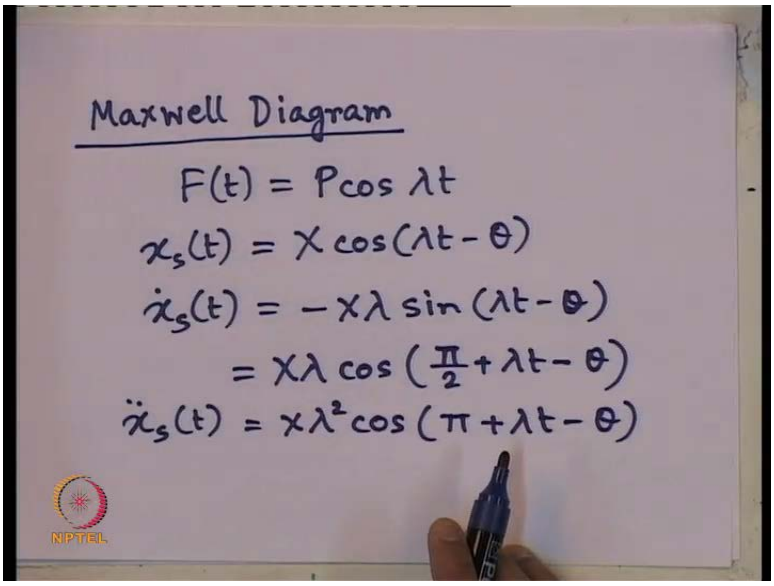
Soil Dynamics
Prof. Deepankar Choudhury
Department of Civil Engineering
Indian Institute of Technology, Bombay

Module - 2
Vibration Theory
Lecture - 9
Maxwell`s Diagram of DMF
Discussion on Phase

Let us start today's lecture of soil dynamics. We are continuing with module two of vibration theory. So, we have understood the physical significance of high value of r or the DMF less than one. When it can happen? Now, let me give its mathematical validation. I have given you the physical significance to understand the problem. Now let me give you the mathematical validation. Mathematically how we can prove this in a better way if I give you?

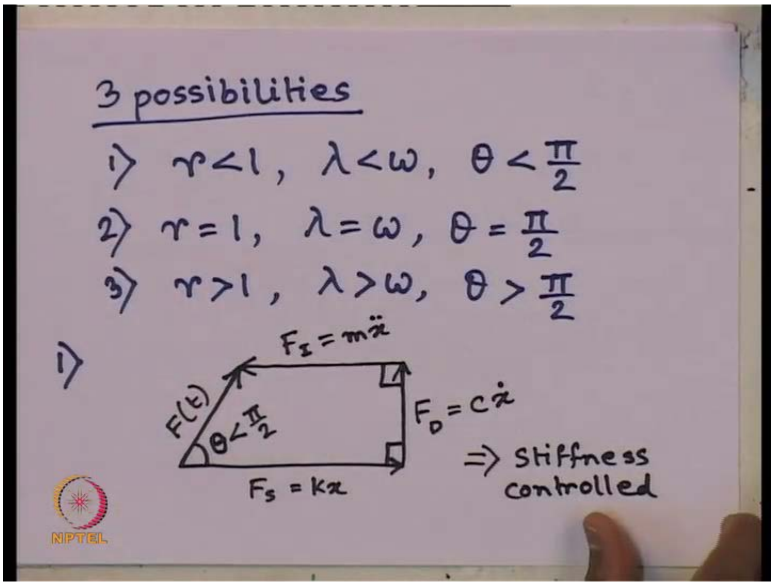
(Refer Slide Time: 01:11)



So, using Maxwell diagram, we can again prove mathematically which zone is controlled by which parameter. We have F of t equals to p cosine of lambda t and for which we got the steady state response x of t is x cosine of lambda t minus theta. That was the form of the solution some cosine functions. So, the velocity is nothing but minus X lambda sin of lambda t minus theta, which we can write as X lambda cosine pi by 2 plus lambda t minus theta.

So that means, the velocity is in the phase with the displacement with 90 degree, which is known to us; and what about the acceleration X lambda square cos pi plus lambda t minus theta. So, acceleration is at a phase of 180 degree with respect to displacement. So, this is known to us from our high school physics. So, now, what we can have? We can have three conditions.

(Refer Slide Time: 03:00)



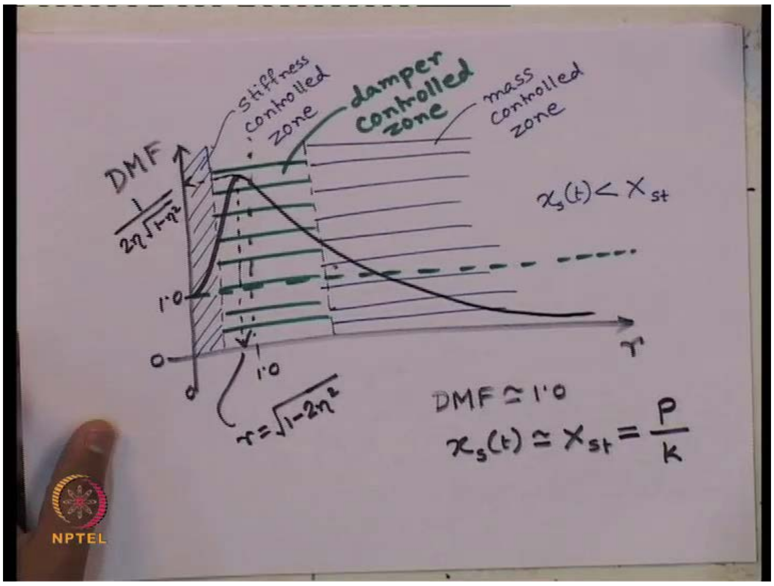
So, three possibilities can arise. The first one is when r less than one that is frequency ratio is less than one. That means lambda is less than omega. We have theta less than pi by 2 that phase angle. The second case r equals to 1 that is lambda equals to omega theta equals to pi by 2. And the third case can be r greater than 1 means lambda greater than omega theta greater than pi by 2. Let me draw the Maxwell diagram for all the three cases. The first case if I want to draw the Maxwell diagram what it will look like? Suppose this is our F s which is equals to K x. I know drawing the force diagram in Maxwell force diagram. This is our F x F s that is stiffness force equals to K x.

Now, damper force F D will act ninety degree of this force right because that is a function related to the velocity the damper force F D is nothing but c times x dot. So, that x dot is at 90 degree phase with respect to the displacement function.

So, we need to draw. Now let us draw this. So, this is now F D the damper force equals to c x dot it should be 90 degree phase with respect to the displacement function. Now, let me draw the inertia force F I inertia force will be m x double dot. Now, that x double dot acceleration is 180 degree phase with respect to the displacement function. So, this is our F I which is m x double dot this is also 90 degree and it has to be a four closed polygon.

