Structural Dynamics

Prof. P. Banerji
Department of Civil Engineering
Indian Institute of Technology, Bombay

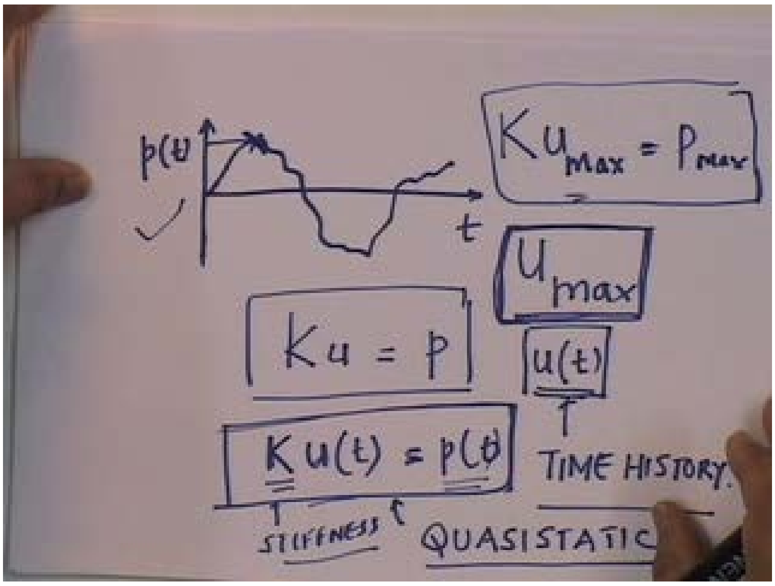
Lecture - 1
Introduction

Hello, I am Pradipta Banerji from the department of civil engineering, IIT Bombay and I will be spending the next 42 lectures talking about Structural Dynamics. Now, what is structural dynamics? Essentially it is the response analysis for structures subjected to dynamic loadings. In this course, I am going to essentially look at, start of with how to develop equations of motion. Then I will look at specifically, what we call as single degree of freedom systems, which are essentially where one displacement quantity can define the entire response of the structure.

The single degree of freedom is being studied essentially, so that we can develop the equations of motion, develop the methods for free vibration of one degree single degree of freedom systems. And then move on to what are known as response analysis for dynamic loadings, and in the beginning for the single degree of freedom, I shall essentially be looking at different types of dynamic loads, that we see in practical situations. Then we are going to move on to more complex structures, which we refer to as multi degree of freedom systems.

In multi degree of freedom systems, we are going to start of with again, development of the equations of motion. Then we are going to look at the free vibration, which is essentially when the vibration of the structure, when it is not subjected to any load and finally we are going to be looking at the force vibration. In other words, the multi degree of freedom response analysis, when they are subjected to dynamic loads. So, this will be in overall be the course content for this particular course, so let us first start off.

(Refer Slide Time: 02:36)



What do we mean by a dynamic load, actually any load that you have is always a dynamic load. By definition, you do not have where p which is loading varies as a function of t. Now, if p varies with the function of t, if we look at our old classical methods, which is a static methods you know that they essentially are be solved, this is the equation that we solve. So then why do not we just look at this, in other words why do not we get u of t directly from this equation, and of course understand also the structural dynamics. One aspect is looking at, this is known as the time history of the displacement, very rarely are we interested in the time history of the structure. We as a structural designer, what we are really interested in is, the maximum value of any response quantity.

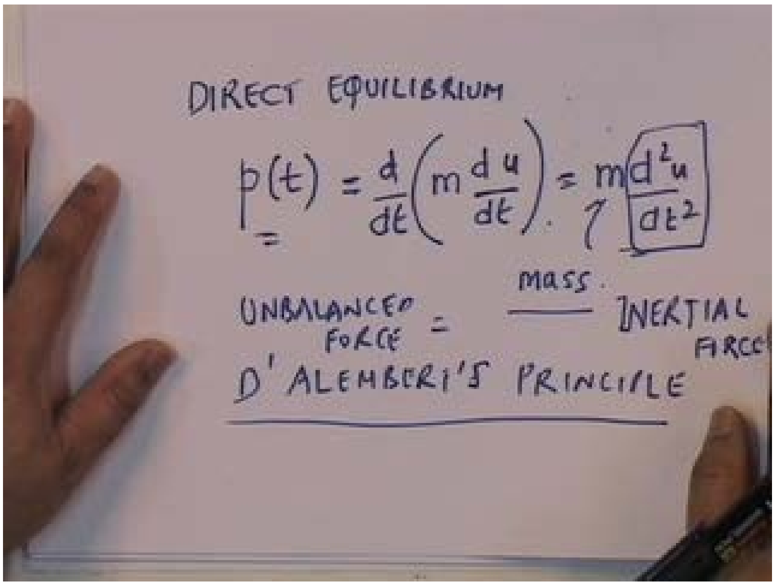
So, in other words, if you really look at it, the rest of the course we are going to be spending discussing, how to develop this peak response, because if you look at it by enlarge, why do we find out the response of a structure, response of a structure we always find out because, we want to find out we would not be able to design a structure to with stand the load it is subjected to. And essentially, unless you are looking at fatigue related problems where, the time history of the system becomes very important, we typically are as designers, interested in knowing what is the maximum value of the load that can come on the structure and what is the maximum response of the structure.

