Structural Dynamics
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Lecture - 12
Frequency Domain Response Analysis

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Hello, today we are going to be looking at a continuation of what we looked at last time, and I will call it as the classical Frequency Domain Response Analysis.

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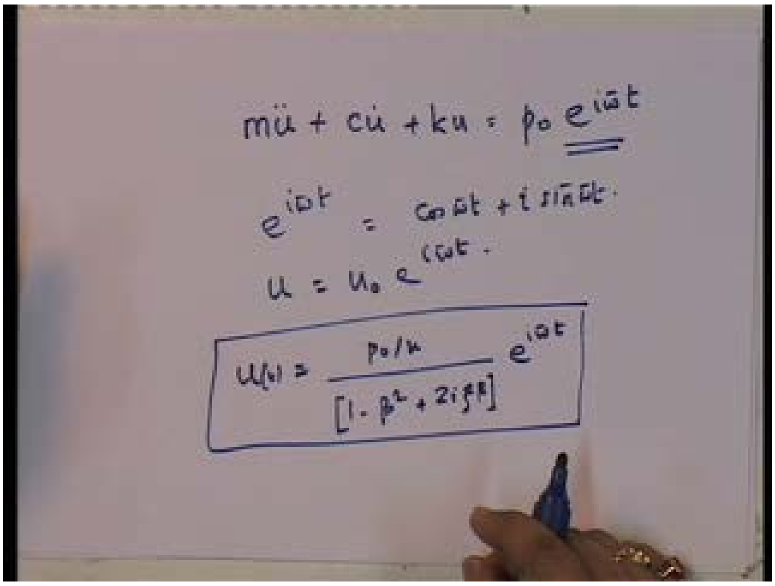

So, the last time if you looked at it, we started with the derivation of the response of a single degree of freedom system to a harmonic response, which is defined by this function e to the power of i omega bar t. If you really look at it, e to the power of i omega bar t is nothing but e to the power of i omega bar t, which is equal to cosine omega t plus i sin omega t. So, in a way, if you look at it, it is a complex loading function but it essentially is a harmonic loading function.

So, if you do that, we saw that the u is equal to u 0 e to the power of i omega bar t and substituting this, we get ultimately that, u was equal to p naught upon k into 1 minus beta squared plus 2 i zi beta, e to the power of i omega bar t. This is the response of course, it is obvious that, we are looking at the steady state response only, either no point in mentioning this that, an harmonic response. We always assume for now that, the harmonic load was there for all time, and so therefore there is no question of any initial conditions in the problem, so it becomes a steady state solution. So, this is the response of a harmonic load now, the next step to look at is that, even a periodic load can actually... so this was the harmonic response and then if we look at a periodic loading.

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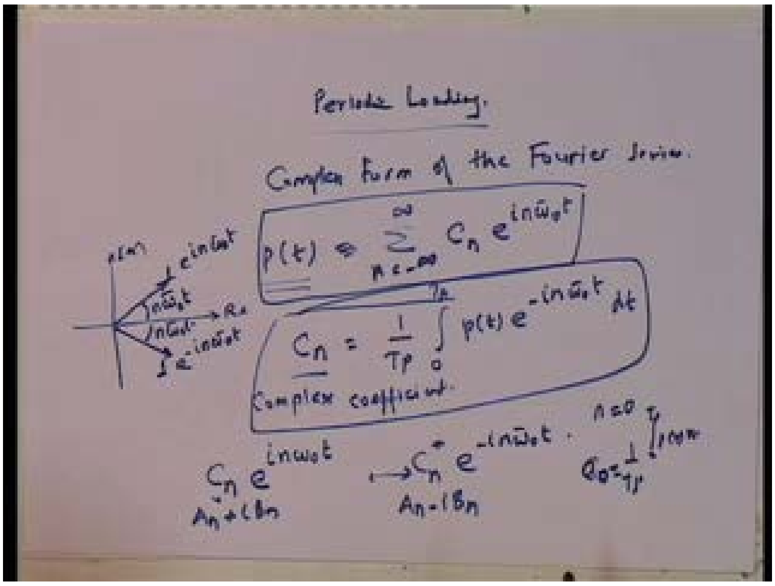

A periodic loading, remember the Fourier series, actually there is a complex form of the Fourier series. The complex form of the Fourier series is actually very elegant, it is of this form p of t is equal to summation n going from minus infinity to infinity C n e to power of i n omega 0 t where, C n is obtain from 1 upon t p, 0 to t, p of t e to the power of minus n omega 0 t d t. So, this is the complex form of the Fourier series. So, in other words, if you look at it, that this is a complex integral and therefore, this is a complex coefficient.

