


**Soil Dynamics**  
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**Module - 2**  
**Vibration Theory**  
**Lecture - 8**  
**Forced Vibrations, Dynamic Magnification Factor**

Let us start today's lecture of soil dynamics. We are continuing from the previous lecture with our module 2 on vibration theory.

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$$\dot{x}(t=0) = \dot{x}_0 = -\eta\omega A + B\omega_d$$
$$+ \frac{\left(\frac{P}{m}\right)\lambda \sin\theta}{\sqrt{(\omega^2 - \lambda^2)^2 + (2\eta\omega\lambda)^2}}$$
$$\Rightarrow B = \left[ \dot{x}_0 + \eta\omega A - \frac{\left(\frac{P}{m}\right)\lambda \sin\theta}{\sqrt{(\omega^2 - \lambda^2)^2 + (2\eta\omega\lambda)^2}} \right] \frac{1}{\omega_d}$$

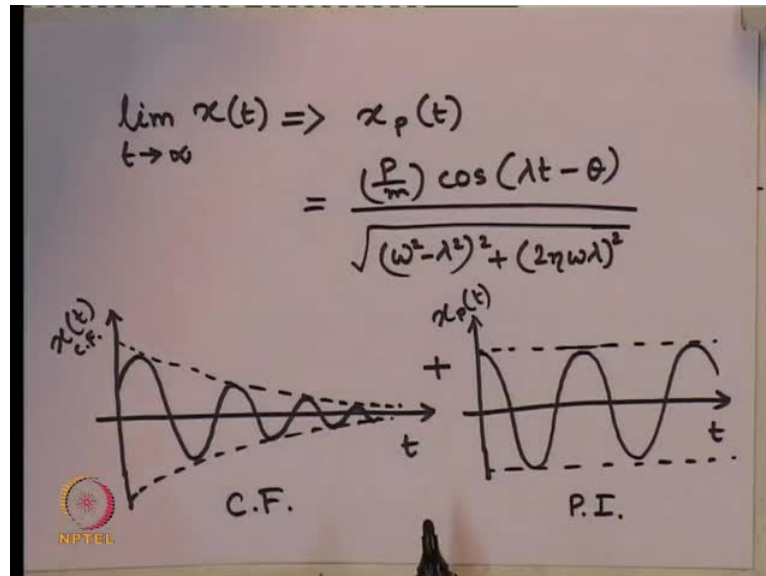


Now, what we are interested in we want to see how the response is looking like. How the total  $x$  of  $t$  when it is varying with respect to  $t$ , how it should look like? We have two separate parts of this solution. The one part is the complementary function and the other one is particular integral. So, if we take  $t$  tending to infinity which part will continue let us look back again in the solution, the form of the solution, what we had written in the complete solution was this. Let me put it here like this, the complete solution was here.

So, when  $t$  tends to infinity what happened to this function of complementary function, it dies down it goes to 0, as we have seen for the case of under damped free vibration. However, if  $t$  tends to infinity this is a harmonic function with some coefficients

something. So, that harmonic function will keep on continuing, it will not reduce. So, in other words if we want to know the response for a longer duration for the system...