So that, we can then design the structure to withstand that maximum load so in essence if you look at it, why do not we solve this problem. In other words, this is actually a valid way of solving, what we are really doing is we are solving a quasi static problem. In other words, we accept the fact that, p of t is varying with time and however, we do not care about any other forces that are acting on the structure. And we say that, look this is the stiffness of the structure and we say we can solve this, so this is known as a quasi static solution procedure.

What is the major issue with dynamics, the major issue with dynamics is that, we cannot solve that problem, in the way that has looked. Quasi static solution is not valid now, in this particular course, I am going to spend some time on looking at a situation where, you can actually solve the quasi static problems, because if you solve that quasi statistic problem, it becomes a simple algebraic equation, which you can solve for the displacements and the peak displacements.

You do not even need to take the peak history, because if you look at this particular thing back again, if you look at this solution, it becomes automatic that, K u max is equal to P max. So, all you need to do is the peak value and plug it into the static equation and solve for u max. And of course, from the displacement, you can solve any other response that you want to find out, that is as far as concerned. And we will show that, in certain kind of situations, in certain depending on the loading characteristics and the structural characteristics, you can actually solve the quasi static problem. Now, we come back to, how is the dynamic problem different from the static problem and for this, we go back to Newton's second law of motion. If you look at Newton's second law of motion, what does it say, classically it says that the unbalanced force on the structure is proportional to the acceleration of the structure.

(Refer Slide Time: 08:30)

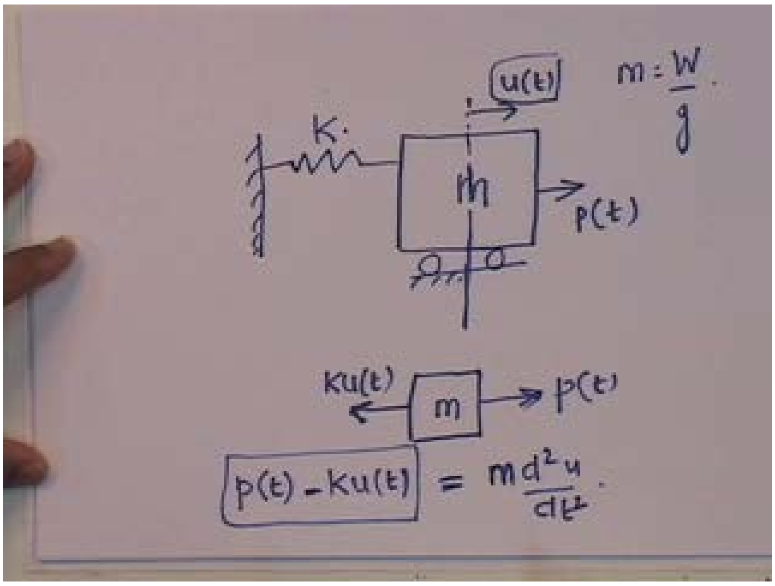


In fact, it also says in this way that, if you look at this, if this is the displacement d u by d t is the velocity, mass into the velocity is the momentum and the rate of change of momentum and that is the acceleration. Mass, this in classical situation, the mass does not vary with time basically becomes and this is the acceleration. So, Newton's second law says, the unbalanced force on the structure is equal to, is proportional to the acceleration and proportionally constant is known as the mass of the structure.

So, if we look at this in fact, this is actually known in our work as direct equilibrium or it is known also as D' Alembert's principle. Now, this procedure is a very, very useful procedure because, what is this direct equilibrium, it essentially says at any instant of time, the unbalanced force is equal to the mass into the acceleration. So, this is in the inner way, if we look at it from structural engineering point of view, this is like a equilateral equation, is not it.

What we are saying is that, unbalanced force is equal to mass into acceleration and this mass into acceleration is really inertia, inertial force and so we are saying that, look the unbalanced force is equal to the inertial force. And so this is an equilibrium, only difference between static equilibrium equation and this is that, this is valid at an instant of time and the equation varies with time. So, that is essentially the difference, so the d' Alembert's principle that I have stated right now is a very important equation, that we look at. And what it states is that, the inertial force, this is the inertia or inertial force opposes the unbalanced force. So, the inertial force opposes, it is the mass that actually exerts an opposing force to the unbalanced force. So now, before I start off with a real structure, I shall take a mechanical structure to explain some of the issues.

(Refer Slide Time: 12:18)



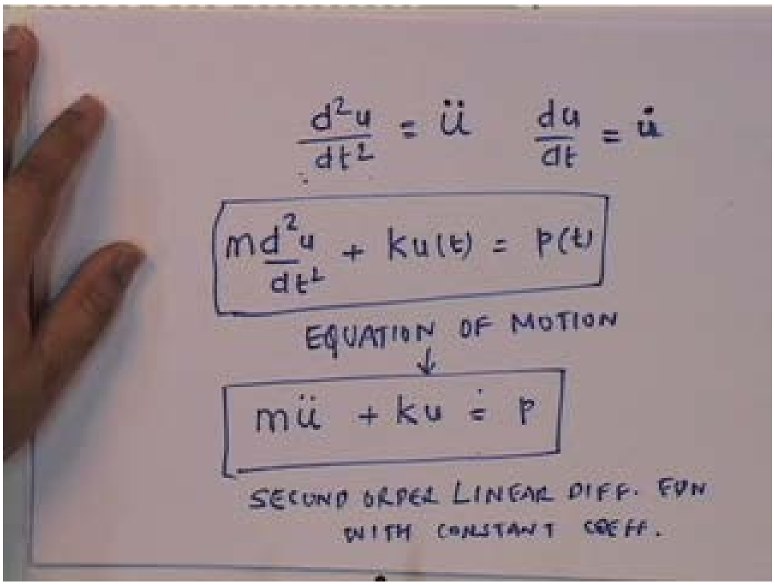
Structure, if you look at this, this is a rigid mass, a rigid block of mass m, note that what is mass, we will come to the definition of mass little bit later. But in this particular case, mass is nothing but the weight upon the gravitational constant. Then we have a spring which connects and later on, I will come to the other concepts and so this here what we are looking at is, if we take this as our inertial system. So, this is our displacement and let us say, this is subjected to and this is a spring with spring constant K.