Now, you may ask that, why do I have a regular loading, a real loading it is equal to a complex coefficient times this form. You will see something very interesting, you will see that, this is n going from minus infinity to infinity, so therefore you see for each e to the power of i n omega 0 t, there is a corresponding e to the power of minus n omega t. So, if you look at the complex frequency response function, you see this Fourier series actually represents, this function is actually minus i n omega t, this is the unit harmonic and this is a unit harmonic which represent e to the power of n 0 t.

So, if you look at this, what this function actually says is that, for each positive loading there is an opposite part. And in fact, what actually happens is, we can actually show is that, if this is C n, this becomes C n complex conjugate. I assume, what are complex conjugate is, if C n is A n plus i B n then this is of the form A n minus i B n. So, if you see here, automatically if you look at this projection, these two if we sum them up of unit, you get a net if you look at it along the real direction.

So, in other words, they have the same magnitude, these two are the same magnitude and that is how it goes, I am not going to derive it from first principles. It is fairly obvious derivation that, only if C n is of this form and the opposite one is A n minus B n, will there together land up giving you a real quantity. The beauty of this is that, you see it becomes a very simple form of the Fourier series. If you looked at that the trigonometric Fourier series, there is A 0 then there is an A n cosine omega n t then there is also a B n sin omega bar t, both of them going from 1 to infinity.

Now, you see everything is in incorporate in this, let us look at the n equal to 0 term, n equal to 0 term obviously, if you look at there is minus 1 to infinity, there is 1 to infinity, these two are there. And then the n equal to 0 term, if you look at n equal to 0, e to the power of i 0 is 1 so that basically becomes C 0. And that C 0 is what, let us look at what c 0 is, if you look n equal to 0, this disappears this becomes 1 and what is C 0, C 0 is equal to 1 upon T p, 0 to T p, T t d t.

So, note that, the C 0 is actually nothing but the A 0 that we have got and it is a real quantity. So obviously, the Fourier series contain both and the point to note is that, e to the power of i n omega 0 t is cosine n omega t plus i sin omega t and e to the power of minus n, if cosine minus i sin. And so all of these point become obvious and that is actually the reason, also you prove that, these two are complex conjugates of each other anyway. So, so much is the complex form of the Fourier series, how do I get the response of this periodic loading. Again we know that, note that, in a periodic load, I take each load, find out it is response and then solve it.

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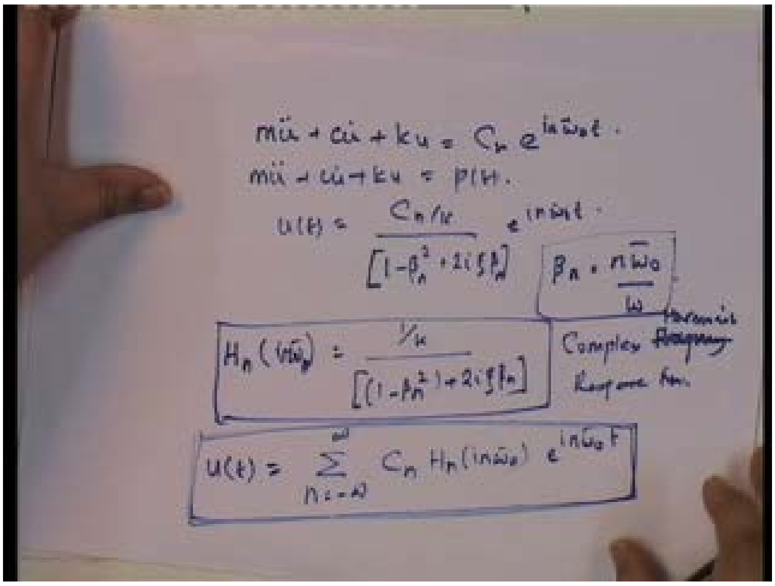

So, if look at each response what does it become, it becomes m u double dot plus c u dot plus k u is equal to C n e to the power of i n omega t. Now, note we have already solved p naught sin omega t, what it will become, u of t is going to be equal to C n upon k into 1 minus beta squared plus 2 i zi beta into e to the power i n omega bar t. Now, and I am going to identify a specific term and this is beta n squared into beta n where, beta n is what, beta n is n omega 0 upon omega, which is the natural frequency of the structure.

So here, H n i omega and this is i omega bar is equal to 1 upon k into 1 minus beta n squared plus 2 i zi beta n. Note that, this is actually n omega 0 and this term is what, this term if you look at it, it is the response of a unit complex harmonic function, this is the response of the structure to this to a unit amplitude complex harmonic. So, this is actually called as the complex frequency response function and this is actually complex harmonic response function and note that this is the function of n omega bar because beta n changes with n omega bar.

