar L square upon 2.

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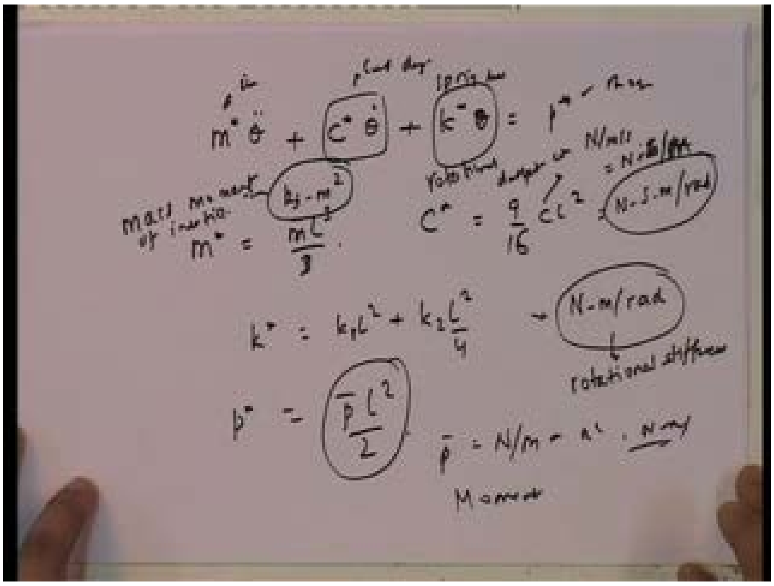

So therefore, I can rewrite this problem in the following format and that is that, m star theta dot double dot plus c star theta dot plus k star theta is equal to p star, where m star if you look at it, is m bar L cubed by 3, c star is equal to 9 by 16 c L squared, k star is equal to k 1 L squared plus k 2 L squared by 4 and p star is equal to p bar L squared by 2. It is very very interesting, let us look at the units, the units are what, let us look at this, this is Newton per meter, Newton per meter into L squared gives me what, Newton meter.

Actually the units of these are Newton meter per radian, because this is per unit theta, so this is the units of this, let us look at this, what is this, m bar. M bar is given as k g per meter, so if k g per meter into L cubed becomes what, k g meter squared. So, this is k g meter squared, this one what are the units of c, it is Newton per meter, c if you look at this is what, it is equal to force per unit velocity. So, force is Newton per unit velocity, so it is Newton per second, so this becomes Newton meter per second is c.

And then Newton second per meter that is wrong, Newton second per meter, because second comes on the top, becomes Newton second per meter into this. Basically what you are getting is, Newton second meter per radian. If you look at all of these, what is Newton meter per radian, moment per unit rotation, what is that, that is like a rotational stiffness. What is this k g meter squared, k g meter squared mass into distance squared, what kind of this thing is this.

This if you remember from your old high school, it is mass moment of inertia, what is the mass moment of inertia, it is if you want to rotate something, it is the mass moment of inertia of that disc to oppose rotational rotation. So, this is nothing but mass moment of inertia for unit displacement. Similarly, this is actually a rotational dash pot constant, and let us look at this load, what is this load, p bar is Newton per meter into meter squared is Newton meter, this is again Newton meter, what is Newton meter, moment.

Let us look at this equation, what should this equation look like, angular acceleration angular velocity, angular displacement. So, if you have that, rotation into angular displacement is, rotational stiffness into angular is the kind of, should we say moment in the spring, this is mass into theta. So, mass moment of inertia into angular acceleration is what, it is like an inertial moment and this one is like... So, this is spring moment, this is moment, this is inertial moment, this is inertial damping force.

So, you see obviously, that the moments are being added up, so therefore, there is nothing inconsistent in this. If theta are rotational degree of freedom then automatically corresponding to a rotation, what is the force, moment. So obviously, the entire equation is a moment equation, now let us write the same problem. I am going to take the same problem here, that I have looked at and instead of theta as my degree of freedom, I am going to say that, the displacement of this tip is my degree of freedom. So, same problem excepting that, I am defining my what am I doing, I am defining my degree of freedom differently. So, let us look at it, this is going to be a very interesting problem.

