Structural Dynamics
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Lecture - 14
Numerical Analysis of Response of Single Degree of Freedom Structure and Time Domain Approaches

Hello there again, just to review we have been looking at the Response of a Single Degree of Freedom system to an general kind of loading. A general loading, arbitrary loading, is a loading which is defined that it has a fixed duration and what we figured out was that, there are two approaches for solving this problem, one approach is the time domain approach where we got the Duhamel integral.

And then we looked at in more extensively over the last couple of lectures, on what are I call as the frequency domain analysis, which is the where you use the Fourier transform and the inverse Fourier transform, procedure to solve the problem. And the essential thing that I discussed last time, was the discrete Fourier transform approach. And again you know the, I did both the discrete Fourier transform and the inverse Fourier transform, inverse discrete Fourier transform.

And essentially I understand that the frequency domain analysis, still remains that you got a digitized record of the of p and you have a structure defined by dynamic characteristics t and xi. So, what is the first step, the first step is to do a discrete Fourier transform and take the load into the frequency domain, once you have taken the load into the frequency domain. Then you can find out the frequency domain representation of the response, by multiplying this load p i omega with h i omega, h i omega is the complex frequency response function, which you already discussed and once you have that you have.

So, therefore, you know when you do the discrete Fourier transform you find out p i omega bar m right. So, now what you do is, you do p i omega bar m, so you are all doing it for omega bar m and that you find out you multiply that with h i omega bar m and you get v i omega bar m, that is the at omega bar m, what is the amplitude of the response. Now, once you found out the v at I omega bar m, you use the inverse discrete Fourier transform to get v of t. So, that in essence is the frequency domain response analysis.

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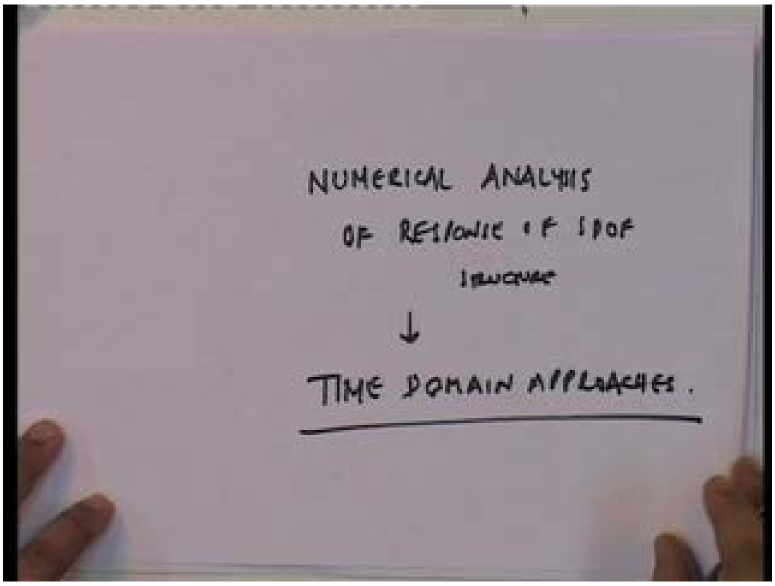

Now, if you remember we had done the Duhamel integral and in this particular lecture, which is the fourteenth in that series of lectures that I am looking at, we are going to look at Numerical Analysis of Response of Dingle Degree of Freedom Structure and I am going to look at Time Domain Approaches. We are going to look at the time approaches today. So, let us look at this, what are the specific thing, I will start off with the classical Duhamel integral approach.

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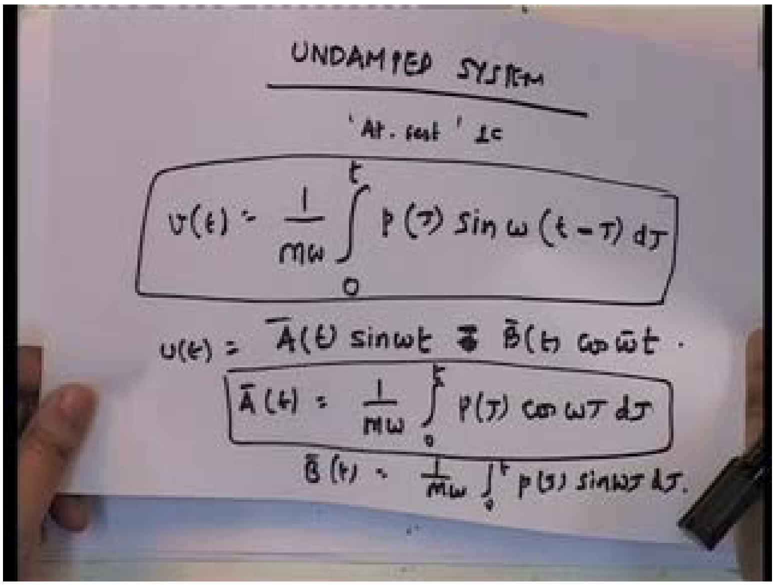

And let me rewrite this, let me start off with an un damped system and we know, that v of t is equal to 1 upon m assumption at rest which is typically before you have a load, the structure is always at rest. So, the Duhamel integral for the un damped system is p of tau sin omega t minus tau d tau, this is the Duhamel integral approach we have already derived this. Now, if I want to look at this, let me look at this I am going to rewrite this I can rewrite this in the following format, I can rewrite this and you know sin omega bar t minus you know, so that part become sin omega bar t, cosine omega bar t minus sin omega cosine omega bar t.