So, once we do that then the solution of to load p t, what does m u double dot plus c u dot of k u equal to p t where, p t is a periodic load then the response becomes nothing but u of t which is equal to each harmonic summed over all of them. So, this basically becomes the following, it becomes n going from minus infinity to infinity, C n H n i n omega 0 e to the power of n omega 0, this is irresponse to a periodic load. So, elegance you see earlier time, we you know if we looked at the trigonometric Fourier series then this response actually contain three different responses.

Here, this is a very, very elegant way of representing the Fourier, the only problem what is the only problem, the only problem is that, you need to solve a complex integral to get C n, that is the only a issue that we have in this particular case. So now, we have done it for harmonic load, we have done it for periodic load now, let see what happens when you consider an arbitrary load. So, arbitrary load with a given duration, so let us take this kind of situation.

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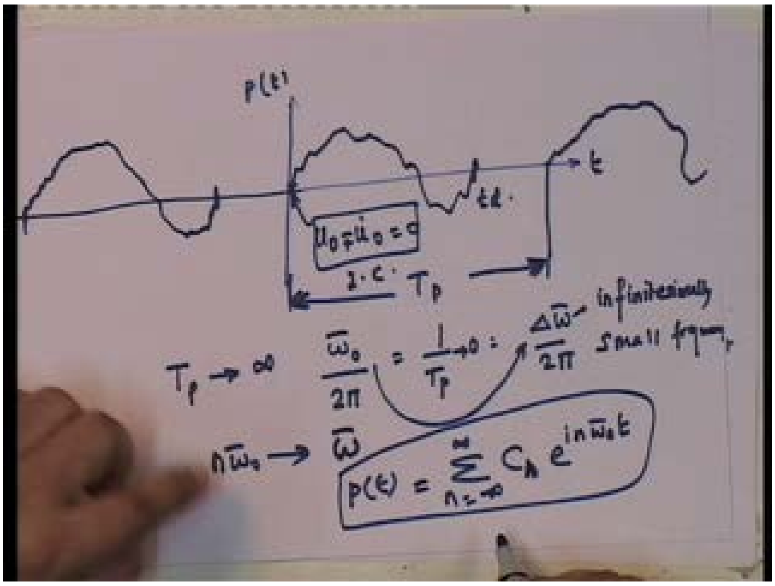

So, I have an arbitrary load p of t and this is t and let say, this is some load of duration t d so this is an arbitrary load of length t d or duration t d. So now, the question is, how do I use the concept, that I have just developed of the complex harmonic response function and also the elegant way, that we wrote down the periodic load. Now, let us take a situation where, I am going to say that, look let me consider this load which is of a particular duration.

Remember, even the generalize loading that I did using Duhamel integral, assume that there was a particular duration, because you cannot keep doing the integration forever. So, these are always generalize loads, but of a finite duration of course, the finite duration is not necessarily a pulse load or an impulse load, it is an arbitrary load of fixed duration where, t d upon t could be anything. It could be n times t or whatever I mean, it does not matter, some significant numbers time t but still a fixed duration.

So now, the question becomes that, why cannot I assume that, note that at this point, I have at rest initial conditions such that, this is equal to 0, initial conditions are 0. So, why do not I assume that, these being repeated after some T p, let me assume that, it is being repeated after T p. Now, understand that, when I say this is being repeated after a periodic, I have say that this is the only load but what I am now saying is, let us assume that this is a periodic load with time period T p, the T p is not the same as t d note.

Now, the question becomes suppose I assume that, T p tends to infinity, what is that mean, that this load, generalize arbitrary load of a finite duration can be considered as a load, which is being repeated with infinite periodicity. Cannot I do that, I can do that, it is truth because all I am saying is infinite periodicity means what, that it is not actually a periodic load. It is just that particular load, it just that I am saying that, after every infinite time, this load is repeating itself now, what is an infinite time, infinite time does not exists.

So therefore, actually when I am saying it is a periodic load with time period of repetition infinity, I am actually saying that, this is the load that I am considering. Now, note that, if T p goes to infinity then what happens to omega 0 upon omega bar 0 upon 2 pi. Let us look at this, omega 0 by definition is what, omega 0 upon 2 pi is by definition 1 upon T p. So, as T p tends to infinity, what happens to omega 0, omega 0 actually becomes an infinitesimally small.