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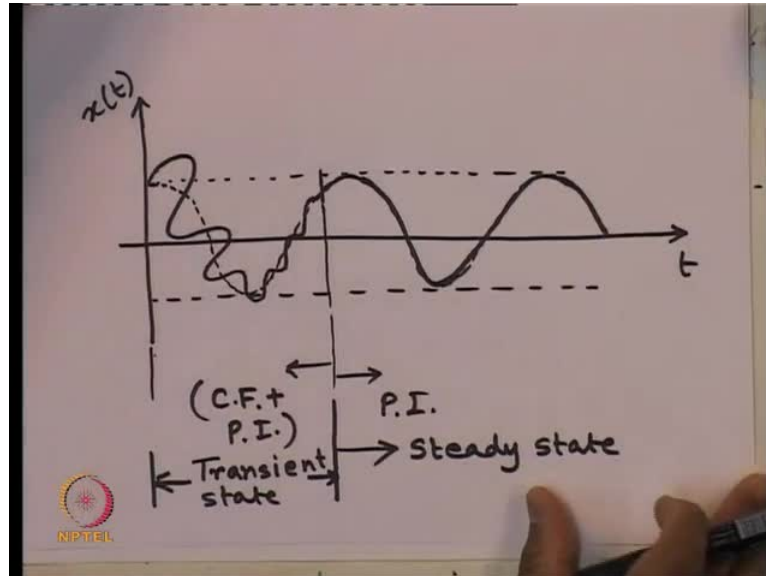
If I want to write say limit  $t$  tends to infinity  $x$  of  $t$  how it should look like? It will give me the solution as  $x_p$  of  $t$  which is nothing but  $P$  by  $m$  cosine  $\lambda t$  minus  $\theta$  by root over  $\omega$  square minus  $\lambda$  square whole square plus  $2\eta\omega\lambda$  whole square. So, it will tend to the result to this harmonic function as  $t$  tends to infinity because the other component has, goes to 0 or vanished. So, this is a periodic function as I said the total solution is having two components, what I am describing you just now. I am just drawing them separately and then we will combine them.

So, I am putting this plus sign here, in this I am drawing  $x$  of  $t$  the complementary function only and in this axis I am drawing  $x_p$  of  $t$  the particular integral part. So, the complementary function was similar to our case of under damped free vibration. That is it oscillates with a harmonic function, but with a decay, because of the presence of exponential decay function. So, this is the part of the solution for complementary function and the part of the solution for particular integral as  $t$  increases, there is no decrease.

So, it will keep on continuing like this like this. So, this is the part of particular integral. So, what will be the total solution? The total solution is nothing but when we are

combining these two that is why this plus sign I have shown here. So, how the combined response should look like let us combine it and see how it should look like.

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The combined response should be something like I am just showing the trend, how it should look like.  $x$  of  $t$  I am now plotting the total response, it will be let me first draw some dotted line with constant amplitude and then let me draw this curve that is the particular integral part that I am drawing. What will happen the initial portion when  $t$  just started from 0 it will now follow the complementary function over this curve as a base line. So, taking this curve as a base line our response of complementary function will run harmonically with an exponential decay.

So, let me draw it now. We will have some functions like this and then it will die down after certain time. So, the total response becomes this firm line or the solid line then beyond it, it vibrates periodically or harmonically following this step. So, clearly up to this point we can see this zone is having complementary function plus particular integral together as a solution, whereas beyond this point the solution we have only the particular integral after a certain time  $t$ . What that means the terminology which we used for this zone where we have this a periodic response is called transient state. Transient state and when this initial effect dies down and the response of the system follows the particular integral solution only, that is the harmonic function in this case only, we call that state as steady state.

So, beyond this point where the initial conditions of the system vanishes or dies down is called the starting of the steady state response. So, this is called the steady state response and this state is called transient state response. So, when we can say that a system has reached to its steady state condition when the response becomes harmonic. Then we can say in this case it has reached its steady state response or if we want to generalize more, it should be steady state of the system has been reached when the initial conditions have died down. So, initial conditions, effect of initial conditions are no longer existing because you look at the solution once again in this solution of  $x$  p of  $t$  there is no  $x$  naught no  $x$  naught dot. So, the initial conditions are not present in this particular integral which is for  $t$  tends to infinity.

So, initial conditions have totally removed that point when it is getting started is called the starting of the steady state response and transient state is that state of the system when the response is aperiodic that is the effect of initial conditions are present because in this case we have that all effects of  $x$  naught  $x$  naught dot everything present and the response is aperiodic.

So, that is the final solution for the forced vibration subjected to a harmonic excitation. We have seen how the transient state, how the steady state they behave and the responses we have drawn. Let us look at the slide for the same summary what I have now derived.