So, we look at this particular so how do we solve this problem now, if you look at it, every problem that we solve in structural mechanics is actually, by drawing a free body diagram. So, here also and look, what do we do, we always draw free body diagram and look at the equations of equilibrium. And d' Alembert's principle essentially is the same, only difference is that, we set up what are known as the dynamic equations of equilibrium in this particular case.

Because, there is only a single degree of freedom, this is the unknown and so if you look at this, I take a free body of the mass, it is been subjected to a load p of t and if it is displaced by u then we have a spring force, which is equal to K u of t. So, if this is what is unbalanced force, then my unbalanced force is equal to p t minus K u of t. And d

Alembert's principle says that, this has to be equal to m d u by d t square, this is what d' Alembert's. See, this is unbalanced force, if you look at it, p is acting in this direction, k acting in this direction and therefore, the unbalanced force in this direction is p of t minus K u of t, which by the d' Alembert's principle is equal to the inertial force, which is mass into acceleration.

(Refer Slide Time: 15:35)



If we rewrite this equation in a slightly different way, what we get is m d u squared by d t squared plus K u of t is equal to p of t, this equation defines the motion of the mass of the rigid mass, so this is known as the equation of motion. And if you look at this, I am going to use some specific terms, I am going to say that, let me d 2 u by d t square, I am going to say as u double dot d u by d t is u dot. And so this is the displacement, u is the displacement, u dot is the velocity and d u squared by d t squared is the acceleration.

So, in other words, every dot that I put in if you look at it, refers actually to a differentiation with respect to time. So, one dot means one single differentiation with time, double dot means double differentiation with time. So, this equation then finally, becomes m u double dot plus K u, I am just dropping the time variation. We know that, all of them vary with time, this becomes the equation of motion of this structure, so this give you the equation. Now, if you look at it, this equation is nothing but a second order linear differential equation with constant coefficients.

So, you see what have we done, you know when we solve a static problem, it essentially is that you are solving a algebraic equation, when you have to consider the response of a structure to a dynamic load that algebraic equation now becomes a second order differential equation. So obviously, the solution of this second order differential equation is an order of magnitude, more complicated than the algebraic equation that you have to solve in for the static problem.

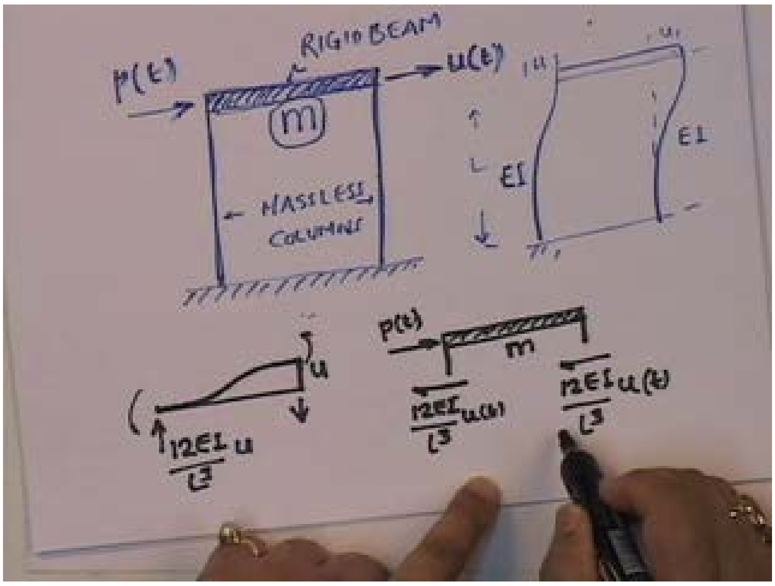
So obviously, what we try to do really is, as far as possible try to solve, try to ensure or may be even try to see if we can increase the load such that, we do not have to solve the linear differential equations. So, in this particular course, you will see that again and again, I will solve the second I mean, I have to solve this linear second order differential equations, for me to able to understand what the response is…

And then once I understand what the response is, I will then use what I will call as fudge factors or you can say as design factors, which I will multiply the static response with, and that dynamic factor multiplied by the static response will give me the dynamic response. So, this in a sense, is what we are interested in and in this course, you will find again and again, I will go back to trying to establish the dynamic factors for different kinds of loads for different kinds of structures.

So, I just want to state it right at the beginning that, this is not a course on solving linear differential equations, you will have to understand how to solve linear differential equations but that is not the focus of the course. The focus of the course is to establish dynamic factors, which we can multiply the static response with to take into account what we call as the dynamic effect of the loading and the structure. So, essentially now I want to go back and tell you that, what I have looked at a mechanical system. I have shown you mechanical system with the rigid mass and a spring and I have solved that problem.