So, as T p tends to infinity, 1 upon T p tends to 0 and therefore, this tends to this where, infinitesimally small, this becomes an infinitesimally small frequency. So, what does n omega become, n omega 0 becomes what, if omega 0 tends to delta 0, this basically tends to omega bar. In other words, what am I doing, if you look at a periodic load, I am saying a periodic load is one, which has a… If you represent as a frequency domain, what it basically becomes is that, a periodic load consists of n harmonics where, the lowest harmonic is omega bar 0 and each subsequent harmonic is...

So, let us look at it this so if I have omega 0, what is omega 0, omega 0 is 2 pi upon T p by the definition because T p is a periodicity so lowest frequency is 2 pi upon T p. So, what we are saying is that, look there is one 0 frequency, which is the A 0 and then I have omega 0, I have then 2 omega 0, 3 omega 0, 4 omega 0, that is what a Fourier series is. Now, you see if the periodicity goes to infinity what happens, the distinct harmonics the step, as I said omega 0 bar essentially becomes what, it becomes delta omega which basically means this.

So, if I do n what am I doing, I am stepping along the frequency axis so n omega 0 is actually omega bar instead of, doing discrete harmonics, I am doing a continuous sweep across the entire frequency domain. And what is the highest frequency, this is the interesting point, we will discuss this in a little while but let us just now put this thing into prospective.

So, essentially if we look at this, I am going to now say that, so therefore, let us look at what happens to the frequency domain, p of t is equal to summation n going from minus infinity to infinity C n e to the power of n omega bar 0 t, this is if the loading was periodic. Now, if the loading is not periodic as T p tends to infinity all this happens, this thing can be written in this form, I will define a new parameter.

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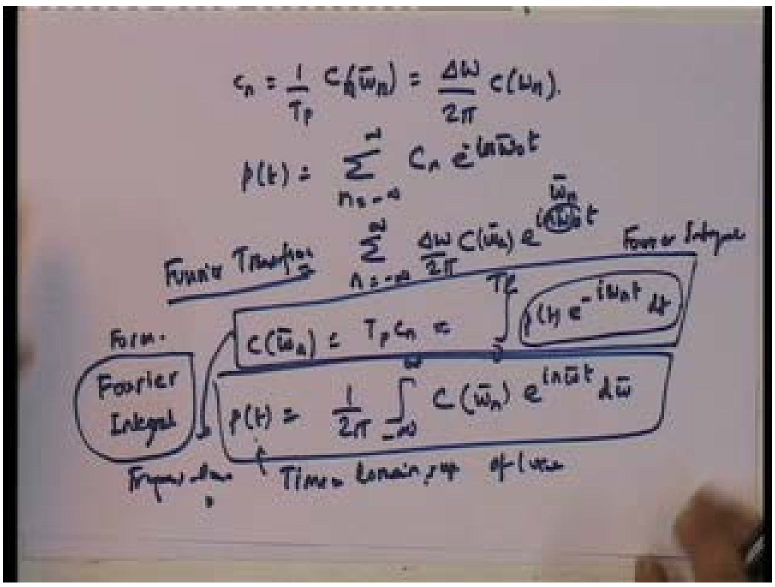

I will define a new parameter which will be essentially, this I will define a small c n, which is equal to 1 upon T p into the C n, capital C n that I have. The reason behind it is the following that, if I write it in this fashion now, call this as C omega bar n. So, this is going to be equal to by definition, delta omega into 2 pi C omega n. So therefore, the p t which is equal to summation n going from minus infinity C n into e to the power of i n omega 0 t basically becomes then as it becomes delta omega by 2 C omega n exponential i n omega 0 t.

Now, if you look at, n omega bar becomes what, it basically becomes omega bar n so p of t is equal to this where, C omega n if you look at this, this C omega n is equal to T p n. And since T p C n and T p C n if you remember is 0 to t p, p t, it is e to the power of minus omega n t d t. And if you look at this, if this is infinitesimally small then actually p of t becomes 1 upon 2 pi minus infinity to infinity of C omega bar n e to the power of i n omega bar t d omega bar.

See these two parts, this actually is nothing but what is known as a Fourier integral form note that, this what does this represent, it say that look p t is of this form where, C omega is given by this. So, this if you look at, this is the form, this is the time domain and if you look at this particular function, this is nothing but the Fourier integral and this is the... So, in other words, if you really look at this, this is the classical Fourier integral and this is known as a Fourier transform. What is that, that this represents the time domain representation of load and this represents the frequency domain representation of load.

So, this is known as the Fourier transform, this is the Fourier transform, it is the inverse Fourier transform and the entire thing is known as the Fourier integral and these are a pair. This represents the time domain representation and note that, this is a continuous function in omega, this is a continuous function and time so this is known as a Fourier transform. And essentially what happens then is the following that, what does this represent, this represents the amplitude at each frequency that is contained in this load p of t.