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**SOIL DYNAMICS**

**Forced Vibration**

**Harmonic Vibration with Viscous Damping**

Including viscous damping, the governing differential equation for the SDOF system subjected to harmonic force is,

$$m\ddot{u} + c\dot{u} + ku = p_0 \sin \omega t$$

Initial conditions  $u = u(0)$   $\dot{u} = \dot{u}(0)$

The particular solution of the differential equation is :

$$u_p(t) = C \sin \omega t + D \cos \omega t$$

where

$$C = \frac{p_0}{k} \frac{1 - (\omega/\omega_n)^2}{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}$$

$$D = \frac{p_0}{k} \frac{-2\zeta\omega/\omega_n}{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}$$

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So, the particular solution as I have mentioned is computed like this for this equation.

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**Harmonic Vibration with Viscous Damping (Contd.)**

The **particular solution**  $u_p(t)$  (steady state response) can be written in the form:

$$u_p(t) = \frac{p_0/k}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \sin(\omega t - \phi)$$

Where,  $\tan \phi = \frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}$      $0 < \phi < 180^\circ$

The **complementary solution** of the differential equation is the free vibration response given by

$$u_c(t) = e^{-\zeta\omega_n t} (A \cos \omega_D t + B \sin \omega_D t) \quad \text{Where } \omega_D = \omega_n \sqrt{1 - \zeta^2}$$

The homogeneous solution  $u_c(t)$  (transient response) disappears after some time. The complementary solution is,

$$u_c(t) = E e^{-\zeta\omega_n t} \sin(\omega_D t + \theta)$$

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And the particular solution is expressed in this form and the complementary function is expressed in this form.

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**Harmonic Vibration with Viscous Damping (Contd.)**

The **complete solution** of the differential equation is given by

$$u(t) = u_c(t) + u_p(t)$$

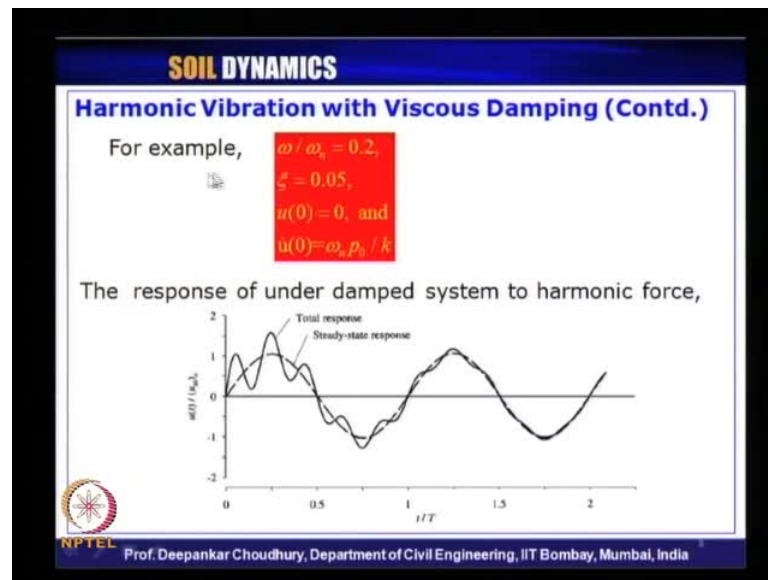
$$u(t) = E e^{-\zeta\omega_n t} \sin(\omega_D t + \theta) + \frac{p_0/k}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \sin(\omega t - \phi)$$

- The total response is  $u(t) = u_c(t) + u_p(t)$ . But after some time  $u_c(t)$  disappears and  $u(t) = u_p(t)$  (steady state response).

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Then we get the total solution as the summation of the complementary function and the particular integral comes out to be like this, where the particular integral gives us the steady state response and the initial condition this will give us the transient state.

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So, for one example problem has been shown here for a particular value of  $\lambda$  in this case this  $\omega$  is applied excitation in our symbol, we have used  $\lambda$ . So, this is  $\lambda$  and this  $\omega_n$  is natural frequency. So, the exciting frequency to the natural frequency ratio is taken as 0.2, the damping ratio  $\zeta$  is 0.05, initial displacement is 0 and initial velocity is given by this expression. For this values the results a typical results plotted is shown here in terms of non-dimensionalized form that is the  $u$  of  $t$  to the static  $u$  of  $t$  at 0 time that is at initial time with respect to the x axis's time to the natural time period.