So, basically then this becomes I will call this as A of t into sin omega t plus B of t and I will just mark them in this way cosine of bar t. And since, this is minus what happens this becomes sin cosine and then it becomes minus I will make that as minus keep the minus in here. And then if I look at this A bar of t becomes 1 upon m omega 0 to t well let us look at if I take sin omega t out, what do I have inside, inside I have cosine omega tau right sin omega t cosine omega tau.

So, then this becomes p of tau cosine of omega tau d tau. And similarly, you have B bar of tau becoming 1 upon m omega 0 to t p of tau sin omega tau d tau, so in other words these terms if you look at them, they are just a function of tau this is also a function of tau and v of t becomes this. So, if I can derive these two terms, in using any numerical integration scheme I will have found out the solution. Well what is the easiest way of solving this, I can say that look I am going to say that if I use any summation any summation approach.

I can write A bar and I am only going to derive for A bar and note that you know, all that happens is you just have to replace cosine omega t, you know because you see the only difference between this is that this here it is cosine omega t in the other for finding out B you just you replace it with sin omega t. So, this is going to be equal to 1 upon m omega 0 to t y of tau d tau is equal to, I mean it is approximately equal to delta t delta tau delta tau upon m omega, into some summation m I will call that as 1 upon zeta I will call this zeta where this is y t.

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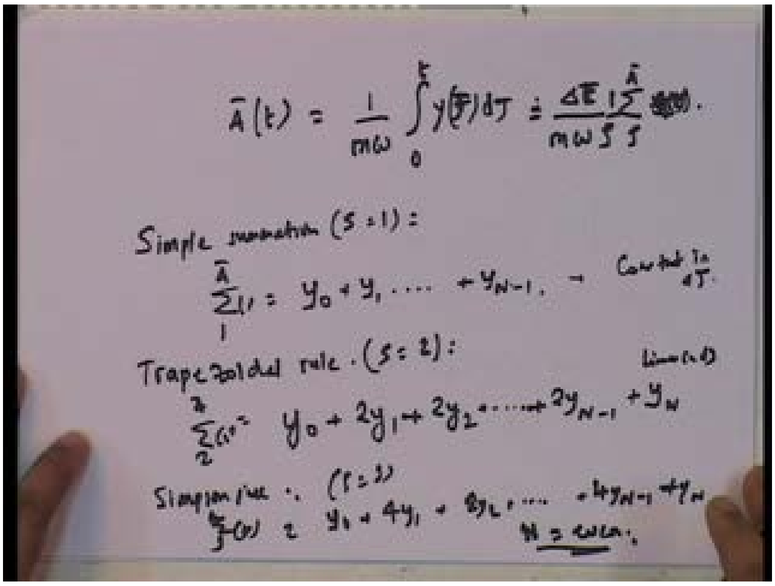

So, now, this is a numerical integration scheme for example, if I use simple summation, this simple summation is where xi zeta equal to 1. So, this basically becomes 1 A bar it becomes equal to y 0 plus y 1 into y n minus 1, if I use the trapezoidal rule, where xi is equal to 2 then 2 A prime is given by y 0 plus 2 y 1 plus 2 y 2 plus, plus 2 y N minus 1 plus y N. And if I use something called the Simpson's rule, for these rules N can be any number.

But, if I using the Simpson's rule, which is actually xi equal to 3 A prime t is equal to y 0 plus 4 y 1 plus 2 y 2 plus, plus 4 y N minus 1 plus y N, where in this particular case N is even by definition. The other two they do not have the definition now, where do the simple summation comes, simple summation is based on constant in delta tau, this is linear in delta tau and this is slightly more you know complicated, Simpsons rule is based on a cubic's prime.

So, these are how the rules come, once you have these rules then using these rules, how do I make the Duhamel integral come out well a bar of t. If you look at it what is the A bar of t.

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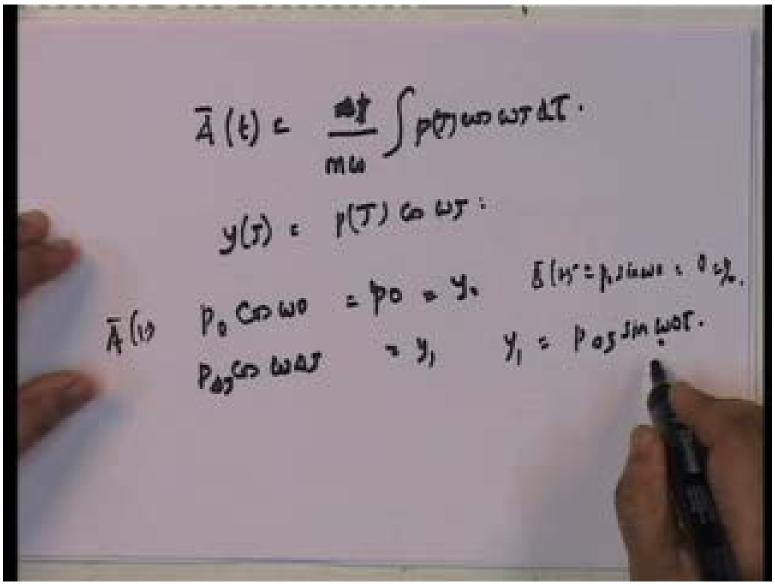