So now, we have a very interesting concept and that is, let us look at that means, even an arbitrary load can be represented in this form where, each harmonic note that, this… What is this, if you look at this and you look at this term, is nothing but saying that look p of t is actually a summation now, integral is also summation. So, p t is a summation of each of these forms so that means, all I need to do is really find the following.

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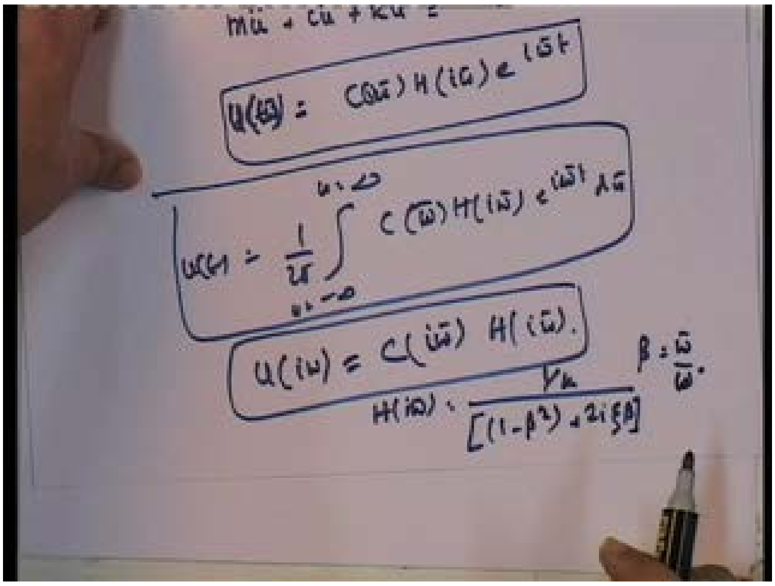

I need to find out the response of this where, in omega n I am dropping because this is really omega bar, it is a continuous scan of the frequency. I know this, I know the response to this, response to this is nothing but C omega bar H i omega bar e to the power of i omega bar t, that is the response. Once I have this response then obviously so this is for each one so what is the total one, the total one will become just this, 1 upon 2 pi minus infinity to infinity, omega going from infinity to infinity, C omega bar H i omega bar into e i omega bar d omega bar.

You see, this is u of t and actually this represents, it is not u of t, it actually represents if you look at it, this is the function of i omega bar, this is also function of i omega bar. So, this if you look at it, u i omega bar is actually, if it drop the e to the power of i omega bar t is C i omega bar H i omega bar. Then i omega bar is just there to represent that, all of them are complex frequency response functions. And what is the H i omega, we already solve this, have not we have solved it and we have called it as H i omega bar is equal to 1 upon k, 1 minus beta squared plus 2 i zi beta. Where, beta is equal to omega bar upon omega where, this is the specific frequency and this the natural frequency of the structure, elegant.

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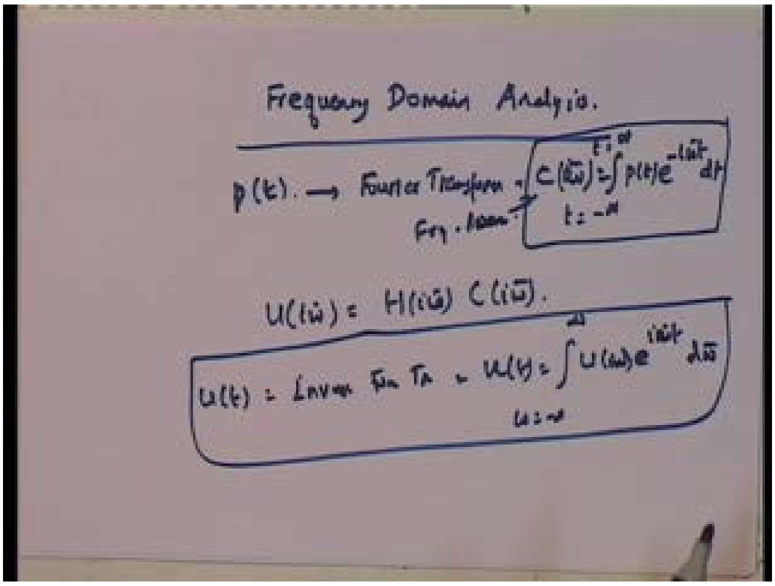

So now, let me write down what this entire frequency domain analysis shows, this is known as frequency domain analysis. How do I go through the frequency domain analysis, I have load p of t, the first thing I do is Fourier transform, which is C omega i omega bar is equal to… And now, I am going to call this note that, if you note 0 to T p where, 0 to T p actually is T p goes to infinity, we can write it as minus T p by 2 to T p by 2.

