# Structural Dynamics
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## Lecture – 13
## Methods of Analysis for General Loading

Hello everybody, we are going to be continuing our discussion on methods of analysis for general loadings. Last time we looked at the last two times we actually looked at the Duhamel integral approach which is timed away in approach and we looked at the Fourier transform approach, which is really a kind of extreme representation of the periodic load excepting where the periodicity which is the time period of repetition. The periodicity is considered to go to infinity. In other words, it is really an arbitrary load, but arbitrary load once it is considered as a periodic load, then you can establish that you can develop the Fourier Transform.

Now these are all integrals and especially the Fourier transform if you look at it, it is essentially a complex integral. So, obviously these although they look very elegant in presentation in reality if we have to solve them, we have to go about some other approaches.

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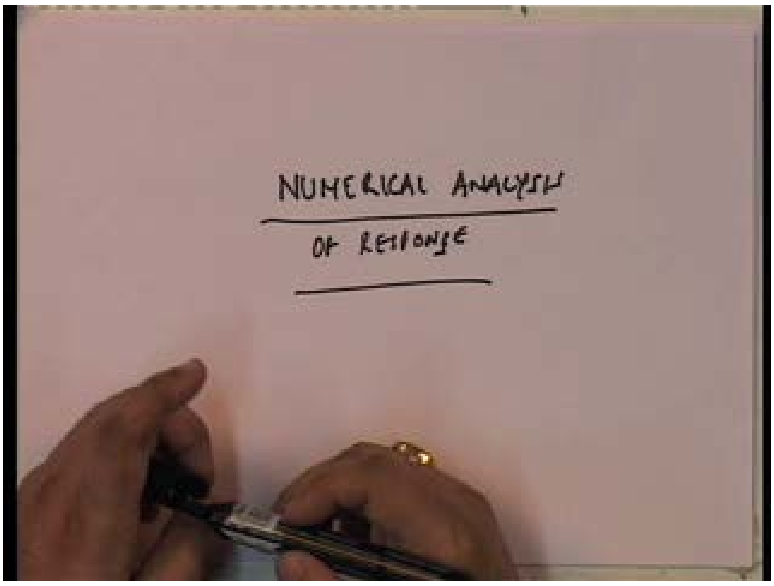


So, what we are going to be looking at in the next lecture is to really look at continue looking at a numerical analysis of response. Because the integrals that I have shown in reality can never we solved excepting unless you know the form of p of t and typically what you have given is essentially nothing but the following this thing is that you are typically given.

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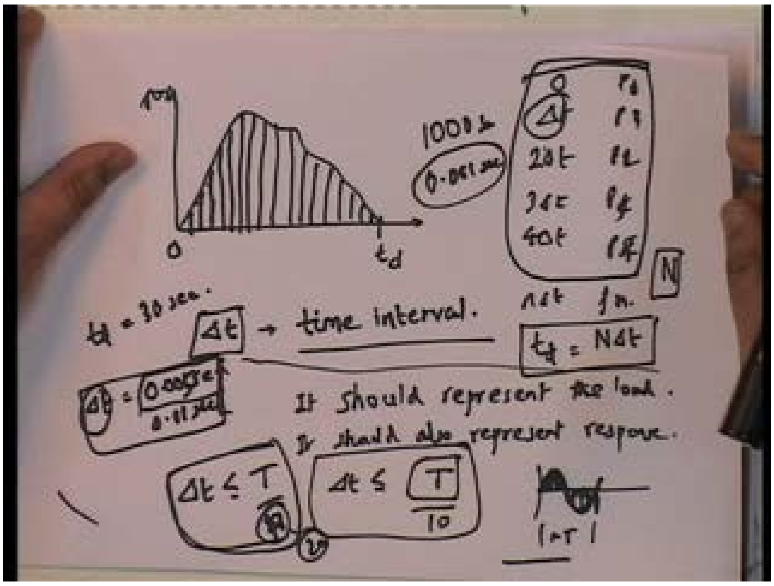


Let us take that this is some form of the load again you know I mean I am showing it on one side, we will see later on that there are situations in which you have both directions. Let us take something of this where this is 0 and time duration. So, what we in general have is essentially. In today’s world, a most data is stored in digital form. In digital form what you have you essentially are given. Look, this is you have delta t, which is like a fixed time interval.

There are two aspects to this delta t and we will discuss this as we go along, but there are two aspects. One, it should represent the load appropriately. So, one thing about delta t is that you should represent the load properly and it should also represent response. So therefore, you know delta t is typically taken in such a way that it should represent both the load and it should represent the thing and what you have is at delta you know 0, delta t, 2 delta t, 3 delta t, 4 delta t what you have is p 1, p 2, p 3, p 4, p 5. You know or you can call this as p 0, p 1, p 2, p 3, p 4 to keep.