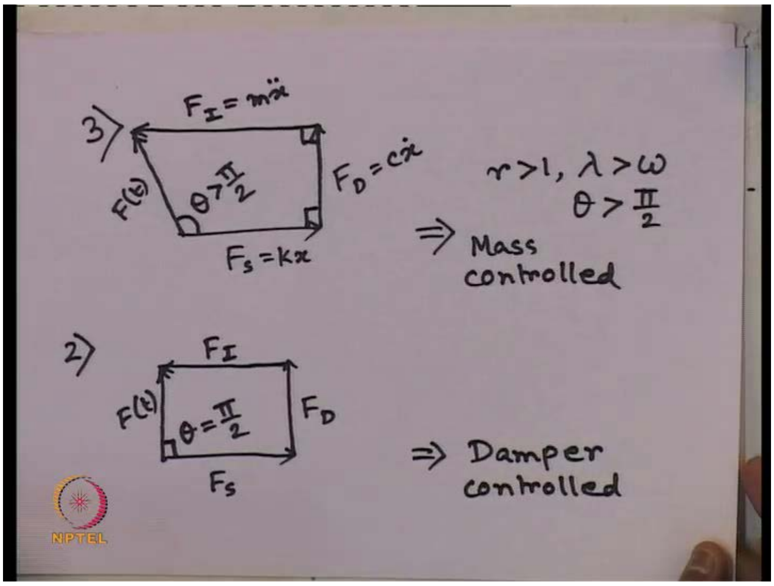
So, this our F of t externally applied to maintain the equilibrium of the system and this angle is nothing, but the phase theta which is less than pi by 2 that is the first case. So, from this Maxwell diagram this force diagram what we can see? Which force is maximum among these four forces F s? Am I right to make this angle theta less than pi by 2 which satisfies this condition? We need this force to be maximum among the three. That is the meaning that this is a stiffness controlled zone. Is that clear now? Why r less than one is called stiffness controlled region because as you go r less than 1 your stiffness force dominates.

(Refer Slide Time: 07:07)



Let me put back it again as you go r much less than one your stiffness region dominates. So, this we had understood physically. Now, we are understanding it mathematically, let me see the other two cases before coming to second case.

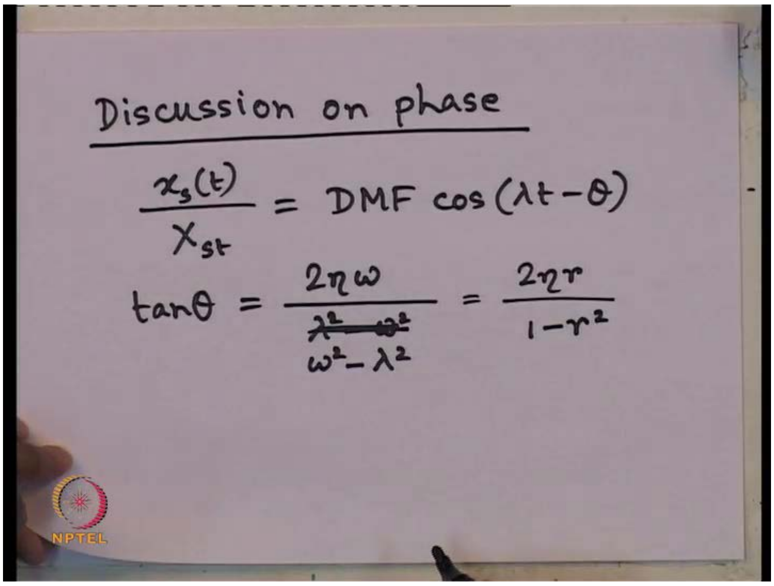
(Refer Slide Time: 07:34)



Let me give the third case. First for the third case, what we have? We have F s 90 degree to it. We have the damper force F D. Now, 90 degree to that we have F I inertia force. This is 90 degree this is 90 degree and closed polygons that will be our F of t with this angle theta greater than pi by 2; so that the third case as we have said phase angle theta will be greater than pi by 2. So, in this picture what we can see? Which force is dominating inertia force is dominating? F I is dominating.

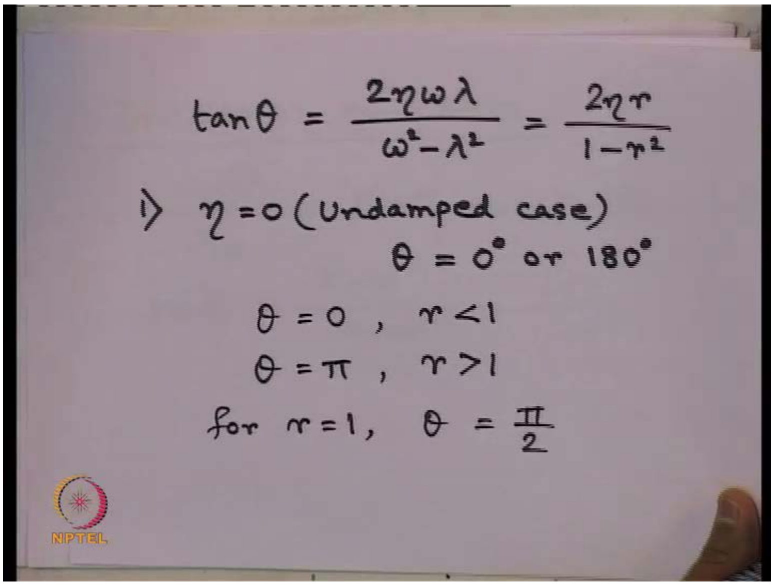
So, that is why we call this case that is r greater than 1 when we have lambda greater than omega; so that theta greater than pi by 2. This is called mass controlled zone. This is the region or this is the proof. Mathematically why it is called mass controlled zone which we had already understood from this picture? Why this region is called mass controlled zone? Physically we have understood the problem. Now we have proved it mathematically. And obviously, the remaining parameter will control the second case which is something like F s F D F I and F t with theta equals to pi by 2. This will be actually the damper controlled as you have a control on this damping force. You will still it remains theta equals to pi by 2. So, the remaining 1 is damper controlled zone. So, this damper controlled guides you the DMF which we have seen here the peek value of DMF or maximum value of DMF is controlled by this damper control reached. Now let me discuss little bit about this phase. The phase I have used in the Maxwell diagram already, but I have not yet discussed about this phase. How it varies with respect to our and also different values of damping ratio.

(Refer Slide Time: 10:52)



So, the discussion on phase angle. What is the solution? We got the steady state response to the static response is given by DMF times cosine of lambda t minus theta where we know that tan theta we have defined as 2 eta omega by lambda square minus omega square or 2 eta r by. I think it is the other way round it was omega square minus lambda square yes. So, tan theta was 2 pi eta by omega square minus lambda square. So, we divided by omega get it 2 eta r 1 minus r square.