So, this also basically becomes minus infinity to infinity where, time goes from infinity goes and C i omega is p of t e to the power of minus i omega bar t d t so that the Fourier transform. These are all complex integrals let us not worry about that, you should be able to I mean, we can solve for it but this is the frequency domain representation of the load. Then next find out u i omega bar, which is H i omega bar into C i omega bar and then do u of t, which is actually the inverse Fourier transform, which is u of t is equal to omega going from minus infinity to infinity, u i omega into e to the power of i omega t d omega bar, that is the way. Note that, if you remember now, let me go back so this is for any arbitrary load, arbitrary load if we look at it, if I use the time domain approach, which is Duhamel integral, the Duhamel integral becomes, what?

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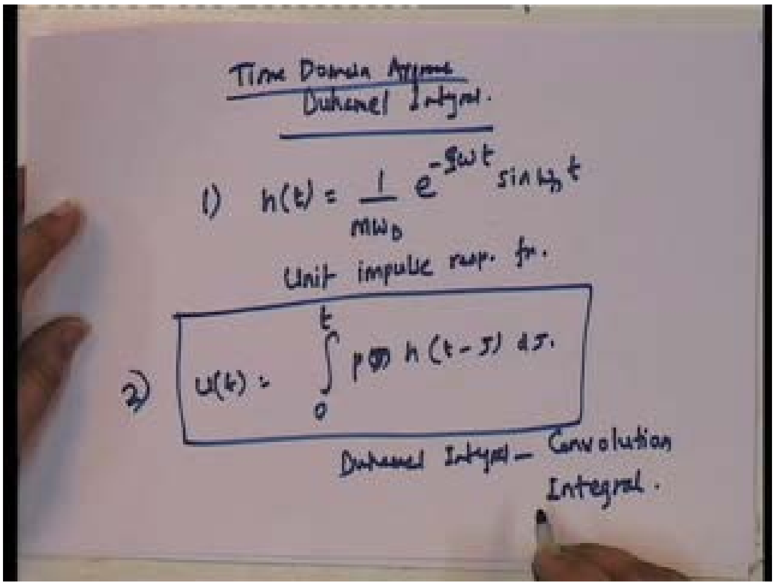

I first find out h of t, which is equal to 1 m omega D e to the power of minus zi omega t sin omega D t, this is what the unit impulse response function. The first for a given structure, you find this out and then u of t is given as 0 to t, p of t h of t minus tau d tau, sorry p of tau h of t minus tau d tau. This is the response using the Duhamel integral, which is essentially a convolution integral. So, this is the, what is known as the time domain approach, first find this out and second just do the convolution integral so that is the time domain approach.

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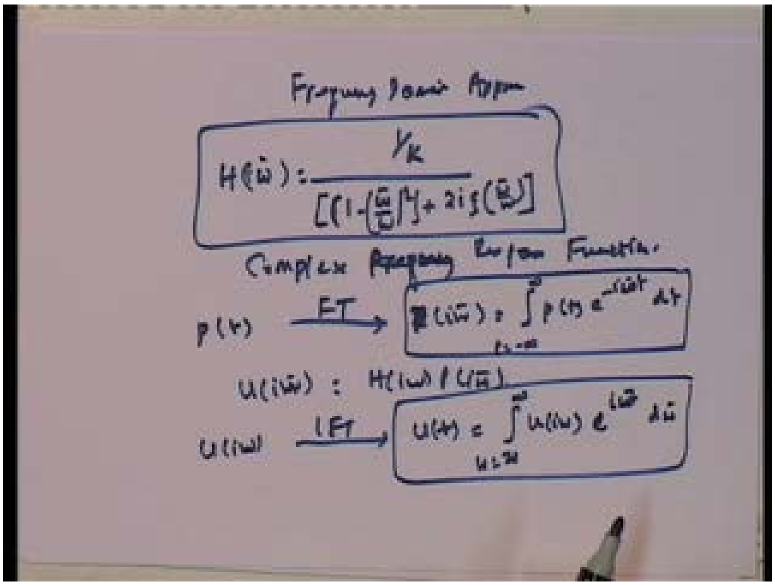

The frequency domain approach is first find H omega, which is H omega, this is the complex frequency response function. First find that out for every frequency then take p of t, transform it to zi or I will call it actually p i omega, what is p i omega, this is time going from minus infinity to infinity, p of t e to the power of i omega bar t d t. So, first do the Fourier transform, this is known as the Fourier transform then u i omega is equal to H i omega into p i omega.