Let us just look at A bar of t is by definition delta tau up on m omega into p tau cosine omega tau sorry 1 upon m omega into d tau. So, this then I am going to make it into, where y of tau basically becomes p of tau cosine omega of tau that is all that I have, so essentially it becomes the following, I have if I use Simpson's rule then this is the term that I plug into y 0. So, this basically then becomes p 0 right, so the first term y 0 becomes cosine omega 0, so that is equal to p 0.

The next term is equal to cosine into omega bar. So, this basically then becomes what, it becomes cosine omega into delta tau, so that is how this entire this is p of delta tau cosine omega t. So, this become this is y 0, this is y 1 and in this fashion you can continue, calculating and using any of the rules that you require to do it, that is all there is to it this is of course, for A bar similarly you know, B bar is going to be equal to p 0 into sin omega 0 that is going to be 0, for b 0 this is your y 0 and y 1 will be equal to p at delta t sin omega delta.

Where of course, what is omega, omega is nothing but 2 pi upon t. So, this is how the entire thing becomes, so that is as far as the un damped system response, the you know the numerical integration of the Duhamel integral for the un damped system. Similarly, if we rewrite the response of the un damped system, what do you get then you get the following thing.

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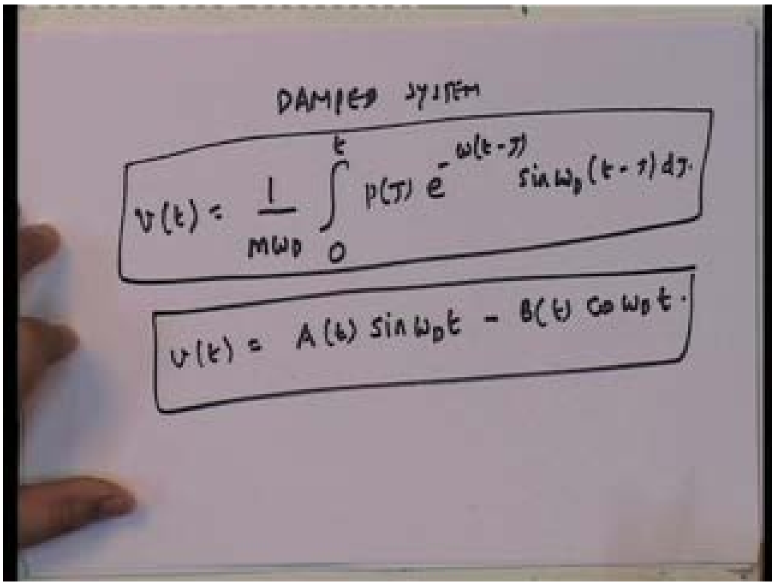

So, if I have the damped system Duhamel integral, the Duhamel integral is the following v of tau is equal to 1 upon m omega D 0 to t p of tau exponential minus omega t minus tau sin omega D t minus tau; this is the Duhamel integral for the damped system. Now, in the same way if we write this, we can write it in this following form and that is that I am going to write v t as A of t into sin omega D and I am going to put minus B of t cosine omega D t. So, if I put this my entire thing essentially becomes that I can find out A at anytime and then you know go ahead and do it.

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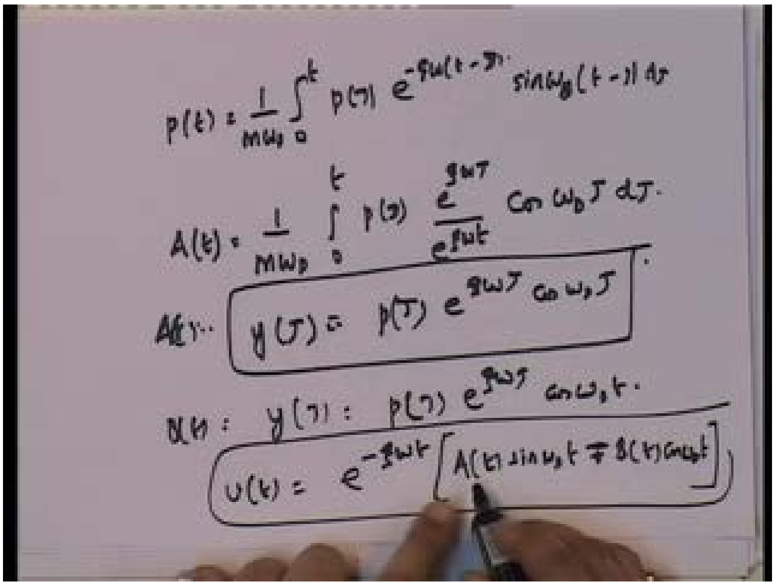

So, therefore, what we get is the following and that is that what is A of t, A of t is equal to it is equal to now you know put down, let me put this down again. So, that you do not make an error 0 of t p of tau exponential now, if you look at this, this essentially becomes the following, it is minus xi omega t minus tau, this one I can write it in the following format I can write it as e to the power of minus, so if you look this, this essentially and then you have sin omega D t minus tau.