Now, you say that is a mechanical vibration problem, you are right that is a mechanical vibration problem but it is very interesting that our structural problem is almost identical to the mechanical vibration problem. And in fact, we will see over and over again that, mechanical vibrations that only different mechanical vibrations and what we call as dynamic structures is the fact that, the loads that structures are subjected to are very different from the loads that the mechanical systems are subject to. But if you look at for the mathematics, the mathematics is identical in structural dynamics, as it is with mechanical vibrations. Same differential equation that you have, this is the differential equation that you have a mechanical system, and we will show that, this is exactly the same that, you have for our structural system.

(Refer Slide Time: 21:49)



So, let me now draw a structure, the simplest structure that I can think of, which is what I had called as a single degree of freedom. What we have here are two massless columns connected rigidly to the waves, it is the foundation and on top, this is a plain single storey single bay frame and on top of it, we have a rigid beam. Then this problem essentially becomes a single degree of freedom where, the degree of freedom is the displacement.

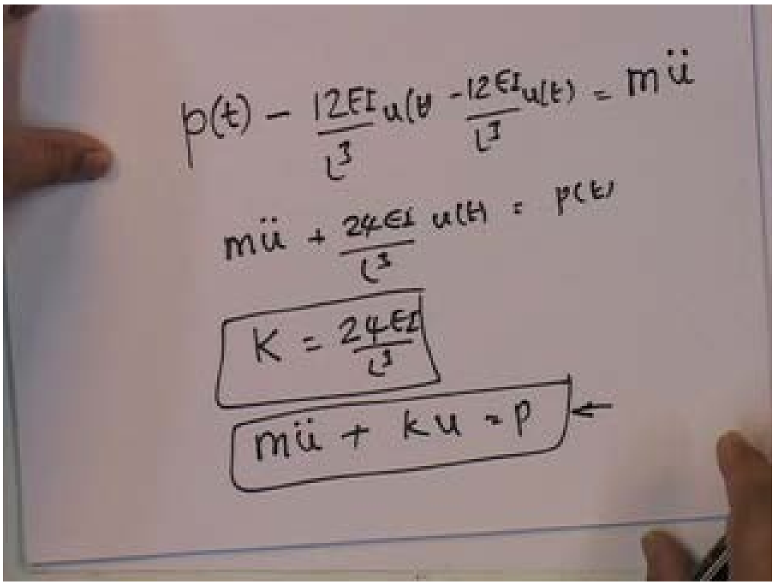
If we look at it, if I find out this displacement, I can find out the shear in these columns, I can find out the shear transmitted to the base, all the other response quantities of the structure we can find out. So now, the question then becomes is, how does this structure with and of course, it is subjected to a lateral load. Note that, if you have actually rigid columns, these are actually rigid massless columns flucturally flexible but actually rigid. And here, you have beam which is both actually and flucturally rigid, if we look and if it is subjected to vertical gravity loads, they will go directly get transferred because, it is all rigid, this is also rigid so it will get transferred.

So, the only load that causes displacement in the structure is really the lateral load and lateral load, the unknown displacement is this u of t. And let us say, that because, these columns are massless, we will say that look this is where, the floor mass is concentrated and the floor mass is m. So now, let us look at the displacement, let us give it a displacement u, that is not right now look at the fact that, u is the function of t. Let us just give it a displacement, how will it displace, it will displace in this fashion where, both the columns will have undergrounds displacement u.

Now, from basic structural mechanics you know that, if this column has EI and this is a fluctual rigidity EI and is of length L, both of them have EI and length L. Then each one when it is subjected to a displacement u, will have a force if you look at it. What am I doing to this structure, I am essentially taking this beam and subjecting it to a displacement u. So, what was the forces that are developed here, the movements that are not interested again, I will tell you why but anyway, we know what these things are.

If you subject it to the displacement, you have moments here and you will have forces here and this force we know from basic structure mechanics, is equal to 12 EI by L cube upon u. So now, if I take a free body of the mass, let us take a free body of the mass and again the mass is a rigid mass, so this is m, here we have one column, here we have one column. Now, if you look at this, if you were in this direction then this column has 12 EI upon L cube because, the force in the column always opposes. So, it is 12 EI upon L cubed u, you have the load p of t and of course, this is undergone a displacement by the way, this is u of t. So, in other words, this is the instantaneous free body for this particular structure.

(Refer Slide Time: 28:18)