And then u i omega inverse Fourier transform shows u of t is equal to omega minus infinity to infinity, u i omega into e to the power of i omega bar t d omega bar, it is very interesting. In the time domain approach, you directly have unit impulse response function and you directly find out, although this is complex Duhamel integral is not normally solvable directly, unless you have specific p of t which is mathematically derived.

The frequency domain approach essentially looks at a frequency Fourier transform, which is the complex integral. Then thus says that, look u i omega is actually equal to product of the complex frequency response function and the Fourier transform of the load. It is a simple multiplication then you do inverse Fourier transform and you get this, it is very, very interesting to note something. And that is, this is also function of t, if you take the Fourier transform of H of t, what is the Fourier transform of H of t, H of t into the e to the power of minus omega bar t d t, t going from minus infinity to infinity.

If we do that, it can be shown that, f of t is actually H i omega, the Fourier transform of the unit impulse response function is the complex frequency response function, very, very interesting. And it can be shown that, these two approaches, the time domain approach and the frequency domain approach are really the same equation. The only thing is that the convolution, look at the convolution integral, it is p tau into H of tau d tau. The convolution integral in the frequency domain, this is the frequency domain.

Convolution integral actually just becomes a product because you are actually doing nothing but p t into H you know p omega, which is p t's Fourier transform in the frequency domain, p t into H t is equal to u. The convolution integral has actually transformed into a product so if you had algorithms to do the Fourier transform in the inverse Fourier transform, actually the frequency domain approach is a far superior approach.

The only problem is that, you should have a procedure of getting the Fourier transform in the inverse Fourier transform. If you get both of these then the problem becomes a very very trivial solution so the solution essentially comes down to Fourier transform. Now, how do you solve integrations, do you know numerically how would you solve an integral of this form. The way you do it, the way any integral is done, if you do a numerical evaluation of an integral, how would you do it?

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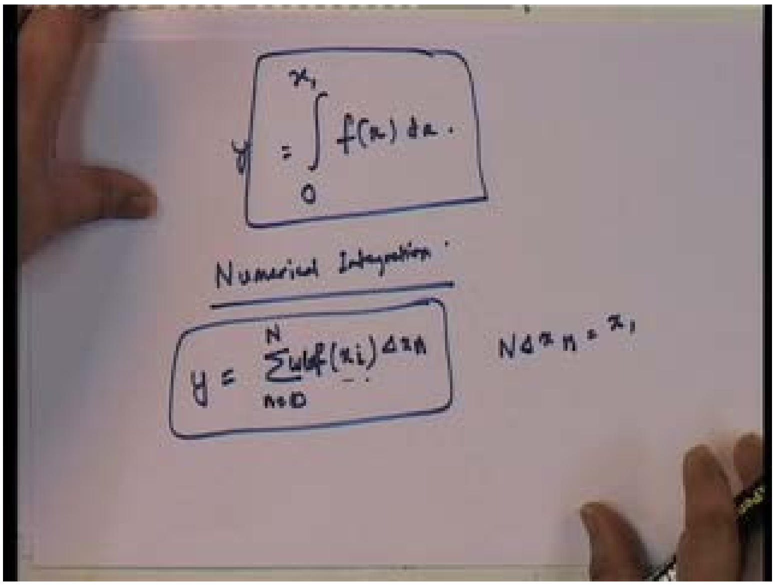

Suppose, I have some integral 0 to some x 1 f x d x, how do we do this in practice, in practice what we do is, we use what is known as numerical integration and you see that, look this is equal to actually in a way, summation. So, let me call this as y, y is actually a summation of some n going from 0 to some value capital N, f x i delta x where, actually N delta x n is equal to the limit. So therefore, n equal to 0 implies, this is the way you do it and of course, this is not done quite this way, there is a weighting function that you do because you cannot just take an integral and make it into f x n.

To go, take d x from, you have always have a weighting function, which is the function of this thing. So, this is the typical numerical integration, that you would do in reality and this is exactly, what is done for these also. Now, I am not going to talk about numerical integration of what kind of numerical integration schemes, this is not a course on numerical integration. But, you have simple terms as the trapezoidal rule, the Simpson’s rule then you have the Gauss quadrature, there are various numerical integration schemes, which I am not going to refer to. But however, what I will do is, I am going to talk about, how to do the Fourier transform this and the inverse Fourier transform. This in mathematically, how we do it is a very interesting concept to look at. And the way we do it, it is called as discrete Fourier transform, it is called discrete Fourier transform.