So in other words at n delta t you have pn and td is equal to N into delta t. In other words, there are total N number of records or n number of values in any record of the loading. That is typically what you are given and let us say what should be delta t well delta t should be small enough to represent the load properly. In other words, if you have a load like this, typically delta t should be small enough that it takes all of you know these represents each part properly and secondly, delta t should also be small enough to represent the now how should it represent the load. Load is not you know typically you know, if duration is of the order of about 30 seconds, delta t is typically of the order of 0.005 seconds or 0.01 second.

So, this kind of representation delta equal to 0.005 or delta equal to 0.01 second essentially typically represents most loads. Of course, if you have blast loads which last for microseconds; obviously, delta t will have to be in blast load you know typically last milliseconds. In such in such situation 0.005 which is essentially 5 millisecond, but delta t does not make sense because if your blast load lasts for 5 milliseconds. Obviously, your delta t has to be significantly lower than that, but for other kinds of loads other than blast loads, typically delta t equal to 0.005 or delta equal to 0.01 is good enough for representing the load. How should we represent the response? Typically, if you look at the fundamental time period of the structure delta t should be less than T upon 10.

These used to be the, in other words, what does that mean; that means, is that one cycle of response. This is like one cycle of response. If it is a single degree of freedom system, one cycle of response is t and if delta t in other words, we should at least have 5 representations. This used to be the old way of looking at it. It is no longer you know valid. This is required for stability for some numerical methods, but you know typically delta t should be a significant number. Well, I will call this number some you know p or something which is of course, an integer, but it should be significant large and this typically you know if you have a single degree of freedom, I would not say ten I would probably say the you should have at least ten in this plot.

So, p should be about twenty, but that for single degree of freedom. So, in other words essentially delta t has to satisfy both the representation of the load and the representation of the response anyway, so that is, but typically this is not in your hand. This is more for the representation of how delta t should be done for numerical analysis, but you know, representation of the load is typically done in a format which is the measuring equipments time interval the maximum for example, let us say that you know a measuring data acquisition system has a maximum of 200 samples per second if it has a

200 samples per second, automatically if it is doing at a 200 samples per second. What does that mean?

It means essentially delta t is 0.005 because you are getting two hundred samples per second. So, delta t is automatically set. You know I have some you know we will show you when we go the lab. We will show you that there are some data acquisition systems that can you know do it at thousand samples. In other words, delta t is measurable at 0.001. So in other words, let me let it be very clear what we are doing here is looking at for the numerical analysis. This part, delta t may be different from this delta t because this is more digitization now how if we digitize should we get this. Well you know there are ways and means of doing it sometimes it is called down sampling. For example, let us say you are doing it at thousand samples per second.

So, you are picking up at 0.001 second right, but you do not want to do the analysis of that you want to do analysis at this. So then what you do is you take this load and you pick up every fifth representation of the load and that is how. So, so those are things that there is a difference between measurement of load. Digitized record of the load and the delta t that you require. So these are two different, but for now let me just not get too much caught up in how the measurement is done. Let us just assume that you know even if I have down sampled it I have got it in this form. This is the form that is either delta t that I need. So, how do I now proceed for this? The first thing that I am going to do before I go back to the time although I started the Duhamel integral and came to the frequency domain, I would first like to do the frequency domain.

How we do the numerical analysis frequency domain analysis because that is typically more used in today’s world purely because if you have a linear system, principle of preposition is valid and then frequency domain analysis is a far superior and quicker approach a more computationally efficient approach for solution of response purely because in the frequency domain, if you remember the convolution integral which is a Duhamel integral basically transforms into a product you know a multiplication orbit it is a complex multiplication, but it still a multiplication. So, that is more trivial approach. So, now, I am going to really talk about how when you do Fourier transform the Fourier transform how do we do it in reality.