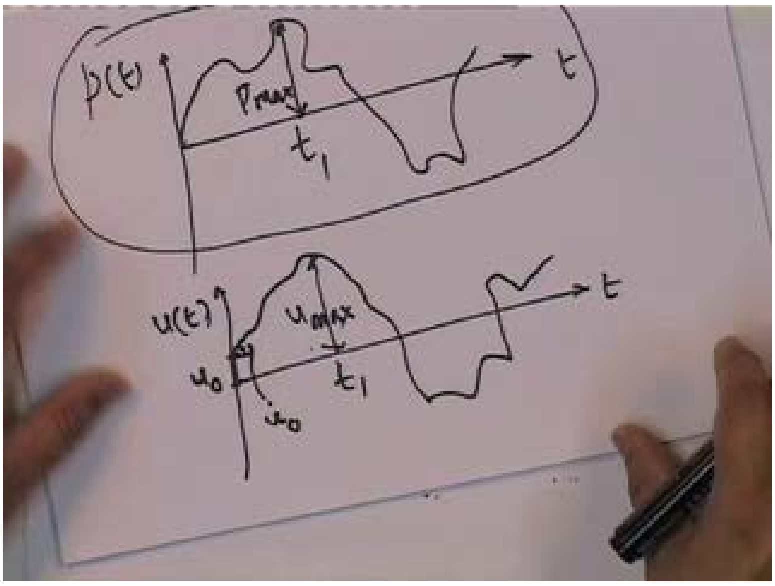
So now therefore, if I look at it, using d' Alembert's principle what do I get, I get that the unbalanced force, which is again p of t minus 12 EI upon L cubed u of t minus 12 EI upon L cubed u of t. This is the total unbalanced force is equal to mass into acceleration, note that this double dot is acceleration. So, if we rewrite this, this essentially becomes and if we say that, K is equal to 24 EI upon L cubed then this equation becomes m u double dot plus K u is equal to p.

You see exactly the same equation, that we got for the mechanical system, our single bay single storey frame gives us exactly the same equation. So, I will wanted to show that, from your here on out as far as single degree of freedom system is concerned, consider this as my single degree of system for a structural engineer. So, once you get this then that becomes essentially, this is the equation that we have, that we have to solve. Now, as I said that, this is a linear second order differential equation with constant coefficients and solving this is fairly trivial mathematical problem.

What do you need to, how can you solve this equation of course, if you look at it, the mathematics says you can solve this problem, if you have what are known as initial conditions. What are initial conditions, initial conditions what was if you have a second order differential equation, you need two initial conditions to be able to solve this and this is u at t equal to 0 at the instant of time when the load started. What was the state of structure u of t equal to 0 and u dot, displacement and velocity once we know what the displacement and velocity at the instant of time, that the load was started to apply then we can solve this equation.

And this u t of 0, I will refer as the initial conditions as u 0 and u dot 0 so these are the initial conditions. So, mathematics says we can solve this problem, I will come back to this problem later on. Now, I would learnt in structural engineering, mathematics is very important but it is not as important as the physics of the problem. So, let us go back to the physics of the problem, essentially the physics of the problem is what.

(Refer Slide Time: 31:50)

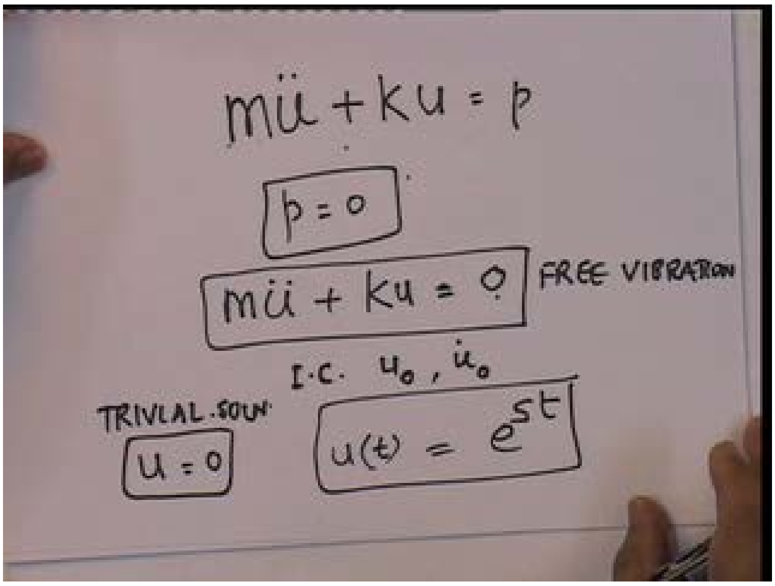


If I have a load which is varying with time, let us give some any arbitrary loading I am here, right now it is not of that importance. This is going to give rise to a displacement, which also varies with time and depending on, now this is where which is important. The loading where it starts, where it ends depends on how the load is defined but here the displacement time history is, this is the initial displacement and note that, the initial velocity is this.

So, we need to define these two and then this displacement will be something else, note that if the displacement exactly follows the pattern of p of t in other words, the u max occurred exactly at the same time that, p max occurs. If u max and p max occurred at the same instant of time then this problem we do not need to solve the dynamic problem, do we. Because, if they occur at exactly the same instant of time then we can say that, look this in essence is nothing but like a static problem.

Because, if this has the same feature as this then this is essentially a quasi-static, we could just take p max and solve it with u max. However, when you solve the linear differential equations, you will see that this did not necessarily be of the same shape as this and the point, at which u max occurs is not necessarily the same time where p max occurs. And even if they do occur at the same time, p max upon k does not give us your u max. So, that in essence, is what we are going to be showing now, the question then comes back is that, note I mean, as I said the initial displacement and the initial velocity are important to define the shape of u of t.