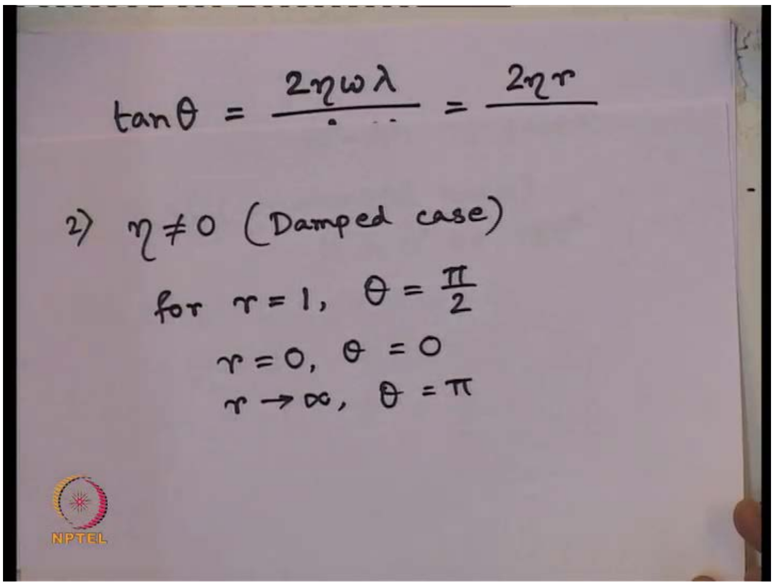
(Refer Slide Time: 12:18)



So, the expression for steady state displacement to static displacement was given by this where tan of theta was defined as 2 eta omega lambda by omega square minus lambda square. That was the expression for tan theta which we can simplify as 2 eta r by 1 minus r square. We have divided both numerator and denominator by omega square. So, here 1 omega 1 omega get cancelled lambda by omega is nothing, but r frequency ratio. So, now, this phase how it varies for different cases. Let us see.

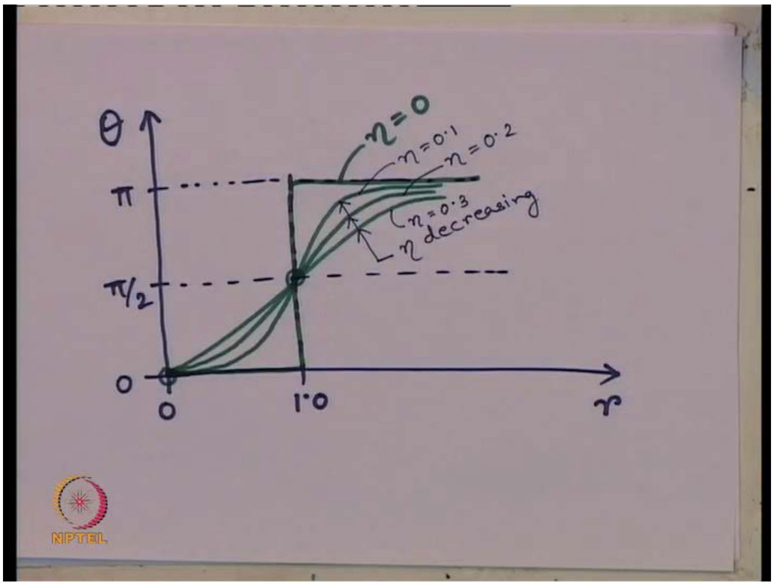
So, when eta equals to 0; that is undamped case. We have the value of theta equals to 0 degree or 180 degree when this is 0 tan theta 0 means it will be either 0 degree or 180 degree. So, theta is equals to 0 when r less than 1 and theta equals to 180 when r greater than 1 and for r equals to 1 theta will be equals to pi by 2. This is for the undamped case. Now for damped case how the variation occurs?

(Refer Slide Time: 14:01)



So, when eta non 0 that is damped case for r equals to 1 what we are getting? r equals to 1. Let us look back to the expression of tan theta when r equals to 1 it is becoming infinity. So obviously, theta is pi by 2. That is why irrespective of value of eta I have written earlier also for r equals to 1 theta will be pi by 2 fine. So, for damped case also r equals to 1 means theta equals to pi by 2 and r equals to 0 means if we put r equals to 0 in this expression. We get theta equals to 0 and r tends to infinity means theta equals to theta fine. So, with this values now let us plot this phase variation of phase with respect to r like what we have seen the variation of DMF with respect to r. Now we want to plot the variation of phase with respect to r.

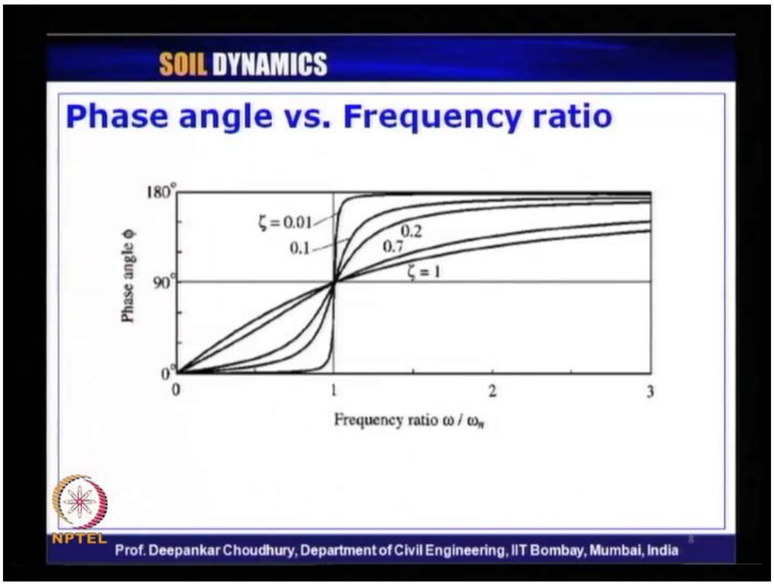
(Refer Slide Time: 15:24)



So, r this is theta at r equals to 1. We have some value r 0 theta varies between 0 then pi by 2 and pi let me draw some parallel lines first. So, it will be easy for us later on to draw different graphs what we got when r equals to 0 theta is 0 and when r tends to infinity theta goes to pi. And for undamped case when eta equals to 0 degree we got the value at theta equals to pi by 2 for r equals to 1. So, what will be the variation? It will follow this path it follows this and this. So, this the variation of phase for eta equals to 0 undamped case. But if eta is non 0 then the variation goes like this it starts at r equals to 0 is theta 0.