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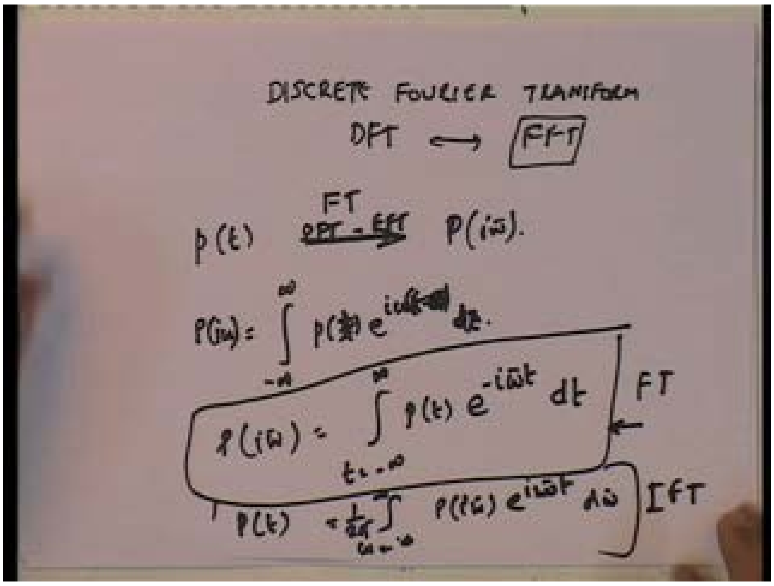


And that approach is called as I mentioned in last time in passing, it is called the discrete Fourier transform. So that is DFT and FFT which is the fast Fourier transform. Everybody would have heard today that what is of FFT of a signal. Well, what is an FFT of a signal? It is actually a Fourier transform. I have done the Fourier transform. I have taken load p t and transferred it into the frequency domain and the reason again remember I put p I omega not because p is just to show that p is a complex function of omega bar that is all I am trying to show.

So, essentially this procedure going from here to here is the Fourier transform and so we use the discrete Fourier transform. I will essentially write down the discrete Fourier transform and the FFT which is developed by coolly talky at you know IBM lab was essentially a faster way of doing this DFT. So I am not going to spend too much time on d on the FFT because the FFT is really a mathematical procedure which does not add to our knowledge.

I will leave it up to you to read the original paper by coolly talky which appeared in you know in 1962. But you know, you really do not need to know it. If you know the discrete Fourier transform, you know the basic information that you need to put into any FFT routine or FFT you know analyzer that you need to put need to you know use today because typically FFT analyzers are black boxes. But there are some fundamental aspects that you need to consider before you do the Fourier transform. So, now the thing is what is a discrete Fourier transform? We are essentially saying that look, this integral which was the integral minus infinity to infinity which is p I omega which is equal to what was it? p of t e to the power of i omega p of tau i omega t minus tau i t d t. So, this there is no tau here it is just t.

The t minus tau comes in the Duhamel integral. So, essentially again I am going to rewrite this I omega bar is equal to t going from minus infinity to infinity p of t e to the power of this is the frequency domain representation of the and the other thing that you have is the this is the Fourier transform and again I want to rewrite that p of t is equal to omega going from minus infinity to infinity 1 over two pi. p i omega e to the power of i omega t d omega t. This is the inverse Fourier transform. So, these so essentially what I am doing right now is, putting down this particular form now. If I do a discrete Fourier transform, what we are essentially doing is, we are going to solve this problem and this problem by doing what? A summation. In other words, this will become delta t this will become delta omega and that is how we are going to do it and how would we do it.

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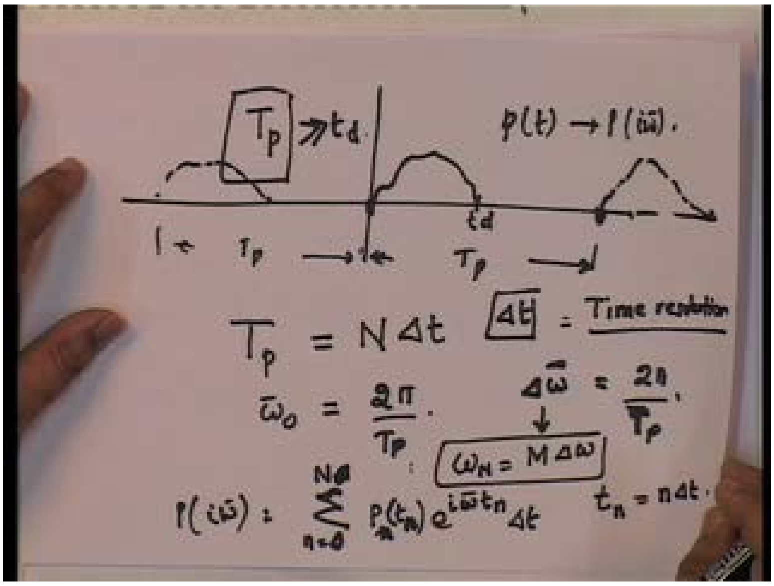


Now the only way to do it is by saying that look T p which we took. Remember periodicity which we took as infinite is not infinite. In other words, what I am saying and this is very very important to understand that if I have any load and I have this T p, where this is t d, duration of the load. T p is always significantly greater than t d and So, I am assuming that there exists another and there existed another and there will be like you know, every T p this is T p also. So, T p is much greater than the duration that is our assumption if we assume that, then essentially what are we saying that if I am doing a digitized, I will say that look T p. I am now going to define it as N delta t. In other words, this duration is put into n. There is another thing and that is delta omega 0 the lowest frequency that I can represent is going to be equal to. That one since T p is large I will call this as my delta omega bar, because if T p tends to infinity, this becomes d omega bar.