And that is that, let us write this, so this problem is now the same problem theta excepting that, this is now my degree of freedom. So, let us put the degree of freedom there, so that means, I am going to say, this is the displacement, this is z of t. If this is z of t, what is it at the midpoint, it is z upon t upon 2 and at the one third point, it is going to be equal to z t 3 by 4 and what is this equal to, this is equal to z t by L.

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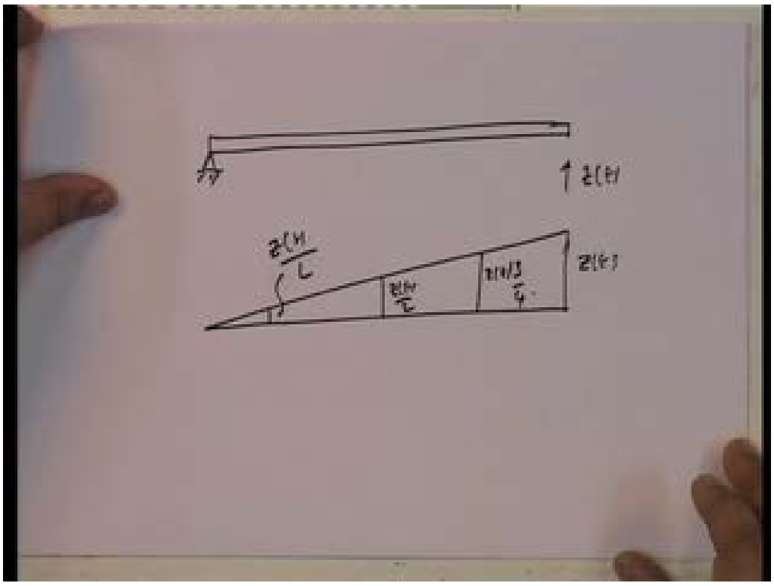

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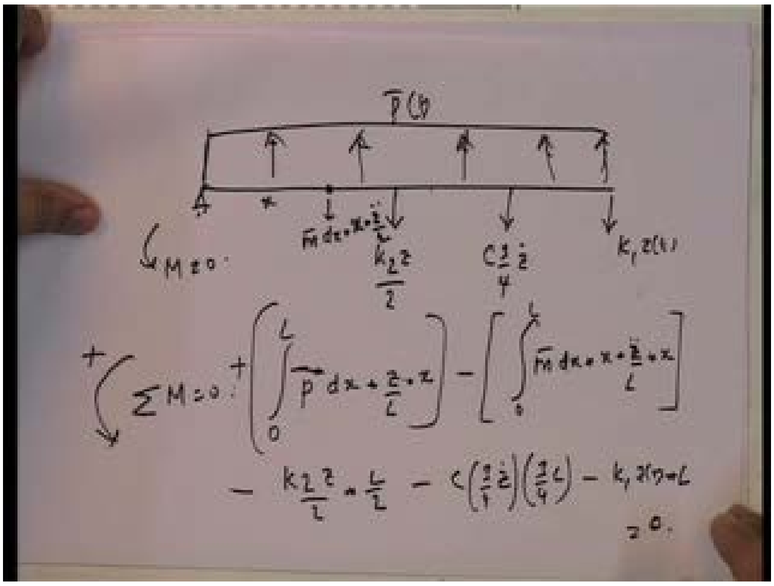

So therefore, if I look at the forces due to this displacement, what do I get, I get the following forces. I am going to have k 1 into z t, this is the force in this direction, in the center spin it is going to be equal to k 2 z upon 2, at the dash pot it is going to be c into 3 by 4 z dot. Because, if this is three fourth z is the displacement, the velocity is going to be 3 by 4 z dot and if I look at of course, I have the other load that is still remains that is, p bar of t uniformly distributed load, upward load.

Now, I look at an infinitesimal length, how much is this, m bar d x is the load or should I say mass, m bar d x is the mass of this infinitesimal length, which is at a distance x. Now, what is this going to be, it is going to be equal to x into z by L, because my theta is z by L, so this is going to be equal to x into z by L, so these are all my forces. So now, again since this is my thing, I take moment equal to 0 and just for doing it, I am going to say that, summation moment is equal to 0 and I am going to take anticlockwise as positive.