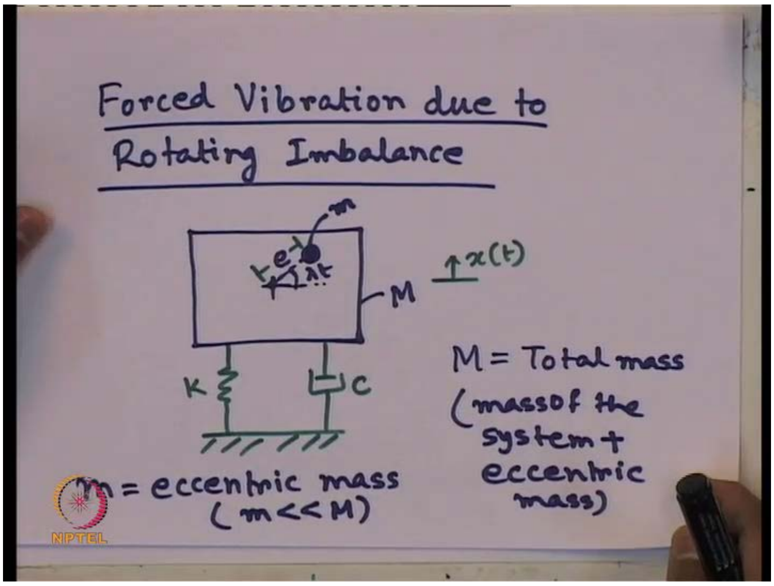
So, from this point it will always start and at r equals 1 it will be always pi by 2. So, this point they must touch and r tends to infinity it becomes pi. So, all the variations will be like this. So, what we can write here? This will be in the order of eta decreasing. That means, it can be eta say 0.1 it can be eta 0.2 this can be eta 0.3 like that. So, 0.3 0.2 0.1 and eta 0; so that is the way the phase is varying with respect to the frequency ratio r; so now, the same thing. What I have now just explained let us look back to the slides here.

(Refer Slide Time: 18:26)



The DMF variation we have seen already this is the variation of phase angle with respect to frequency ratio as I have explained. 0 degree 90 degree and 180 degree it always pass through this point. When r equals to one frequency ratio is one the phase angle must be 90 degree when r is equal to 0 phase angle is 0 degree and r infinity tends to goes to pi. So, this is the variation and this is the decreasing trend of the damping ratio; so this high value going to the lower value. Now, let us come to the next sub topic, the forced vibration due to imbalance of mass. Let us look at the sheet now.

(Refer Slide Time: 19:15)

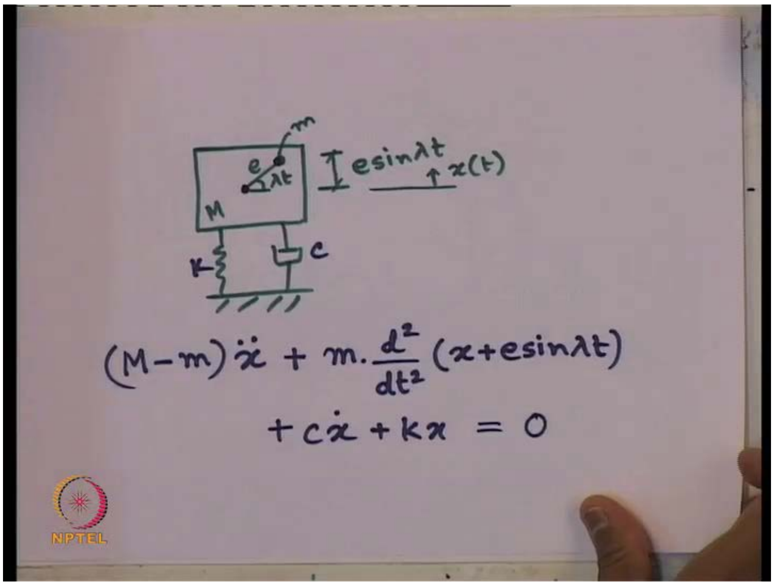


So, forced forced vibration due to rotating imbalance. So, what does it mean, let us draw a picture. That is suppose we have a big machine with amass say capital M and this is the C G of the machine big machine and there is a rotating component of a mass that is a small mass of m small m it is rotating continuously within that machine. So, it is having some eccentric mass with respect to its own C G which is keep on rotating fine.

So, let us say this is the value of the eccentricity e and this frequency which it rotates is lambda. So, this angle I am expressing in terms of lambda t and this magnitude of eccentricity of that small mass with respect to the C G of the main machine is e. So this is the eccentricity with which it rotates. Now, as usual I have the single degree of freedom system. We are considering stiffness and damper we have x of t with respect to the c g of the main mass. And what we said capital m is the total mass of the system? That is mass of the main system plus mass of the main system plus eccentric mass.

So, capital M is the is mass of the entire thing including that eccentric mass. Whereas, the small m is eccentric mass and what is the condition this small m is much, much lower than this capital M; obviously, a small eccentric mass is rotating as I have explained already. So, what will be our equation of motion for this system? This single degree of freedom system, let us see.

(Refer Slide Time: 22:35)

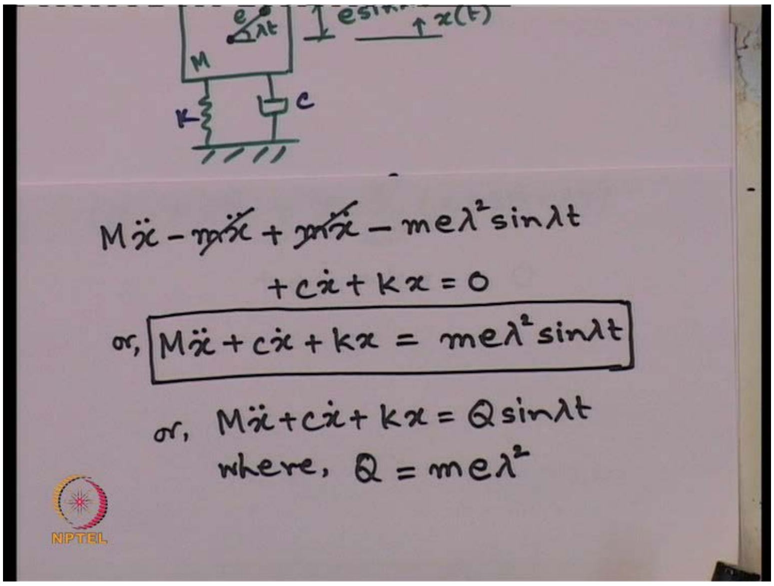


So, if I try to draw the system once again about this C G I have rotating mass here with e this angle is lambda t. So, what is this value is this distance e sin of lambda t. Am I right?

And our x is measured with respect to this reference point x of t. So, if I want to write down the equation of motion for the entire system what it should be? The main system its having how much displacement with respect to its C G x t excluding the eccentric mass.