So, in other words, if T p is large, this becomes my delta omega bar. In other words, if you really look at it, delta t represents my what is it represents? it represents my time resolution and delta omega bar if you note, it if this is delta omega bar you know once in other words I made it a periodic load if I made it a made a periodic load what do I have? I will have the largest frequency is going to be equal to m times delta omega bar Omega n. Because in other words, if you look at it what I am doing is I am discretizing delta t. t from minus infinity to infinity.

Now obviously, it is never minus infinity to infinity. In this particular case, it becomes then what? it becomes 0 to T p and you know if you look at this particular one oh it I mean it could be like minus T p to T p by 2 something of that is what where you know we break it up, but those are not issues you know it is it is typically 0 to T p because that is the kind of time frame that I am looking at.

And this one essentially then becomes what? It becomes now no omega note that omega has to go from minus infinity to infinity. Because for this representation to represent, you require both plus and minus and delta omega essentially represents my frequency resolution for this integral. So this is a very very interesting point to be noted and that is that I have the discrete Fourier transform. I am going to take the Fourier transform. In other words, I am going from p t to p i omega. This is the process that I am doing. If I do that process, what does it happen? It basically is p i omega bar becomes equal to. Now, summation n going from 0 because it starts here. So that is a 0 here and it goes up to n because this is n, but this becomes then what?

It becomes if T p is n, then this becomes 0 to n minus 1. Because if this is the thing, there is n because you start at 0 t and this ultimately becomes n. So there you go from 0 to n actually there are n plus one numbers. So you go to from 0 to n. Now, if you really look at it, it is p at t n e to the power of i omega t n because you note that this is all n. So this is p t n p at t n e to the power of i t n into what? Into delta t that is, what we have. So now, now the thing is that, note that t n is equal to n delta t. So if you really look at it, this becomes what? e to the power of i omega bar into n delta t. So if I really represent this properly, then I get the following form. Let me just write that form for you.

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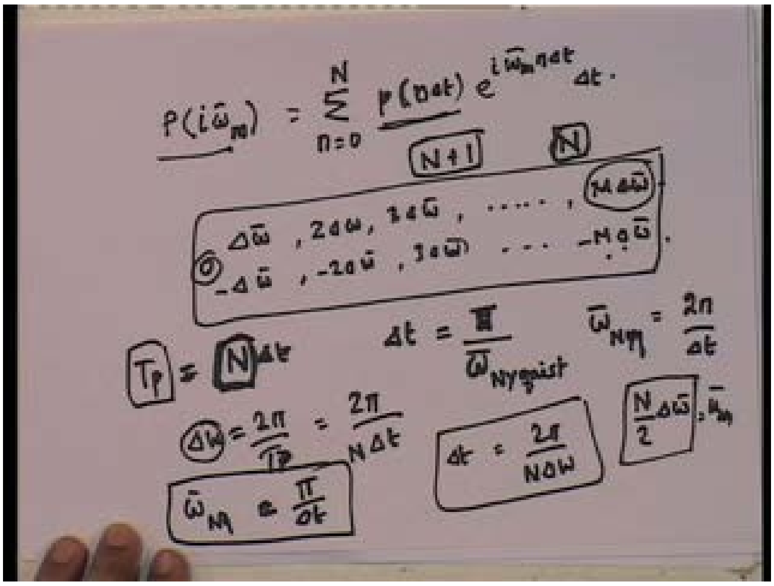

I get the following form and that look at it that I have p at I now the question then becomes that look even this is you know I cannot do p i omega bar because again I am actually finding out p at i omega bar m. Why? Because, you know if I am doing a and I have essentially what do I have? I have n going from 0 to n and I have p of n delta t. That is the p that I am picking up and here e to the power of i omega bar m n delta t into delta t.

Now the question that becomes that you see even, the even the frequency domain is being transformed into this. Now note, so therefore what we have is, frequency domain has delta omega bar, 2 delta omega bar, 3 delta omega bar, blab blab blabla till I get M omega bar. Now note that, there are, there are these issues that for every delta omega bar, I have a minus delta omega bar, minus 2 delta omega bar, 3 delta omega bar, blab blabbla and a minus M delta omega bar.