So, this is not I am not going to explain this is exponential function, but this if I expand I get it exactly the same way. So, therefore, A of t becomes 1 upon m omega D 0 to tau p of tau I am not going to put this equal to e xi omega tau upon e xi omega t that is what it is actually, if you look at it. This term is really this upon this because this is e to the power of minus xi product e to the power of minus and if I put e to the power of minus, it is actually 1 upon e to the power of, so that is what it is and since this is I am doing a of t sin omega D t. So, this becomes cosine omega D tau into d tau.

Now, this term does not depend on tau, so I can take this outside. So, what I have essentially is that this term goes out here and it becomes e to the power I mean I can put this e to the power of minus xi omega bar t up on m D into p t e to the power of omega bar t cosine omega. So, essentially for y of tau essentially becomes p of tau e to the power of xi omega tau and cosine omega D, that is all you have for A of t and the outside term is essentially e to the power of minus xi omega bar t which is in that particular term.

So, you know in essence what we get is the following and for B of t similarly, y of tau is equal to p tau e to the power of xi omega tau cosine omega D. And ultimately I if I rewrite this and I put the something as a summation is 1 upon m, then outside on v of t I can put it in this fashion this is minus. So, this if I do that e to the power of minus xi omega t which I get over here, will go outside and then A of t basically becomes 1 upon m omega t into p tau e to the power of minus cosine and then so that is that is the simple one.

And this of course, you can always, derive using any of the rules that we have already talked about, so much of for your this thing. So, essentially one approach is in the time domain approach is to essentially look at the Duhamel integral, both I mean and here I have specifically listed out the Duhamel integral for the un damped and damped. And all that happens is that, that between the damped and un damped system is that the y of t which is the term, that is like the function that is being integrated.

Essentially, becomes if it is for you know the sin omega t function the you know the A which is in front of sin omega bar t, that part essentially is becomes then you know y of t becomes p of tau cosine omega D tau, in there is an exponential xi omega tau that is the term that is for a damped and when xi goes to 0, this become p of tau into cosine omega tau. So, that is all I mean you know if you look at it, the un damped system is really you know a special case of the damped system and even the y of t that you have for integral you know is a given function.

And you can use either the what were the things that we looked at, simple summation which assumes that the p over a delta tau remains the constant that this y remains the constant. So, that is why it is all done on the basis of y 0 constant, then rectangle you know y into delta t plus then the next one y 1 you know, so you are always taking it to be constant. A trapezoidal rule assumes that you have y 0, you have y 1 and it is linear between the two and, so that is why you have that second term.

And then you have Simpson’s rule, which assumes that the relationship is a cubic’s plain and if you take it to be a cubic’s plain and if you take it to be a cubic’s plain they have to be even numbers and, so Simpson’s rule is follows that. So, these are essentially you know in a way finding out the area under a curve, that is you know the Duhamel integral essentially becomes like finding out an area under a curve. Now, in time domain there are other approaches, which are known as time marching approaches. And for these what a time marching approach, what this does is it does not go on the basis of the Duhamel integral at all, it goes on the basis of the following.

That it looks at m u of t at time tau c u dot of t at time tau k u of t is equal to p of t. So, this in essence is and then it says that look, at time t plus tau what you have is, so this approach is the classical note that I am not actually trying to find out the Duhamel integral at all. What I am going to do is, I am going to take the original equation of motion and I am going to develop, what are known as time marching scheme. In other words I am given, what am I given, I am given where delta t is the time interval and what I am going to do is, I am given all of these and what I am going to do is the following. I am going to develop and we develop what is known as an incremental equation of equilibrium.

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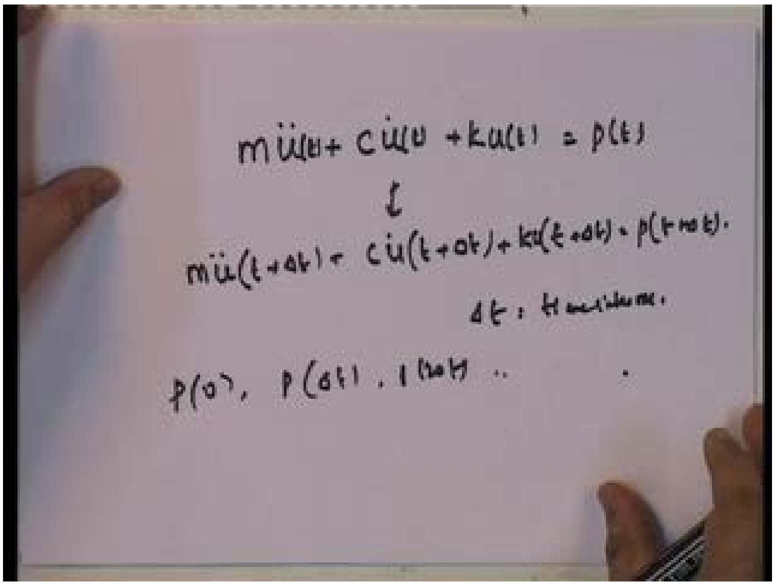