(Refer Slide Time: 34:43)



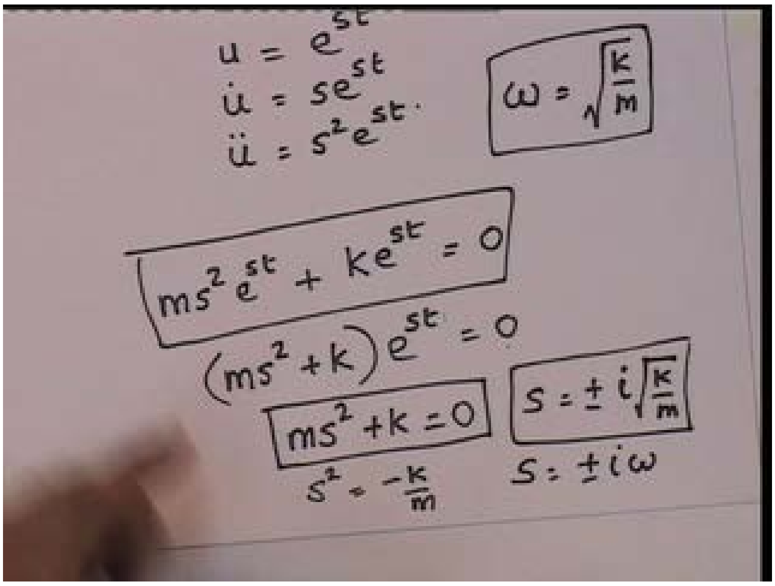
So now before we start solving the response of a structure to the load, let us look at what we call as free vibration equation, which is solution of this equation where, p is equal to 0. So, this where p is equal to 0, this problem becomes m u double dot plus K u is equal to 0 and this is what, we call us free vibration. Obviously, the free vibration is with initial conditions, so you still have the displacement and velocity of time. If you look at this problem, this is a linear differential equation, a homogeneous linear differential equation, second order differential equation.

What is the trivial solution to this, a trivial solution to this is u is equal to 0, this is the trivial solution. What does that mean, physically what does that mean, in other words if a structure is not subjected to a load, why should it vibrate, that is the key point. Why should a structure, which is not subjected to any load vibrate, the only reason why it vibrates is because, if you give it an initial condition. In other words, let me just say that, I have a structure standing and so this is a structure and if it was left in this fashion with no load being subjected to it, it will stay where it is?

Because, any structure is like any human being, unless we have a force on us, we do not move. So, how do I solve the free vibration equation and not get the trivial solution, if I take this and give it a initial displacement now what, if I give it the initial displacement, I have disturbed it. If I have disturbed it, what is it trying to do, it tries to come back to it is original position. Now, what happens, put it here, it comes back here but the problem is that, here at this instant of time, the velocity is 0.

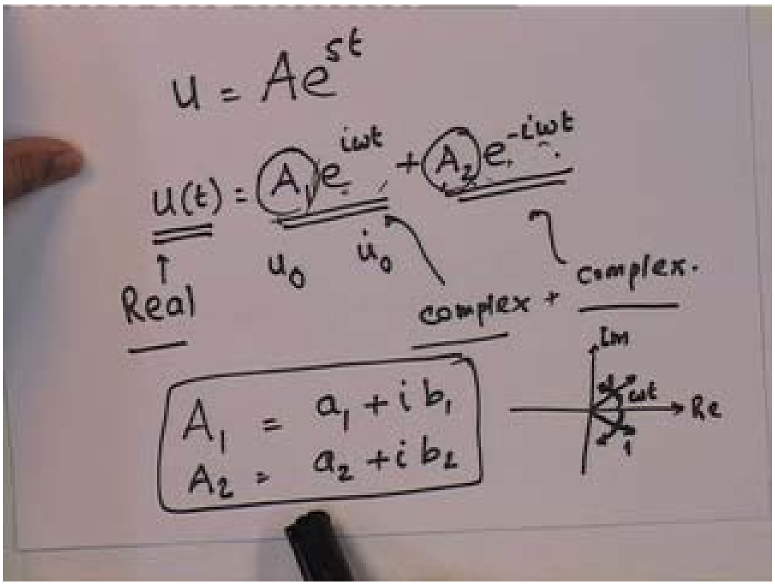
Now, when it comes back, the problem is that, at this point the velocity is not equal to 0 so it goes back and then it comes back. So, this is known so if I give it an initial displacement then because, the structure wants to come back to it is original position, it starts doing this and that is the free vibration of a structure. So, how do I solve that problem so this is the physics of the problem, that it does this. Now, how do I mathematically solve it, mathematically it is trivial because, we know that the solution to this is a very, very simple solution. This can be solved by taking u of t as of this form so let us see, if we substitute this into this, what do we get, what we get is the following.

(Refer Slide Time: 38:40)



I get u is equal to s of t so u dot which is the first differential with respect to time, if you differentiate this with time, this becomes s e s t, u double dot becomes s squared e s t and if I substitute this, I get m s squared e s of t plus K e s of t is equal to 0. So, this becomes my equation and this if you look at it, is can be written as m s squared plus K into e s of t is equal to 0. Now, note that, u is the function of e s t, so therefore if this was 0 then u will be equal to 0, that is the trivial solution. So, this is not 0, if this is not 0, for this equation to be solvable it implies that, m s square plus K is equal to 0. In other words, s is equal to plus or minus i root K over m where, i is imaginary because, what you get over here, this is s squared is equal to minus K upon m and essentially, s square minus means, s has to be plus minus imaginary. Now, if I define omega is equal to root K upon m then I am defining this term root k over m by a term omega. So then this basically becomes s is equal to plus minus i omega.