So I have to have all of these. Now the question then becomes is the following that can I just look at one side? Because you see, I have certain aspects which are very very important and that is, that you know if I have both real and complex, then this part is the complex term. But it actually goes and represents a real term and since this one has. So, how many how many values does it have? if you look at it, there is a 0 value how many how many discrete discrete terms do I have? if I have n, see T p represents N delta t. So, in other words, I have n T p terms I mean sorry n delta t terms. So this time domain is this and if you look at it, we can we can look at this particular one how many terms are there? There are n plus 1 terms in the specific one.

Now if I take the 0 term out, you are essentially looking at n terms. Now in the n terms, you will have to put in these many terms. So if you really looking at it, delta t in a way represents pi upon the nyquist frequency. We call this the Nyquist. This is the largest delta omega bar that can be represented in this system, so that if you look at, it is equal to what? You have 2 m terms. So therefore, omega bar nyquist 2 m terms has to be equal to n. So, the nyquist then becomes essentially 2 n terms have to be represented. Now look at it. If you really look at it, strictly it should be 2 pi upon delta t. But you see, because you have to have plus and minus. Essentially this becomes you know, you would think of the largest if there was no minus right and these terms N and M have to be the identical values because you know they represent the same.

Then what would happen? You would have a situation where if N was equal to M. Then what will be the highest frequency? Well, if that would be n into delta omega bar now n is you know the lowest frequency delta omega bar is equal to 2 pi by t 2 p. So, 2 pi by T p t p is equal to what? T p is equal to 2 pi upon n. T p is equal to yes n delta t. So, if I put this up, what does delta t becomes? It becomes 2 pi upon n delta omega bar. Now, if you really look at it, if you only had one of them. m is equal to n, then what would be the highest frequency? Well the highest frequency would be n delta omega bar. There is no doubt about it. But see, because you have both. It is actually n by 2 delta omega bar which is the largest frequency that can be represented in this and this is the one that is called the nyquist frequency.

So, if you just substitute that in here, this is the largest frequency. This is the nyquist omega bar and if you really look at it, omega bar nyquist therefore, substituting this into this, n delta omega bar becomes 2 pi n delta t and upon 2. So, this becomes pi upon delta t. So, so one aspect is so the n what should be T p is determined by delta t and the number that you want and how do you determine this number? Well this number has two parts to it and you know one part which is that look in the frequency domain. Delta omega bar is equal to 2 pi upon n delta t. So, 2 pi upon T p should give me. So, T p should be large enough such that delta the frequency resolution is good enough. On the other hand, the delta t that I choose you know, automatically determines the maximum nyquist frequency that I have.

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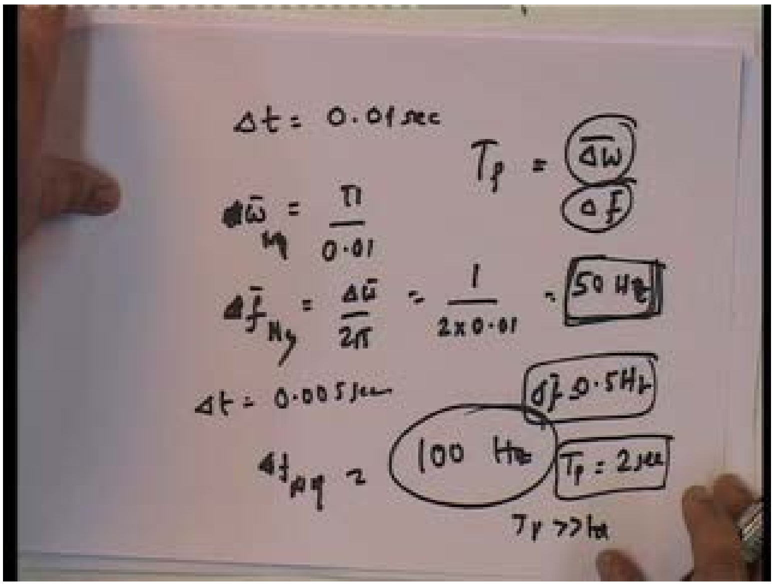


So if you look at it, you know I talked about essentially the following thing and that is that look delta t is equal to let us say 0.01 second. If delta t is equal to 0.1 second, then delta omega bar becomes pi upon 0.01 and the frequency resolution which is equal to delta omega upon 2 pi essentially becomes one upon 2 into 0.01. So it becomes 50 hertz. In other words, if you do delta equal to 0.01, the maximum frequency that can be considered in the analysis is 50 hertz. Now if you know that, that is not good enough. What you have to do? You have to take a smaller delta t. For example, if you take delta t equal to 0.005 second, then this is not delta f. This is the nyquist, f nyquist. So then f nyquist would be equal to 100 hertz. So therefore, T p determines delta omega bar.