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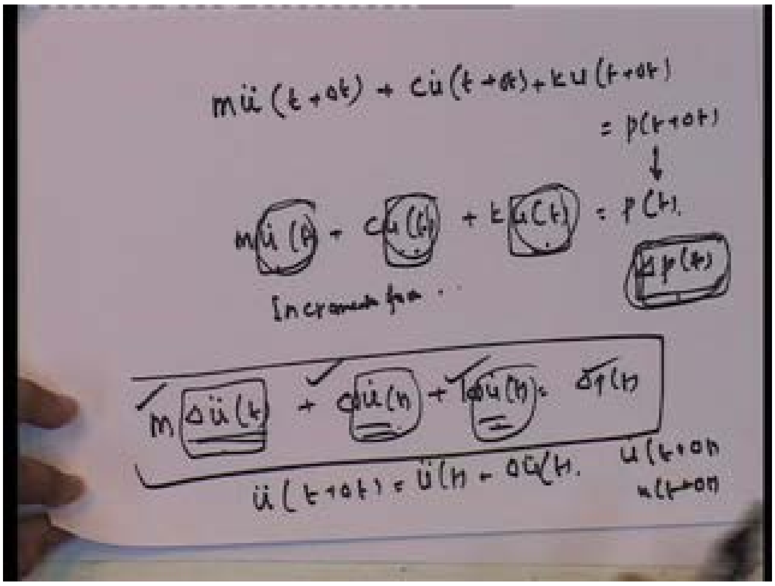

So, note that what I have is m u double dot t plus delta t c u dot this is at the end, at an instant of time t and at instant of time delta t of tau I have k t of t is equal to p t of tau and the previous time I have it as. So, if I look at it, this minus this gives me what it gives me delta p t, at time t how much of delta do I have, so if I write this in incremental form, I get it as m. Now, u I am going to put that as delta u at t because this delta u understand that I am actually, I know what are my I know these are known and I also know this because this is something that is given.

So, from knowing this and this and this I want to be able to find out these out, knowing these. So, my problem becomes knowing these and this I want to find out this, this, this and see for finding out this I need to just solve this equation. And if I know these, if I can find these out then my u double dot at time t equal to tau is equal to u double dot plus delta u of t. So, I found out the next step, so I have found out all now, this is u double dot similarly I can do for u dot and I can do for u once, I find out u at the next time step now, that becomes my initial time step.

So, that becomes these and I find out the next time you, so you understand how the whole procedure goes, I go from this the whole approach goes in the following format if I knew how to do that. So, this is known as an incremental equation of equilibrium, where I know this, I know this, I know this and I know this, if we can find out somehow all these terms then I have solved the problem.

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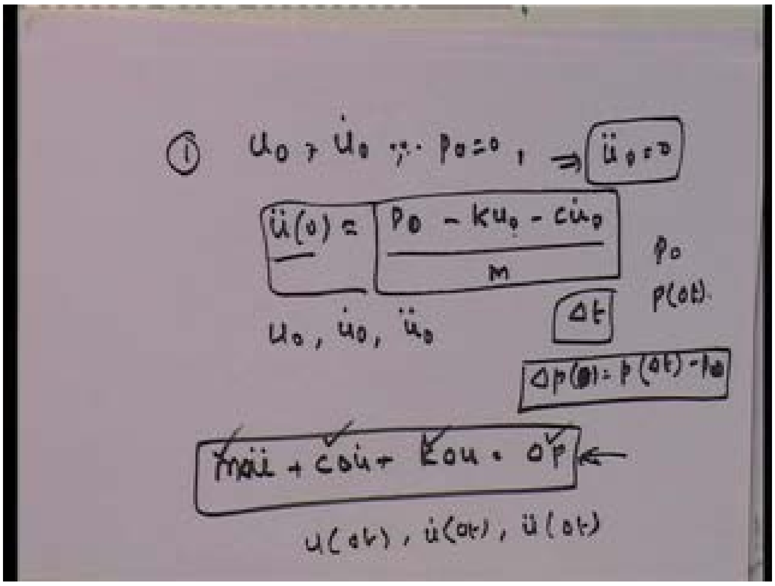

So, essentially what is the problem then look like, how does it how do I start, I start off by look I know u 0, I know u dot of 0 and I know p 0 typically p 0 is always equal to 0 do I know the acceleration. Well you know, if I know these and if these are also 0 and if I plug it into the original equation, which says m u dot plus c u dot plus m u double dot plus c u dot plus k u is equal to p 0.

So, if all of them are 0 automatically, this implies that this is equal to 0 automatically, if these are not equal to 0 you can always derive this because u double dot at 0 is going to be equal to p 0 minus k u naught minus c u naught, all upon M you see this, I can derive this. Now, if this is 0 these are two zeroes and automatically this will be 0, otherwise I can always derive this. So, this is my initial conditions, where I know u 0 u dot 0 and u double dot 0.