(Refer Slide Time: 41:10)



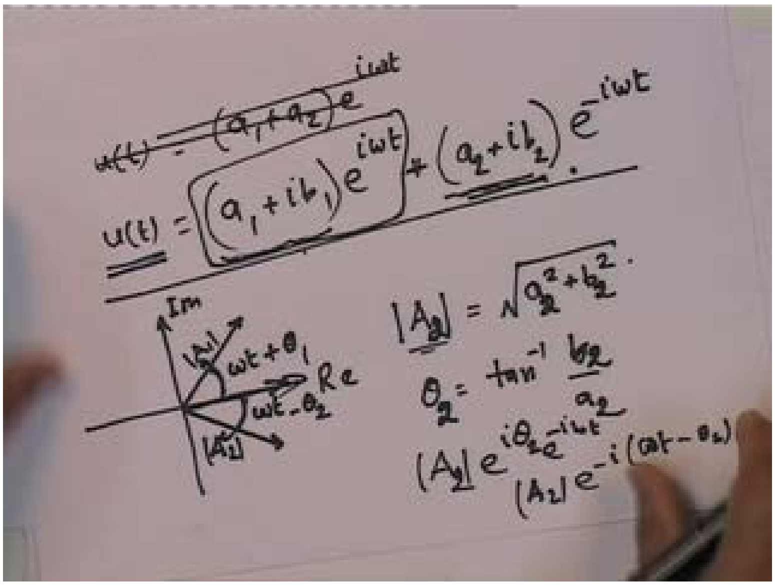
E of s t, there is a amplitude to it actually A e of s t but all the things remain the same so if I substitute this that means, that if I solve that free vibration equation, I get u of t is equal to A 1 e to the power of i omega t plus A 2 e to the power of minus i omega t. This is the solution of the linear second order differential equation and these are subjected to u 0 and u dot 0. Now, substituting now here, how would I solve it, let me before I solve it for A 1 and A 2, it is an interesting problem.

Note that, u of t is real but both these functions are complex so in other words, what I have is a real quantity, this is a displacement, a displacement is equal to a sum of two complex quantities. How can it be, it can be if these amplitudes are also of a special form because, in other words, two complex quantities sum up to be a real quantity, imposes certain characteristics on these. Now, let us look at this, now these are also complex because, if they are not complex, this is just a pure imaginary number and there is no way that, by getting real coefficients, this would become real.

So, automatically A 1 is equal to a 1 plus i b 1, A 2 is equal to a 2 plus b i 2 so if you put this in then put in these two together, what we get is that, sum it up. If you look at e to the power of i omega bar t, what is this e to the power i omega bar t, what is the form. The form, if I look at the complex plane, e to the power of i omega bar t is a unit vector which is rotating anti clockwise and e to the power of minus i omega t is a unit vector rotating clockwise.

So, these two are actually unit vectors which are essentially rotating, this is the complex plane real, imaginary. So, these two represent just unit vectors rotating in opposite directions in the complex plane. So, basically then let us look back at these, if we put these back and look at what we get as the… I am just going to put these in and just solve it and if you get that particular thing, what essentially happens?

(Refer Slide Time: 45:31)



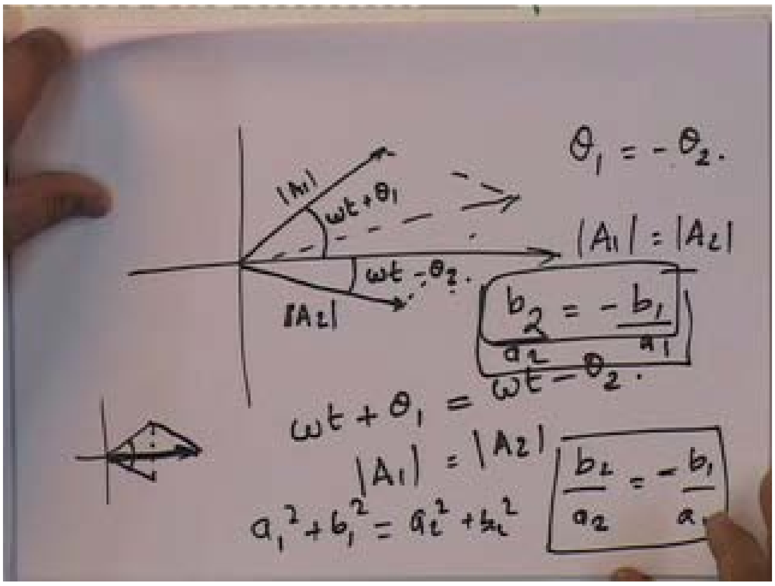
Let us go to the mathematics and then I will come to the physics of the problem, the mathematics of the problem becomes u of t is equal to a 1 plus a 2 e to the power of i omega t, I made a mistake let us let us go back, let us cancel this. U of t becomes equal to a 1 plus i b 1 e to the power of i omega t plus a 2 plus i b 2 e to the power of minus omega t. Now, if I look at it we saw that, e to the power of i omega bar t is nothing but a union factor.

So, now, if I look at this, these have amplitudes associated with them so what are these vectors, what does this vector, what does this represent. This if I look at in imaginary in the complex plane, is the real is the imaginary, this thus is nothing but, a vector. What is this, let us look at this a plus i b 1, I can represent a plus i b 1 in this way, a plus i b 1 can also be represented as, A 1. Where, A one is nothing but square root of a 1 square plus b 1 square, that is known as the magnitude of this and theta which is nothing but tan inverse of b 1 upon a 1.