So, excluding eccentric mass the main system is having an acceleration of x double dot. So, I have the inertia force M minus m x double dot. This is the total mass of the system. If I exclude that eccentric mass that is subjected to acceleration of x double t x double dot. So, the inertia force is M minus m x double dot plus this eccentric mass is also rotating. So, it is also subjected to an acceleration. Now, this eccentric mass is subjected to how much of displacement not x of t, but x t plus e sin lambda t. So, the acceleration this eccentric mass is subjected to will be. So, the inertia force if I want to write small m times d 2 by d t square the displacement which is x plus e sin of lambda t fine because this eccentric mass is subjected to this displacement and I can get the acceleration by double differentiating it. So, acceleration times mass will give me the inertia force coming because of this rotating imbalance mass in balance plus the remaining things as usual c x dot plus k x because they are subjected to with respect to the c g of the system. So, that is why see x dot only and k x only. This is equals to how much? There is no other externally applied force to the system. So, it should be equals to 0. So, now if we want to simplify this expression what we can get? Let me do it put it little above. So, that it will be easy for us to follow.

(Refer Slide Time: 26:07)

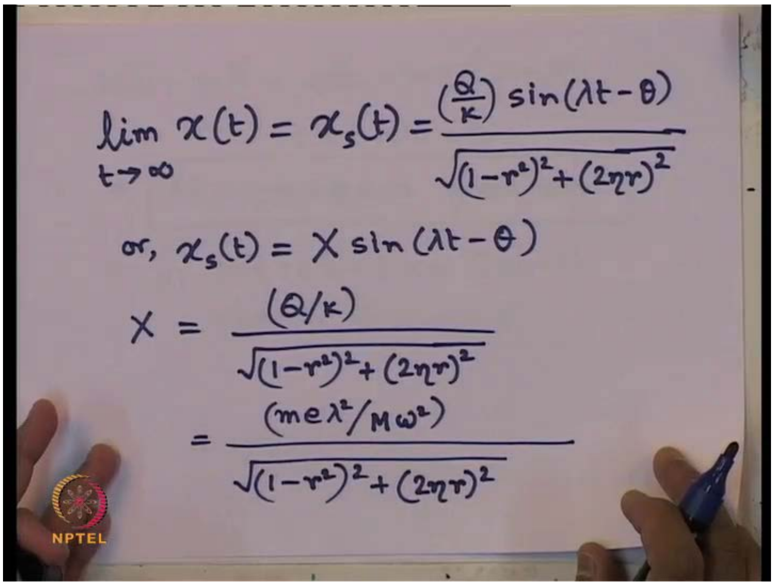


So now, I am simplifying it further capital M x double dot minus small m x double dot. Plus from this I am getting small m x double dot minus I have to differentiate this twice with respect to t. So, I am getting sine to cos cos to sine with minus sign; so m e lambda square sine of lambda t plus C x dot plus K x equals to 0. Now, this gets cancelled. We can simplify it further M x double dot plus C x dot plus K x equals to its come to right hand side m e lambda square sine of lambda t. So, this is the governing equation of motion for a system of single degree of freedom system subjected to eccentric mass rotating imbalance. So, look here the dynamic load is coming from the moving parts of the machine or the rotating parts of the machine.

So, that is why we have mentioned in the very first class that the reason of generating dynamic load one of the reason is this rotating part of a machine machinery body or something like that. So, this is nothing, but again a harmonic load. So, what does it mean? Already we have solved what is the solution for a single degree of freedom system subjected to harmonic load. So, this we can further simplify that M x double dot plus C x double dot plus K x equals to Q sine of lambda t where this amplitude Q is measured as that rotating mass small m eccentricity of the mass with respect to the C G of the entire system and the frequency which the rotation is occurring external frequency.

So, m e lambda square gives us the magnitude of this Q. So, solution of this is already known to us. No need to go through again the solution because a single degree of freedom system subjected to harmonic load. What is the final solution? We know now let us use that and we are interested about the steady state displacement as we have already talked about.

(Refer Slide Time: 29:15)

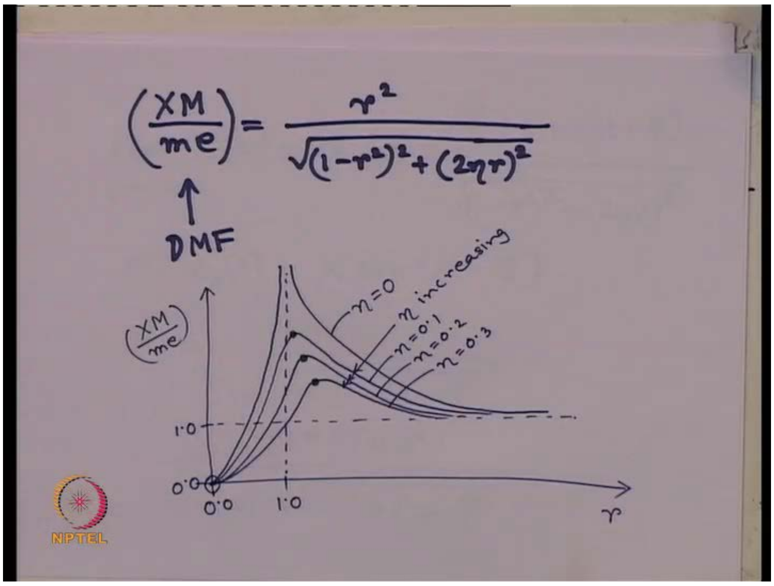


So, I am writing limit when t tends to infinity our x of t the total dynamic response. It is written as steady state displacement which comes out to be earlier it was P by K right. So, now, it is Q by K sine of lambda t minus theta. Earlier we had taken the cosine function harmonic. Now, it is a sine function. So, it will be sine of lambda t minus theta fine. Let me take this equation once again for you yes this divided by root over the same thing 1 minus r square whole square plus 2 eta r whole square.

So, that is our x of t which we are we can rewrite again simplified further like this or x s t the steady state response is x sine of lambda t minus theta what we did earlier also. So, the same thing we are doing here where this capital X where this capital X is nothing, but Q by K by root over 1 minus r square whole square plus 2 eta r whole square. Now, let me put the expression for this Q and K Q is how much? m e lambda square.

So, m e lambda square and how much is K in terms of a by omega K is m omega square fine. In that case it has to be the total mass of course, divided by root over 1 minus r square whole square plus 2 eta r whole square. So, this if we want to simplify further what we will get? Look at here lambda square by omega square. So, this is the r square and the rest of the term small m e and capital m we will take on the left hand side. So, what I am telling you?