Now, I have my delta t for marching because why because I already have my I have defined my p 0 then I know p at delta t I know all of those terms right. So, now, what is delta p of t is equal to in this delta p 0 is equal to p at delta t minus p of 0, so this is my delta p 0. Now, my incremental equation is what m u dot now here, this is delta u plus m delta u plus k delta u is equal to delta p and these all are at time t right this is incremental equation.

Now, here I know this, I know this, I know this, I know this. So, if I can find out these then having found these out I can find out u at delta t, u dot at delta t, u double dot at delta t and once I know these, these become this I know, I can find out delta p and delta t you know, so you keep stepping keep stepping. So, in a sense you see all it boils down is to solution of this equation. Now, how do I solve this equation, this whole thing depends on the following that I can solve this provided, I make an assumption and that assumption is known as the acceleration.

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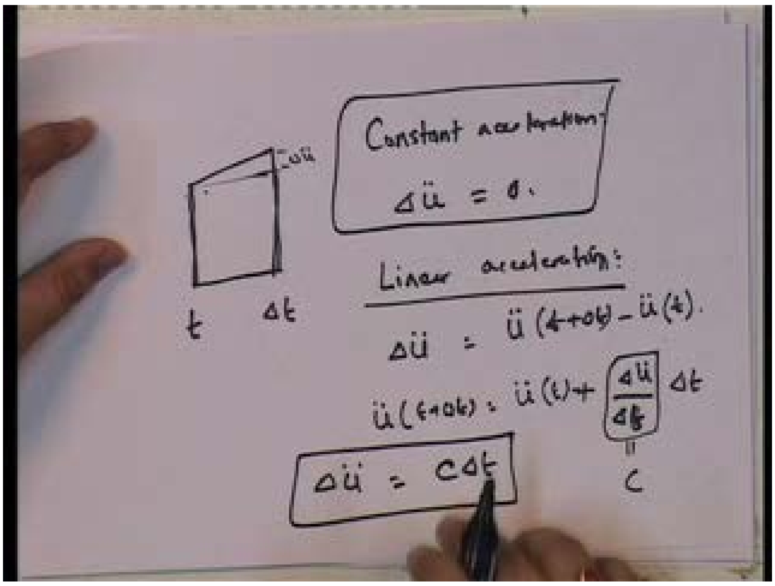

How is, this is an assumption how does the acceleration go between time t and time t plus delta t, during you know in this time step how does the acceleration vary, this is an assumption. So, you have constant acceleration, where you say that look delta u star is equal to 0 or you can take it as linear acceleration, where you say that look delta u is equal to u double dot at t plus delta t minus u double dot at t. And you know, this varies linearly.

In other words u double dot of t plus tau is equal to u double dot of t plus delta u double dot upon delta tau that is the slope, note that delta this is, this is your delta u double dot delta u. So, that is the slope and this slope is a constant this is a constant, so then this into what do you get delta t. So, essentially you are saying that look, delta u double dot is equal to a constant, what is that this is a constant right into delta t that is your assumption when you are taking linear acceleration. So, now, I am not talking about constant acceleration I will follow you know, this can be done easily because all that happens is that, what does that give me this essentially gives me the following aspect, it gives me that you know I have defined it as a function of this thing.

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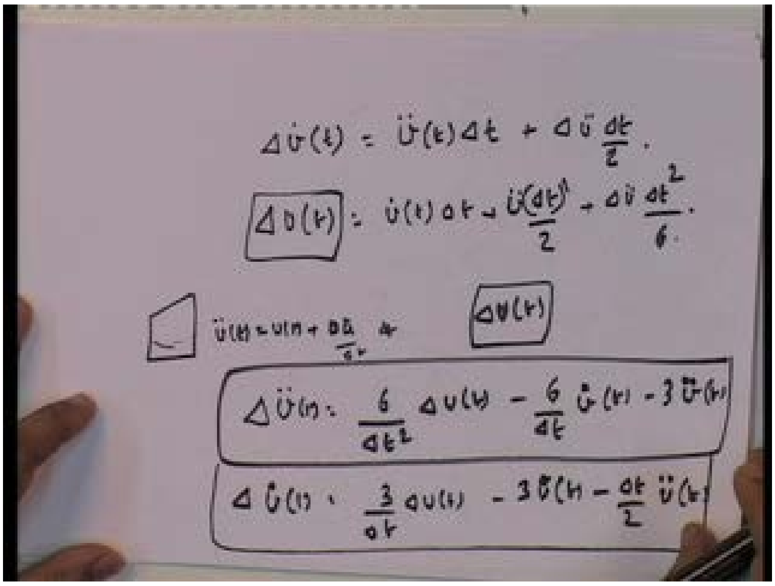

So, therefore, that c is know if I do that then from that it is very easy to put together your delta v t, delta v t basically becomes then v t plus delta t plus your delta v t into delta t by 2 that is a really you are just doing integration. And then delta v becomes equal to v dot into delta t plus v double dot into delta t squared upon 2 plus delta v this squared upon 6.