For example, you know I mean I would call it you all the delta omega bar is circular frequency. I will represent it by f, because this is something that we know. So now, T p has to be large enough. Let us say how long should it be? Well, what kind of frequency resolution do I want? Well, if I am going to have a nyquist frequency of hundred hertz, I would probably like a 0.5 hertz delta f. If delta f is point five hertz, what does T p have to be equal to? T p by definition has to be two seconds long. So typically you know if I am looking at you know the typical kind of things in which you need these, the duration, time duration. See I mean, note also that T p by the way has to be greater than t d.

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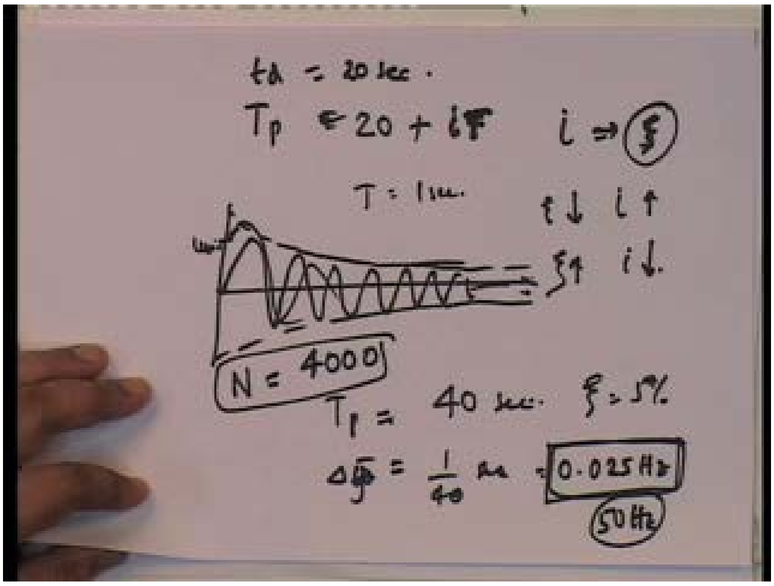


So therefore, what do we have? Well, we have this situation where T p is typically if t d is of the order of above 20 seconds, T p should be 20 plus.

Now, what additional things have to be there? Well, let us go back to that. Remember well you know, this is initial conditions at rest ant that is the one that you are doing. Now when you are taking this, it implies that, this will give you certain aspects. So therefore, now if it is a single degree of freedom that you are looking at I know that I will require some i times T, where T is the free vibration time period of the structure and what i? Well, i will depend on what? On xi. Because remember this is free vibration. At free vibration, remember that e to the power of I omega t. So I need to see that the previous one's free vibration with a particular t d you know u t d and this thing does decrease down to small enough. So i is always dependant on xi.

If xi is small, i goes up. If xi increases i goes down. So therefore, T p is typically of the order of about you know, if you have a time period of about one second, T p would typically be of the order of about 40 seconds for xi equal to 5 percent. Why? Because 20 cycles of 5 percent brings it down to sufficiently small value such that errors are not introduced. So if T p is 40 seconds, then automatically delta omega bar is equal to what?

One over sorry delta f bar is equal to 1 over 40 seconds which is equal to what? 0.025 hertz. So in other words, your this thing is automatically set at 0.00 hertz and if you are doing 0.01 second and you have got 40 seconds, N is equal to what? N is equal to 4000. Well, what is the nyquist frequency? Well, we know delta 0.01. It basically means nyquist frequency is 50 hertz. 50 hertz 2000 points. Well, 2000 points and minus 2000 points that makes it 4000 points.

So you see how the entire thing matches up. So essentially you have to worry about the frequency resolution and the. So your T p is determined by frequency resolution and delta t to a certain extent is determined by the nyquist frequency. Nyquist frequency means well, how many frequencies do I want to consider and those are issues that you know come from practice and also knowing what kind of structure there is, what kind of load that I have, you know what Nyquist frequency should I take, those are not issues.