So, in other words, what we have here is because then if you look at this can be written as A 1 e to the power of i theta and if this is multiplied by e to the power of omega t then it becomes omega t plus theta. And similarly, this particular equation similarly can be written by, all we need to do is, just change this, this becomes a theta 1, this is theta 1 originally so next time it is theta 2, 2, 2 and what we have is theta 2, omega minus. So, if you look at this particular thing, what we get is that look then this becomes A 2 omega t minus theta 2, because if you look at this, if I plug this in, this becomes i omega t minus theta 2, A 2 e to the power of minus.

So, it basically so what we have is, now if I put all these together, I get a vector which is moving in this direction, a vector which is moving this direction but this is a very specific part. And that is that, this vector plus this vector, which are represented by this one, A 1 and A 2, the sum of them has to lie on the real plane because, this is a real function. If this is a real function then we have this that, these two have to add up so if I do vector addition, if you look at vector addition then what I get is nothing but the following.

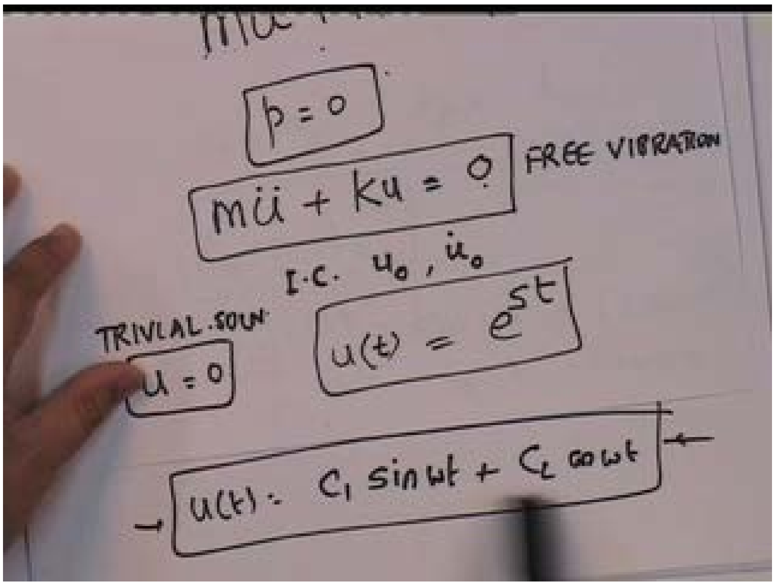
(Refer Slide Time: 50:33)



And that is, this represents A 1, this represents omega t plus theta 1, this represents A 2, this represents omega t minus theta 2. And so if I add over here, if I look at this, you see there is an issue and that, when I add these two because, this is omega t minus theta 2, this is omega t plus theta 1 and so then when I add up, it goes over here. So that means what, that the only way these two can add up to a real is, if the following is true and this can be shown mathematically and that is omega t plus theta 1 equal to omega t minus theta 2.

These have to be true and if A 1 was equal to a 2, if these two were the same in other words, this angle and this angle were the same and these two have the same magnitude just see, if you take things like this. In other words, you take two exactly same magnitude and same angle, when you add two of them up, they will add up like this. And then the resultant lies along the real axis and since u of t lies along the real axis, these two have to be true. In other words, theta 1 becomes minus theta 2 and A 1 is equal to A 2, this if you look at the solution, the only way this can be is, if b 2 is equal to, that is the only way abated to upon. If b 2 upon a 2 and b 1 upon a 1 were equal to each other, that is the thing. If these have to be there it means that, a 1 square plus b 1 square is equal to a 2 squared plus b 2 squared.

(Refer Slide Time: 53:43)



And if that is true then what happens is ultimately what we get is the following, that a 2 is equal to a 1 and b 2 is equal to minus b 1. In other words, what we have here is that A 1 is the complex conjugate of A 2, is the only way we can get the two to be together. So now, if that happens now, let us see what happens, note that e to the power of i omega t can be written as cos omega t plus i sin omega t, e to the power of minus omega t can be written as cos omega t minus i sin omega t.

Plugging all of this in, ultimately what we get is, u of t is equal to c 1 sin omega t plus c 2 cosine omega t, this is the solution of this equation. And how we get c 1 and c 2, these depend on the initial conditions and that we will look at in the next lecture. So, today what have we done, we have looked at how to develop the equations of the single degree of freedom system. And we showed that, our structural equation and the mechanical vibration are exactly the same, they are m u double dot plus K u is equal to p.

We looked at then what happens when we have free vibrations where, p is equal to 0 and we try showed that, essentially the free vibration if you look at it, is a harmonic motion because, sin omega t and cosine omega t are nothing but harmonic functions. So, in other words, free vibration is a harmonic vibration problem, how do we solve it and all of those, I will look at in the next lecture. Now, before I end this lecture, I would like to just say that, there are two very good books currently available in India, that I would like to suggest as books, that you can follow.

I used to follow what I am being doing in this course, the first book is the dynamics of structures by Clough and Penzien, which is printed by Tata McGraw Hill and right now, it is second addition is available. And the other one is dynamics of structures, theory and applications to earthquake engineering by Anil Chopra, which is a decent economy edition by Prentice Hall India and currently that is in the third edition. These are two very good books you may want to know, refer these books to help in understanding what I am going to be doing in this particular course.

Thank you very much.