So, this term squared becomes and this term, then becomes upon 6, this is essentially nothing but an integration.

What we are saying is you know, linear acceleration means v of t is, in this part a you know v double dot is a function of delta v in you know delta t upon d t once you do that that is how you get this these two functions. So, you know if you rewrite this, we get the following and that is I want delta v of t as my basic function, it is a basic variable delta v of t is my basic variable because understand that I know displacement and I want to find out the displacement of particular point, all that velocity and acceleration are only to solve my problem.

Essentially this you know I have this displacement I go to this displacement, that is you know I am tracking my displacements. So, therefore, I will take consider my delta v as my basic variable and if I take delta v as my basic variable, then the whole thing can become in the following delta v of t this is v of t becomes 6 upon delta t squared into delta v of t minus 6 of delta t into v of t minus 3 v dot this is for delta v and delta v t becomes equal to 3 upon delta v delta v t minus 3 upon this is v dot t minus delta t by 2 into v double dot t.

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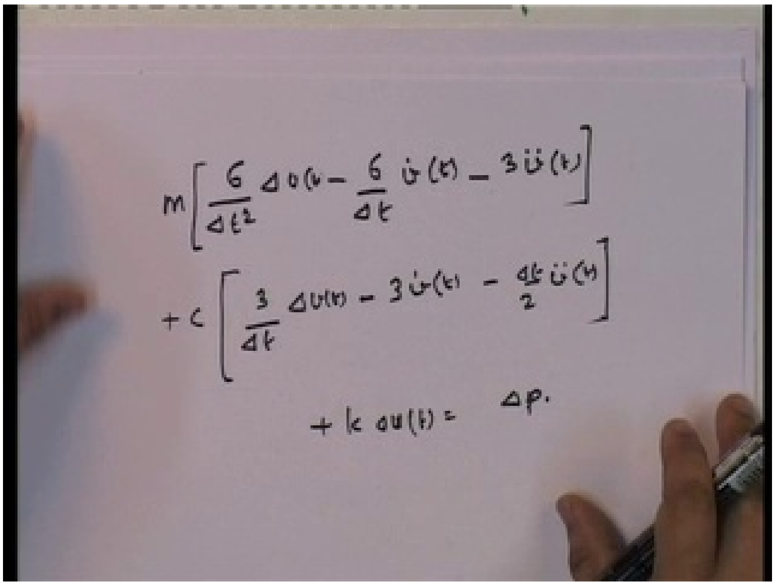

This is delta v yes this is what I get this is v dot mind you v dot, this is v dot, this is v double dot, v double dot the acceleration this is what we get. And, so therefore, if I substitute this into that original equation what do I get, I get the following terms I get the term to be equal to now, I am going to substitute these values that I have computed into the original incremental equation.

And my incremental equation was what m into delta v. So, I am going to delta v double prime, so I am going to substitute for that, so this becomes 6 delta t squared into delta v minus 6 upon delta v, v dot at time t this is also at time t minus 3 into delta v at time t then plus c into, now I am going to put delta v naught. So, that is equal to 3 into delta v at t minus 3 v dot of t minus delta by 2 v double dot of t plus k delta u is equal to delta p. Now, what I am going to this delta you know, so delta v of t.

Now, what I am going to do is, if you look at these terms there are certain terms which have delta v in them and there are certain terms, which have the previous you know v dot and v double dot in it. So, what I am going to do essentially is in the known terms, all the known terms I am going to put in the right hand side and leave the left hand side in terms of delta v.

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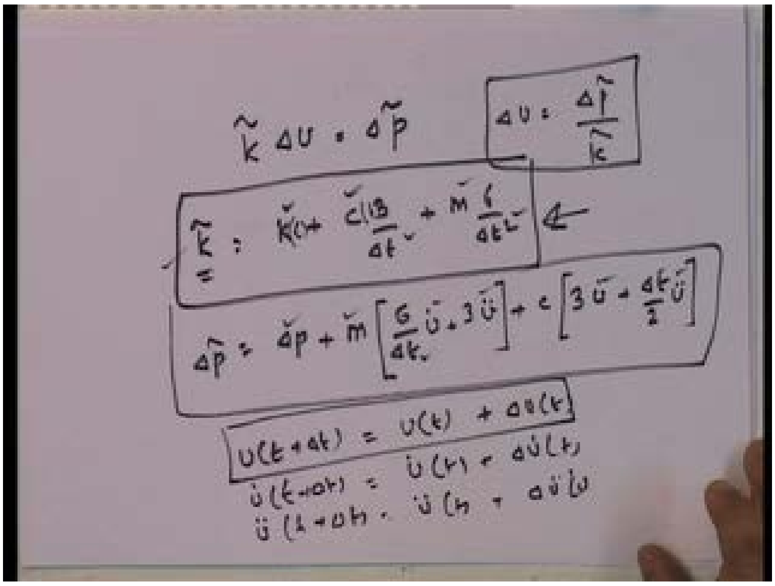

So, then if I rewrite this particular equation and I am just rewriting them and I am going to say that this is equal to k into delta v is equal to delta p tilde, where k is equal to k plus c into 3 by delta t plus m into 6 by delta t squared that is my k and my delta p star becomes because the delta p star essentially becomes delta p, the delta p is already there plus and now I am going to put down all the terms, which are known 6 upon delta t v dot plus 3 into v double dot plus c into 3 v dot plus delta by 2 into v double dot. Note that I know this, I know this, I know this, I know this, I know this, I can find out this.