Just to show you that, it does not matter how I take it, you are going to get the same equation. So, I will take anticlockwise, remember the last time I took clockwise as positive, this time I am going to take anticlockwise as positive. So, let us now, write down the moment equation, let us take the first one and that is, p bar, so this going to be equal to p bar d x. So, this is what, p bar d x, so this is at some point x I am taking and that into, z by L what is z, that is theta.

So, this p bar x into theta, this theta into x, that is what I have here and that integrated from 0 to L is my first term, that represents the moment due to this. And what is this, p bar d x is an infinitesimal length and the moment by that infinitesimal load p bar x d x is going to be equal to x into z by L, that is the moment and that moment is anticlockwise positive, so this is plus. Then I have this moment, this going to be minus, because this gives rise to clockwise.

Let us look at what I have here, I have m bar d x into x into z by L, that is the load and that into x and then integrated from 0 to L. By the way, this was z double dot, because this is the inertial acceleration, x into z double dot upon L is my inertial acceleration, so this is this. Then what do I have then I have k 2, so this is also anticlockwise, so this going to be negative. So, negative is going to be equal to k 2 z by 2 into what, into L by 2 minus three fourth z dot into three fourth L minus k 1 z of t into L, that is equal to 0 and that is my equilibrium equation.

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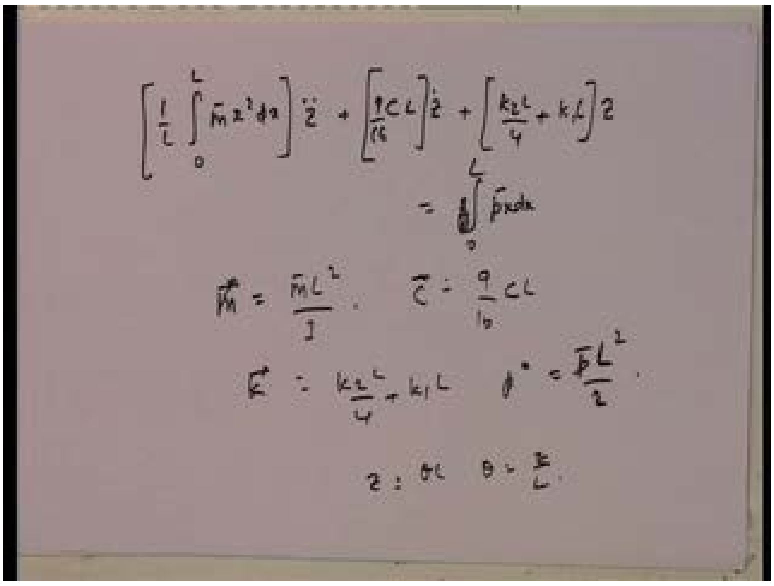

So, if I rewrite this in the proper format and putting all the terms together, what do I get, I get the following. I get 1 upon L 0 to L m bar x square d x into z double dot plus, all I am doing is, I am taking all these minuses onto the other side. So, it is going to be equal to c into 9 by 16 L into z dot plus k 2 L by 4 plus k 1 L z is equal to 0 to L p bar x d x 1 upon L.

So, what I am going to get, so if I rewrite this then I get m bar is equal to, this becomes m bar L cubed by 3, so this becomes m bar L squared by 3, c star is 9 by 16 c L ((Refer Time: 54:01)) m star, k star is equal to k fourth L k 1 L and p star is equal to p bar L. Now, you have x squared, so you have p bar L on 2, so that is what we have, if I get back to this, I just want to ((Refer Time: 54:37)) reestablish this, this is p bar x d x and z by L does not come into the picture, p bar x is the force and that into x is my string.

So, that is why, this becomes p bar x, there is no 1 by L, so I get p bar L squared upon 2, so that is my p star. Now, you will ask me that, this looks different from the other one, if you substitute the fact that, z is equal to theta L and put theta equal to z by L, you will see that, it is same equation. So, it does not really matter, what we define as our generalized displacement. As long as you have a single degree of freedom, you have this situation irrespective of, whether you take theta or z it does not matter, you get the same equation of equilibrium.

Thank you very much, bye bye.