(Refer Slide Time: 32:02)



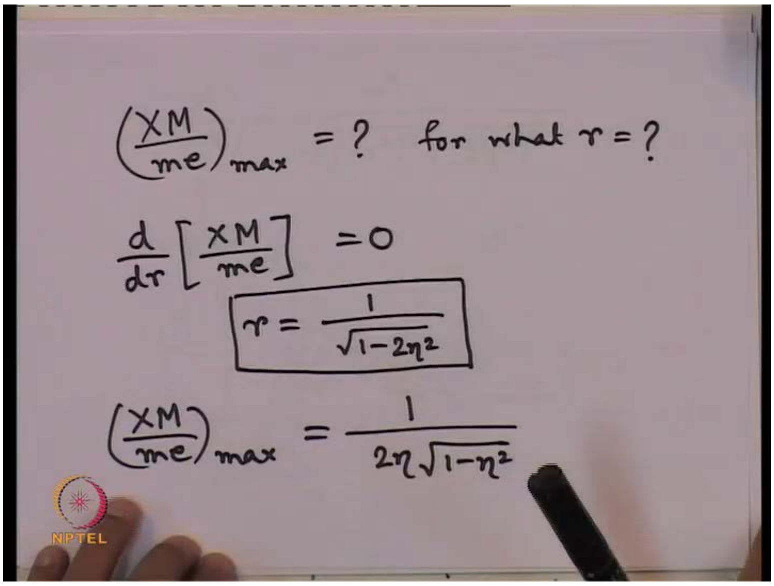
So, I am simplifying it like this X capital M by small m e equals to r square by root over 1 minus r square whole square plus 2 eta r whole square fine. This term X M by m e is known as DMF dynamic magnification factor for this case of rotating imbalance. That is a moving rotating body which is eccentric with respect to the C G these terms.

So, this is our expression for dynamic magnification factor for a rotating mass where as the earlier expression was one by this. So, here there is a slight change in the numerator we have now r square. So, that is difference of DMF with respect to the previous case. Now, let me plot this DMF with respect to r again. What we did for the previous case? We want to do here also. Now, r this is I am plotting X M by m e which is expressed as DMF here at 1 0 0 1. Let me draw these parallel lines first. It will be easy for us to draw the graph and understand that condition. Now, when r is 0 what is the value of this X M by m e irrespective of the value of the damping ratio whenever r is 0 this is 0. So, it is start from this value 0 zero. Always when r is very, very high tends to infinity what happens to this? X M by m e whatever be the value of eta this is infinity by infinity. So, it will tend to approach to 1.

So, it is start from here goes to 1 and when eta is 0 at r equals to 1. What is the value of X M by m e when eta 0? This is vanished r equals to 1 to this is 0 I get infinity. What does it mean for eta 0? The undamped case the variation will be something like this. It approaches infinity. It comes from infinity and here it approaches one that is the variation for eta equals to 0 undamped case and if eta is non zero; obviously, this value will not be 0 even if the r equals to 1.

So, we will get some variation like this. So, starting point is always same 0 0 and end point approach is always same for approaching to X M by m e equals to 1. These are the in the order of eta increasing that is say eta 0.1. This is the eta 0.2, then this eta 0.3, like that and where this maximum value of X M by m e occurs. The same case like previous one we want to find out where the maximum DMF occurs. If you look at the figure carefully already we have drawn this occurs little after r equals to 1. So, let us see where it occurs now mathematically. So, we want find out now maximum value of X M by m e where it occurs.

(Refer Slide Time: 36:52)

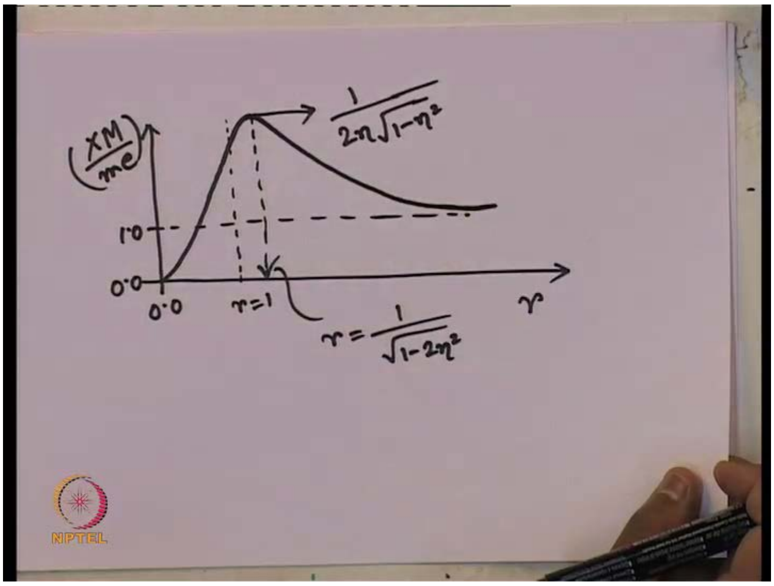


So, X M by m e max for what value of r that we want to find. That means, we do this operation again equals to 0. If we do this from the expression of this X M by m e differentiate this term with respect to r and equated to 0 on simplification we get r equals to 1 by root over 1 minus 2 eta square; that is the value of r, where this becomes maximum, and what is the maximum value of that X M by m e? If we put in this expression the value of r equals to this much whatever X M by m e you will get that is the maximum value like previous case only.

So, X M by m e max. If we put this you will get 1 by 2 eta root over 1 minus eta square, if you compare this two values what you will get? The maximum value the magnitude of dynamic magnification factor is same whether it is subjected to a forced vibration with harmonic excitation or it is subjected to the force vibration due to a rotating imbalance in the system maximum dynamic magnification factor remains the same. But the location that is for what value of r that maximum occurs that differs earlier it was little before 1 r equals to 1 that is little less than 1 r equals to 1. Now it is little above r equals to 1.

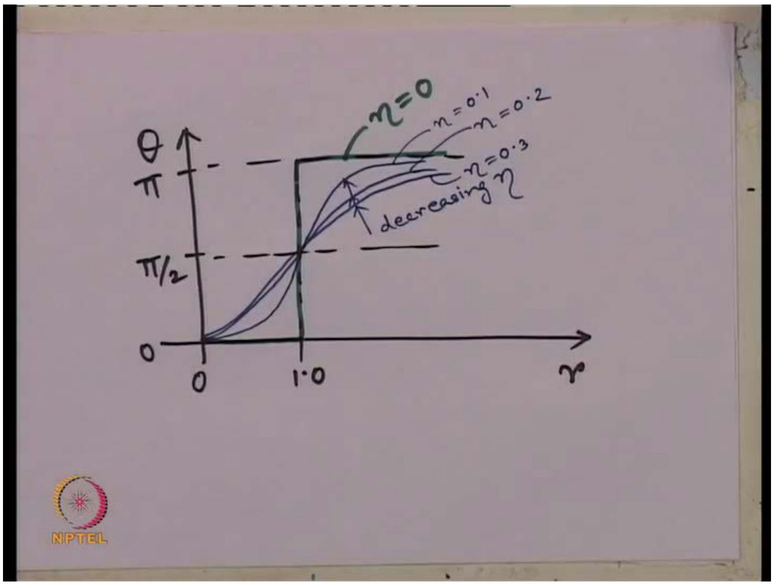
So, r little bit greater than 1. It occur say it is again the reciprocal of the previous case. Earlier r was root over 1 minus 2 eta square and now it is 1 by root over 1 by 2 eta square as all our cases have eta are less than 1. So, obviously, this r is becoming little greater than 1. So, that is why I have plotted already that these peaks are occurring at little after r equals to 1 in this graph.