Then here, I know this, I know this, I know this, I know this, I know this, I know this, I know this and I know this. So, I can find out this once I find out this, I can solve delta v is equal to delta p upon k star and note, if you look at this these terms, for a given k c a constant k c and m, this is needs to be evaluated only once. And these terms also need to be evaluated at every time step, that is all that needs to be done.

And multiply it by c and m and you add it to the delta p that you have, delta p is what p t of delta tau minus p of tau, you know both p at t plus delta tau and you know that p of tau. So, therefore, there are no issues did not solved for this and once you know, delta v understand that v of t plus delta tau is equal to what, is equal to v of t which you know plus delta v of t. Now, you know you already got the expression for delta v and delta v when you know delta v.

So, you can find out delta v dot and delta v dot also. And, so therefore, from those two terms you can solve for v dot at t plus tau which is equal to v dot plus delta v dot, which is in terms of known this is also now found out. So, you can find this out, you can find this out, so you got your mean v double dot t plus delta v t and you got it. So, once you solve just find this out once, and you do 1 upon k into these every time step you find these out, get delta v and then find out this and then next go back because for the time step you can do this.

So, this is how you solve what is known as time marching and it can be shown, that these are known as implicit methods. Because, why are they known as implicit methods. because you know you always need to know the previous times solving and you know, these terms include you see the incremental these include the previous times steps. So, these are known as, what are known as implicit methods, there are other methods which are known as explicit methods, the most obvious one is the central difference.

The beauty of you know implicit methods, which is one of these is this a linear acceleration method, the beauty of the implicit methods is that they as long as delta t is small enough and you I have already discussed this, what delta t needs to be, I do not need to go back. If delta t is small enough, this always is stable because you see you are making an assumption on delta you know v double dot.

And, so the stability of this, once it is ensured it is a you know the functions that you get, that kind of displacement time history, the acceleration time history, the velocity time history and once you got the displacement time history, you can derive anything which I have already discussed earlier you know. So, I do not want to go into that, the point becomes the following, the point becomes that these are inherently stable methods. And the linear acceleration method is if you people have heard of the New mark beta method.

This is actually the New mark beta method, where you know alpha is equal to one third and you know beta is equal to one sixth you know these are these are the things. So, this is actually the linear acceleration method this is a lot larger class of properties, but the essential point then becomes the following that these are, you have to ensure that these are stable methods. And the stability of a linear implicit methods are inherently stable, as long as you keep delta t to be sufficiently small.

The problem with explicit methods is even if you have delta t small because of certain assumptions you know, they can actually diverge you know I mean, a stable means you converge to a solution. So, that is why these methods are good, there is also another important point and that is, that this method remember that the frequency domain approach is based on for linear systems. The advantage of this is that, also valid for non linear systems, all that you need to do is incorporate k of t and c of t as being instance of time t.

So, every time you find out delta v, you know based on delta v you can find out what k of t is and in the next instant you use the k of t. So, this method this time marching schemes, are also equally valid for non linear, in fact these are the only methods that you can use, when you have a non linear structure, where k and c are functions of time not because they are functions of time, they are functions of v and v dot.

So, therefore, you know, so every time you find out, delta v you find out the next v, you know the next v, you know what k is and you for the next step you use that k and, so then you can you know, you can converge to a solution. Of course, when you have non linear your delta t becomes much smaller because you know if your delta v is too large, then you know you might have stability problems in those kind of situations. So, essentially to revisit the kind of problems that I have done in this lecture, a I have looked at Duhamel integral. And I have shown that, you can actually numerically integrate a integral you can numerically integral using standard simple summation, trapezoidal, Simpsons rule, gauss quadrature I am not listener of the gauss quadrature you can use any of these methods to solve the problem.

So, that is based on Duhamel integral you know. So, both damped and un damped I evaluated it and I shown that it really is the same, next what we did was we completely did not look at Duhamel integral, Duhamel integral is based on linear system. Then we developed a method, which we initially developed for a linear system, but and then we showed that, this is based on incremental you know equation of equilibrium and find out finding out, the incremental change in the displacement velocity and acceleration and proceeding in a time marching scheme, you start from 0 go to delta t you keep marching in time.

And you know there are methods which are explicit implicit, I specifically looked at an implicit method, and specifically looked at the linear acceleration and developed equations for solving. And then the advantages of time marching schemes, is that it does not depend linearity is not an assumption, you can have non linearity built into the time marching schemes. I think that is all I have and I am done with, the numerical analysis for single degree of freedom structures.

Next time I will start with a specific kind of load, I will introduce you to something called earthquake load. In fact, it is interesting that entire field of structural dynamics became very, very important once earthquake engineering developed. So, that is why we are going to look at the earthquake load as a specific kind of load, in the next lecture.

Thank you very much, see you next time bye.