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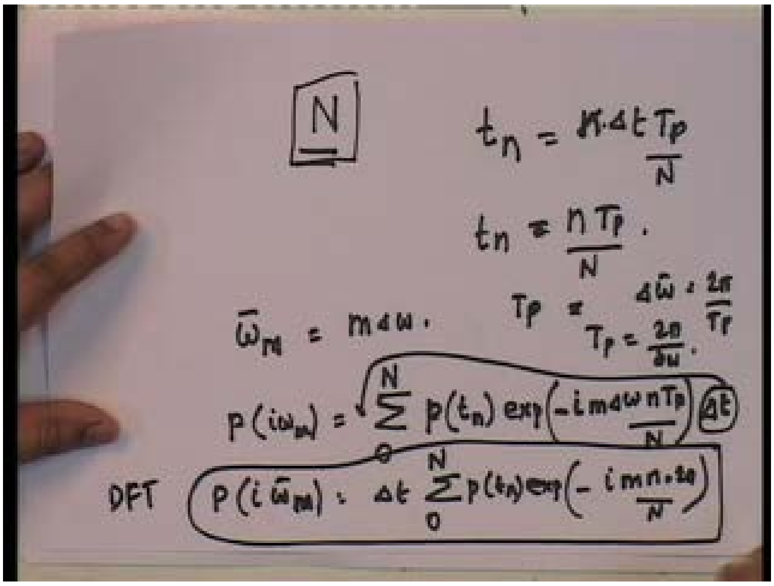


You know, all I am trying to say is that typically the value N is known and also understand that what this means is that t n is equal to n times delta t and you see, so therefore, n times delta t and delta t is equal to T p upon n. So if you really look at it, this becomes n T p upon n that is my t n. Similarly, if I want to look at omega bar m, that is equal to m into delta omega bar. So I have a situation where my p into i omega bar m is equal to. Now doing the summation 0 to n and what do I have here? I have p of t n. Now p of t n is equal to what? p of t n is equal to t n where t n is this, it does not matter, but more importantly is the exponential term. Exponential term is what? It is equal to. I have minus i. Now omega m is equal to m delta omega. Now delta omega into t n. t n is equal to n T p upon n. Now let us look at what? T p is equal to T p is equal to what? We had already said it delta omega bar is equal to 2 pi upon t p.

So, T p is equal to what? 2 pi upon delta omega bar. So, this is my term into delta t. Now, I can take this delta t outside, because this delta t is not a function of n. So, this can go directly here and note that T p is 2 pi upon delta omega. So, if I plug it in here what do I get? I get 2 pi upon delta omega delta omega and delta omega cancel out. So, what do I have? I have p I omega bar m is equal to delta t m going from 0 to n p of t n into exponential of minus i m n and there is a 2 pi. So, into 2 pi upon n. So, this becomes i 2 pi m n upon n and that is my omega. You see, this is the discrete Fourier transform now. So therefore, n is defined and once n is defined note that how many terms I have will be determined automatically by this and if I look at the flip side, if I look at my p t the inverse. So, this is my discrete Fourier transform. Now, I want to look at the inverse Fourier transform.

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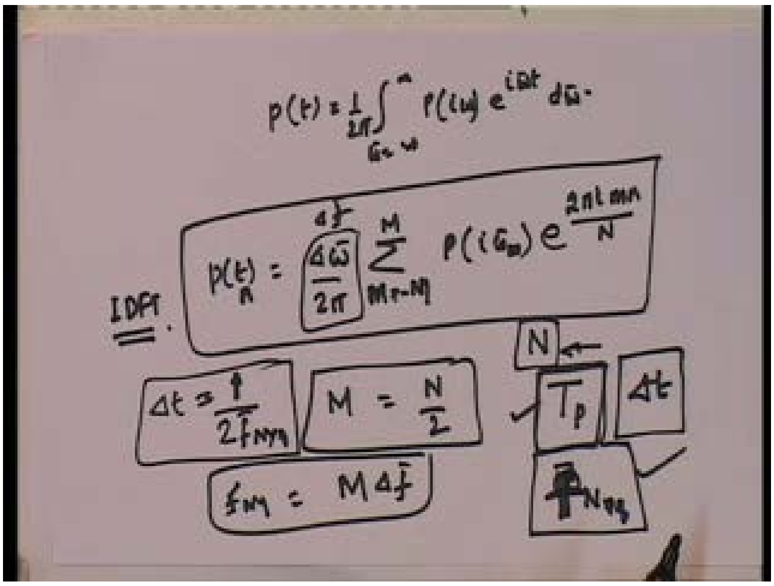


If I look at the inverse Fourier transform, what do I get? Well, the inverse Fourier transform is what? p of t upon 2 pi omega bar minus infinity to infinity p of i omega bar into e to the power of i omega bar t d omega bar. So, this becomes then p t is equal to delta omega bar comes out. So, it becomes delta omega bar upon 2 pi integral. Now, if you look at this, this becomes m going from minus n no m going from minus m to m because frequencies have both sides. p i omega bar m which I know is m into delta omega into e to the power of i omega bar t. Now, I already know what this looks like? This is 2 pi i m n by N from the previous term. This is my inverse discrete Fourier transform and I know that look M is equal to N by 2. I know that M is equal to N by 2 because note that you know I mean this is t n by the way; obviously, this is at t n. So, I have a situation where t n is equal to p i omega bar. So, this is the inverse discrete Fourier transform.