(Refer Slide Time: 39:57)



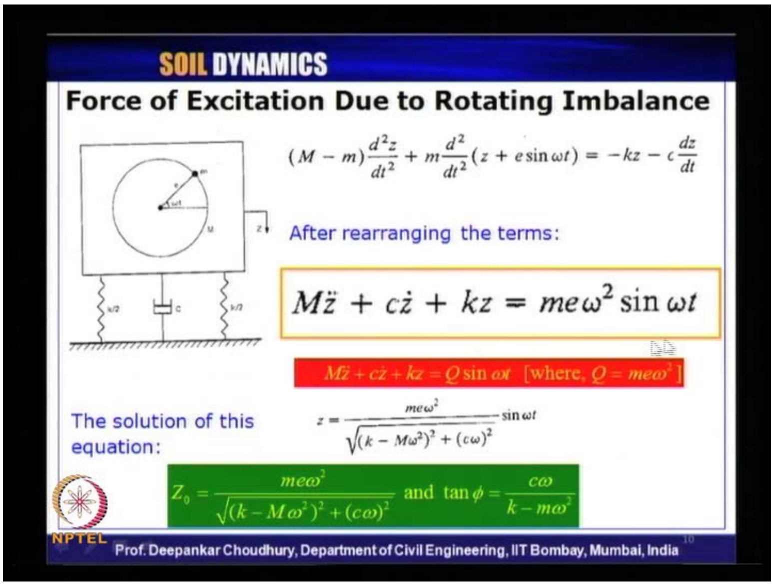
So, if I want to plot it for a damped case it should look like this that the variation of r X M by m e I have this as r equals to 1 this as 0 this 0 here its 1. So, the variation for any damped case is like these were I have this maximum value occurs at r equals to 1 by root over 1 minus 2 eta square. And this magnitude of maximum value is 1 by 2 eta root over 1 minus eta square and how the phase varies if we look at the expression? The phase will vary in the same manner because this also the function is harmonic nature. Let me put back the equation once again for you. So, this variation is again harmonic. See, if you the evaluation of the phase. Let me draw it here itself.

(Refer Slide Time: 41:25)



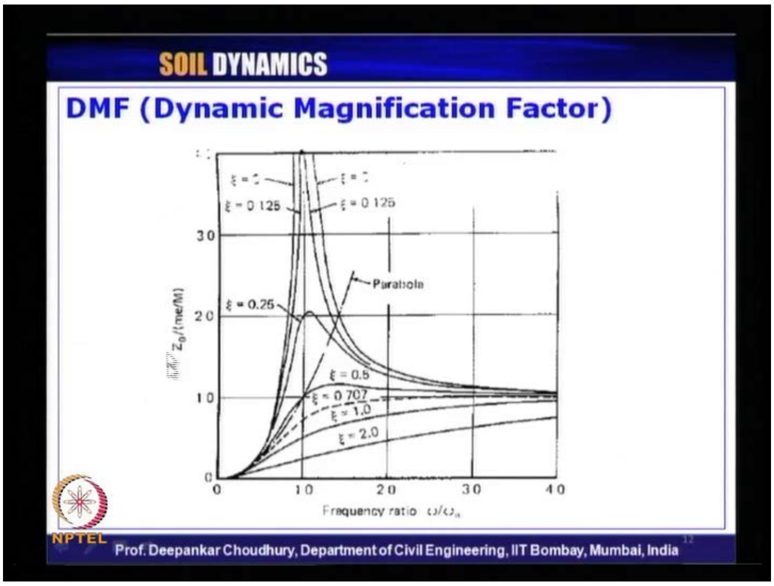
The variation of the phase will be something like similar one that r theta 0 r equals to 1 this is 0 pi by 2 this is pi at eta equals to 0. That is undamped case it follows this. So, this is my curve eta equals 0 and for eta non 0 it follows this trend. So, these are decreasing eta with say eta 0.1, here eta 0.2, here eta 0.3 like that. So, variation of phase remains same like the previous case.

(Refer Slide Time: 42:39)



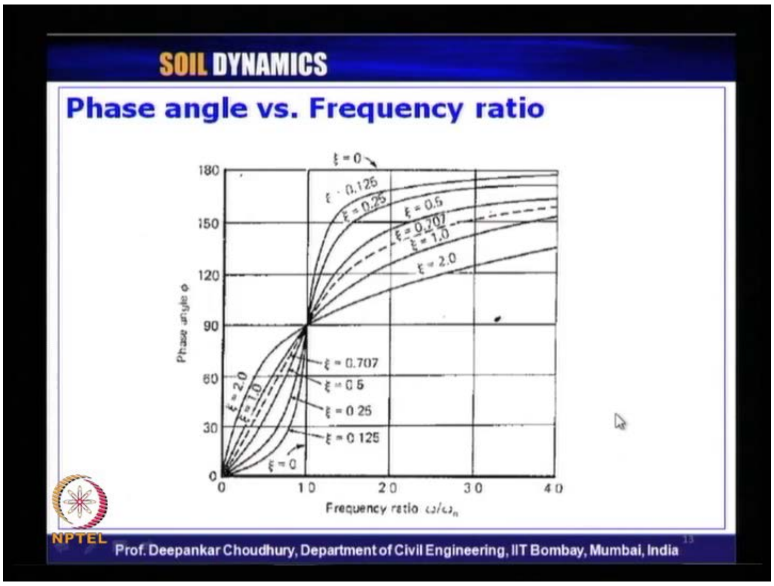
So, now, let us look at the slide what I have explained, derived just now. Force of excitation due to rotating imbalance picture shown this the rotating mass which is rotating about the C G with the eccentricity of E M is the mass of the total system. The equation of motion governing equation of motion expressed like this from which the solution we get like this.