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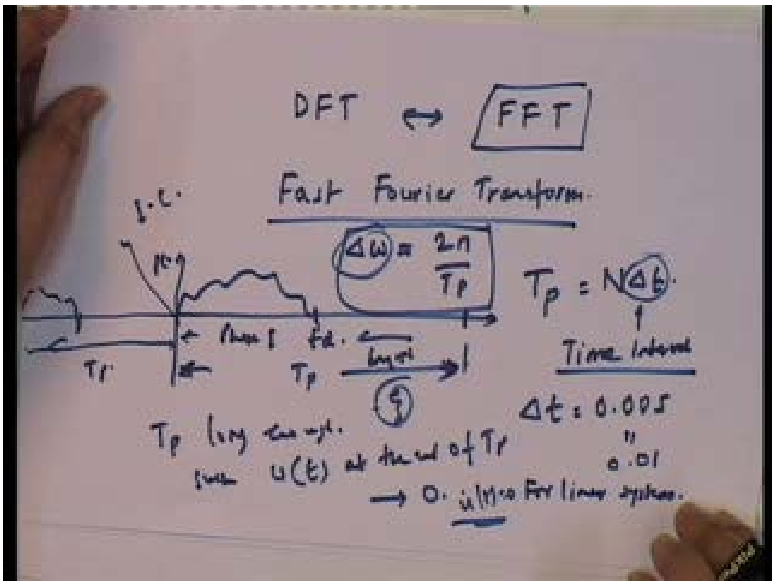

So, the Fourier transform procedure that frequency domain analysis becomes viable today basically because we have the discrete Fourier transform and that actually by people from IBM called Cooley and Tukey. They actually transform the discrete Fourier transform into the fast Fourier transform and all of you must have heard of the fast Fourier transform. Actually it is nothing but the discrete Fourier transform where, the coefficients, you do not need to find out all the coefficients over infinite time.

What you do essentially is, do it over a discrete times steps and by using that, the number of times steps that you have to distinguish a particular time interval, you do it as that n, as of 2 to the power of n, the fast Fourier transform. What it does is this, let me just go through this steps now, I will discuss this in detail in the next class. But, right now I am just giving you the overview of the concept, the concept is let us assume that, I have a loading of t d.

Now, the Fourier transform implies what, that T p is infinite, the discrete Fourier transform actually takes a finite T p and the way this is done is, you do a digitization. A digitization of this p of t and the p of t is done such that, T p is N into delta t where, delta t is the time interval. So, you are actually digitizing that thing where of course, t d also is a function so typical delta t that we take in practice, goes anywhere from 0.05 to 0.01 for linear systems.

For non linear systems, this becomes even a smaller but anyway it does not matter, that is the typical delta t that you take so that, you always have this. Now, the question then becomes, if how long do I take T p, the trick is when we take a finite T p what you are assuming, you are assuming that before this there was a load where, the load duration is there for T p. Now, note at this point, we have initial conditions and if you look at the frequency domain approach, there is no place to incorporate the frequency domain part.

And so therefore, what we do is, what we say is that, let us take T p long enough such that, u of t if you note. Now, note that, you have a situation where you have, this is like if you have T p what is this, this is like both having a phase 1, which is a loading phase and then a free vibration phase. And if a structure has zi what happens here, phase 1 loading the u response then ultimately after this, it becomes free vibration. And after free vibration what happens, if it damped free vibration, after a while it will go become 0.

So, what we say is, we take T p long enough such that, u of t at the end of T p tends to 0, in other words it is become negligible, both u of t and if u of t becomes negligible, u of t automatically become negligible. What is that mean, why because you see this one, which is there before this one because it is periodic loading now, finite time period. Discrete Fourier transform is finite time period, you cannot do infinite time period and do a analysis.

So, what we are saying is that, look if we take this and take T p long enough such that, over this free vibration phase, the u and u dot p t becomes 0 then what we have done. At this point, we have incorporated the initial conditions automatically and then we do the response during this time and again it becomes 0 and anyway, it does not matter to us what you do after that. So now, the question then becomes, what is delta omega now, note that, D omega becomes delta omega, delta omega is nothing but 2 pi upon T p, that becomes the delta omega.

So, you have discretization in time, discretization in frequency and so what you have is, these two integrals basically become series. So, next time, I am going to talk more about the discrete Fourier transform and how you go about it, thank you very much. I hope you have an appreciation of both the time domain approach, which is the Duhamel integral and the frequency domain approach, which is the Fourier transform concept. And how to tackle generalized loading response of a single degree of freedom to generalized loading, thank you very much, bye bye. See you next time when we will talk about discrete Fourier transform.

Thank you, bye.