Now, the only thing that happens is that, remember that T p delta t and omega bar nyquist as the delta t is not in your hand. It is T p and these which are typically the design parameters. Once you have this and look this is you know we always look at this as delta f. You look at, if you look at this, this is nothing but delta f. So, you are looking at nyquist frequency and you know I mean if you look at f nyquist is equal to M times delta f. So, these are the nyquist frequencies at say 50 hertz, 100 hertz. This is something that I know. This is something that I can define. If I define this, automatically delta t becomes equal to 1 upon 2 f prime nyquist. Automatically, this is something that delta t we know once we know delta t I can find out N now.

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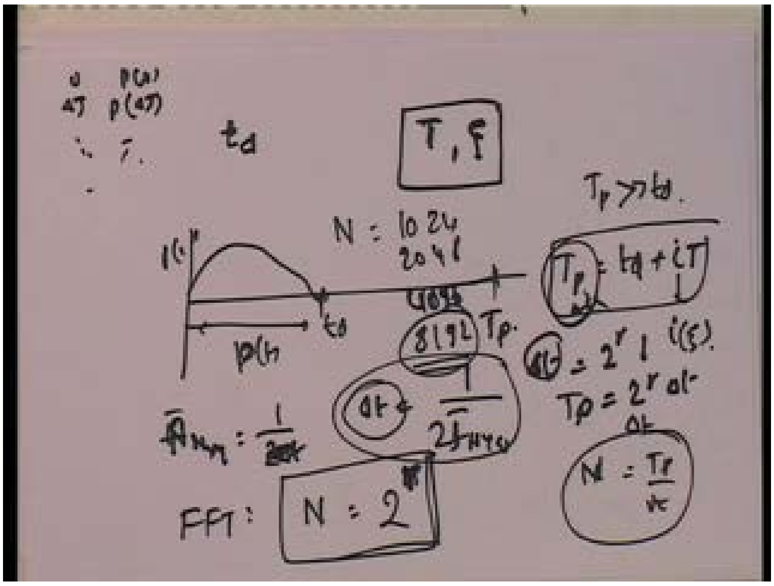


So in other words, if I want to review the frequency domain analysis, it is that look what are the things that I know. I know t d. This I know, because I know the duration of the load and I also know the digitization. That does not matter. I know you know some other term I will use most probably. I will use delta tau. You know p delta tau 0 p 0 these are the things that I have, but delta tau is really the digitization record, then I know time duration. Once I know time duration, look I am going to be doing it for a structure. So, I know t and I know xi. I have a load with duration, fixed duration and I have a structure where I know the t and xi. Once I know t and xi, I know that because I am going to make it a periodic load I need a sufficient number of zeroes. So in other words, I am given this record up to t d. This record I have. This is the record of p t. I have this record.

Now, what do I have to do? T p is not t d. T p is greater than t d and typically T p is equal to t d plus i T where i is a function of xi. So from that given t p, I can find out my total time that I need to put in. I also know my nyquist frequency. How I know my nyquist frequency? Let us not worry about it. Now there are there we will come to that towards the end of the course. I know my nyquist frequency. I know my delta t is equal to sorry, this is wrong delta t is equal to 1 upon 2 f nyquist. T p delta t once I know that n is equal to T p upon delta t. So, I can find out my distance now. I just want to end with the fast Fourier transform. The fast Fourier transform is nothing but the discrete Fourier transform where n is a function of is a power of 2.

In other words, n can be 1024, 2048, 4156 or you know, typically you know up to 4001, 4096 or 8192 anything more than 8192 you are dealing with numbers which becomes very very large. So, that is all that is you know and then what happens is, you know this is the minimum T p and then what you do is, you know delta t. You know delta t from this. So, once you know delta t, all you do is, you take two to the power of r. I mean T p is equal to 2 to the power of r delta t where this is the T p that you define a total power of r delta t is more that this minimum value that you require and that in essence, is the discrete Fourier transform.

So, two parts one is, that you should know your nyquist frequency. Once you know your nyquist frequency, you know delta t. If you are using the fast Fourier transform, you have to use a T p which is slightly larger than delta T p minimum. So, if you know t d and you know i of t which is a function of xi. You know i, the minimum number of cycles to you know reduce the error and I mean because remember the previous one has to die out initial conditions have to be as close to as rest as possible for the Fourier transform for me to give me an accurate estimate. So, this in essence is the procedure that you follow. So, you always have to put in what are known as trilling zeros to build up a period after the duration. This is most important.

Thank you very much.