(Refer Slide Time: 43:07)



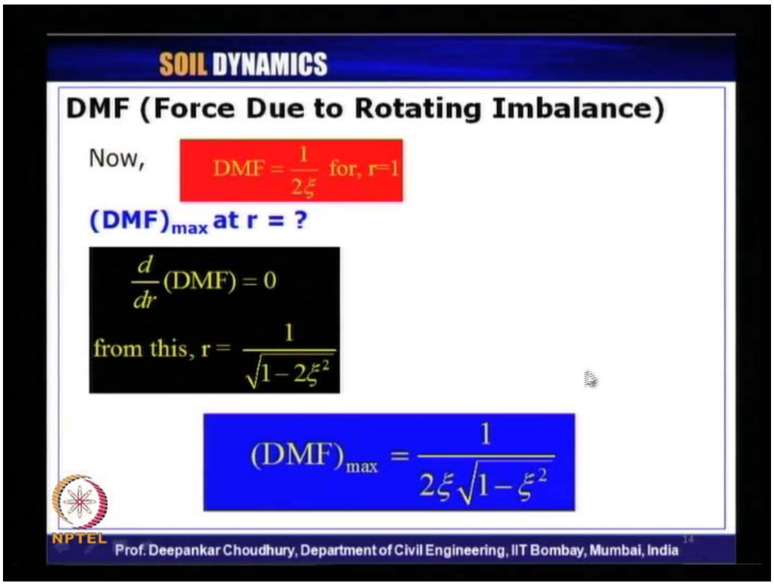
And the variation of that x m e is given like this with peak values coming here with respect to the frequency ratio for different values of the damping ratio.

(Refer Slide Time: 43:21)



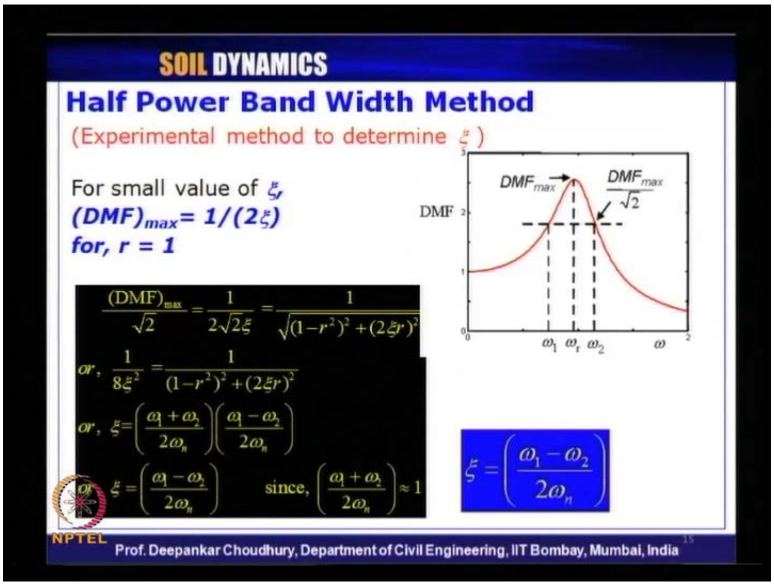
Also the phase angle how it varies? The same way like the pervious case with respect to the frequency ratio.

(Refer Slide Time: 43:31)



Now, for DMF is equals to 1 by 2 eta for by r equals to 1 when the maximum occurs. Just now we have derived it occurs at 1 by 1 minus eta square and that maximum value of DMF is the same like the pervious case. That is 1 by 2 eta root over 1 minus eta square.

(Refer Slide Time: 43:57)



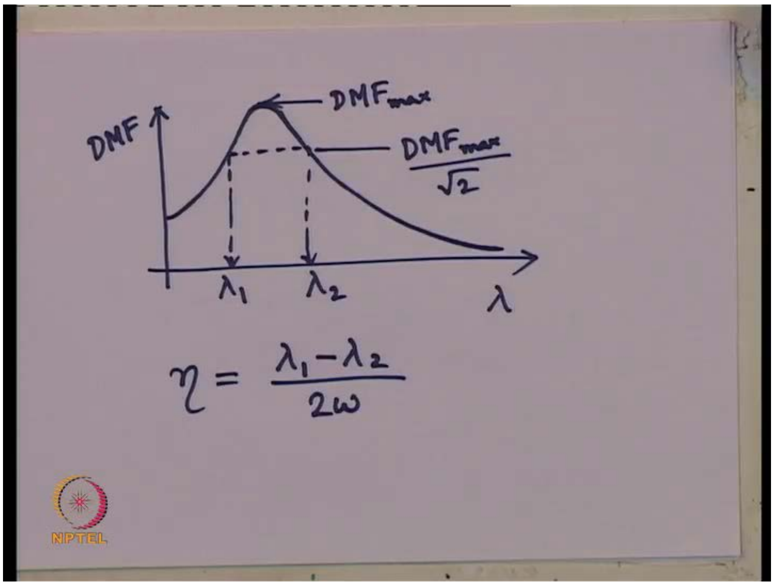
Let me come to now another sub topic which is called half power band width method. This is another experimental method to determine the damping ratio of any system. Earlier we have seen the logarithmic decrement has been used to measure the damping ratio of any system subjected to damped free vibration. If any system subjected to free vibration and we do the testing in that case; we measure the damping ratio using the concept of decay of motion logarithmic decrement. Now, using the concept of forced vibration also we can measure experientially the damping ratio of any system. This is the way the half power band width method by which experimentally we can obtain the damping ratio of any system. How? Let us look at the slide.

So, DMF versus different applied frequency; this is actually our lambda. What I am using in my derivation these are exciting frequencies. We get a plot like this it starts from one goes back to here subjected to a harmonic excitation. We have a maximum DMF here and we can identify one value of DMF which is equals to DMF max divided by root 2. So, that value will obviously, occur at two externally applied frequency values. Let us say one is lambda 1 another lambda 2 in terms of our notifications. This is for very small value of damping ratio for very small value of damping ratio. We can approximate the maximum damping a dynamic magnificent factor is equals to 1 by 2 eta right.

Let us look at this sheet the maximum value of damping ratio for very low value of eta we can approximate it as 1 by 2 eta only fine. So, that is what it is done in the slide. Lets us look here again. Now, for r equals to 1 when frequency ration equals to 1. What we can express? The dynamic magnification factor maximum value. Now, we are dividing it by that root 2 term. So, that is given by 1 by 2 theta. So, now, we are dividing it by root 2. So, in denominator we are we getting root two which is given by this expression.

So, for two different values of r it can occur. So, from this second order expression for r what we will get? Two roots of r. If we solve this we will get two values of r which is obvious that this value will occur at two values of r. So, that is what on simplification if we express this almost equals to 1. So, on simplification we are expressing lambda 1 plus lambda 2 by 2 times of natural frequency almost equals to 1. Then the damping ratio is given by that lambda 1 minus lambda 2 divided by 2 omega.

(Refer Slide Time: 47:55)



So, in our terminology what we are using? Let me express it once again for you. What we were mentioning the DMF. This is say applied frequency of excitation lambda. We have from here to a maximum value than slowly comes to almost 0 we have a DMF max here and we have these two values lambda 1 lambda 2 for DMF max by root two and putting in the expression, what we got? That damping ratio eta is given by that lambda 1 minus lambda 2 divided by 2 omega natural frequency.

This lambda 1 minus lambda 2 this is called half power band width. So, this is that is why the name came. Half power bandwidth method to obtain experimentally the value of the damping ratio. So, with this half power bandwidth method we will complete our lecture today. We will continue our lecture in the next class.