So, as the initial displacement was 0 so its starts from 0 you can see and this is the total response as I have mentioned and slowly with time it dies down, but this keeps on continuing because this is the steady state response which remains harmonic. That is what I have also drawn. So, this is another example problem has been shown here.

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**SOIL DYNAMICS**

### Steady State Displacement

After sometime, the structure vibrates with the same frequency as the applied force. It is the steady state response  $u_p(t)$ .

The amplitude of the steady state vibration is,

$$u_p(t) = \frac{p_0/k}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \sin(\omega t - \phi)$$

The static deformation due to a static load  $p_0$  is  $(u_{st})_0 = \frac{p_0}{k}$

The frequency ratio  $r$  is  $r = \frac{\omega}{\omega_n}$

Now,

$$u_p(t) = \frac{(u_{st})_0}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}} \sin(\omega t - \phi)$$

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Now, from this steady state displacement let us now see what are the other things we can derive or we can get from the steady state displacement.

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### Steady State Displacement

$$x_s(t) = \frac{\left(\frac{P}{m}\right) \cos(\lambda t - \theta)}{\sqrt{(\omega^2 - \lambda^2)^2 + (2\eta\omega\lambda)^2}}$$

$$= \frac{\left(\frac{P}{m\omega^2}\right) \cos(\lambda t - \theta)}{\sqrt{\left(1 - \frac{\lambda^2}{\omega^2}\right)^2 + \left(2\eta\frac{\lambda}{\omega}\right)^2}}$$

$r = \frac{\lambda}{\omega} = \text{Frequency Ratio}$

$$\frac{P}{m\omega^2} = \frac{P}{m\left(\frac{K}{m}\right)} = \frac{P}{K} = X_{st}$$

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So, let me write down steady state displacement on which now we want to concentrate. Why? Because for any system we as an engineer what we want to know how it performs in long run, that is when  $t$  tends to infinity in future how the system or how the structure or how the sub structure or foundation etcetera will behave. So, we are more interested on the steady state displacement because transient state will vanish within few minutes.

That also is important for the design of course, but for a long term behavior of any structure or sub structure we are more interested on the steady state response.

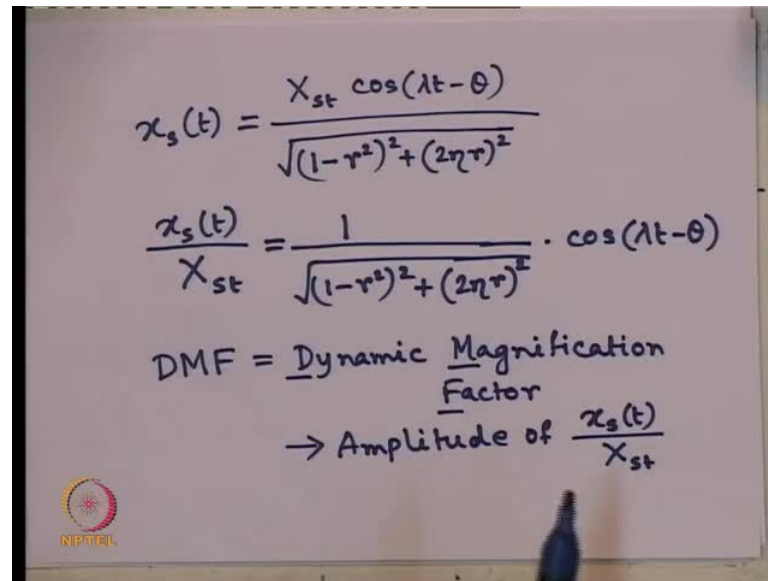
So, let us concentrate on that steady state response. The steady state displacement let me denote it by  $x_s(t)$  this small  $s$  denote as steady state. The solution we got as  $\frac{P}{m} \cos(\lambda t - \theta)$  by  $\frac{1}{\sqrt{\omega^2 - \lambda^2 + 2\eta\omega\lambda}}$  whole square, that was the expression for the steady state. What we can do, we can divide both numerator and denominator by  $\omega^2$ . So, what we will get?  $\frac{P}{m\omega^2} \cos(\lambda t - \theta)$  and the denominator it becomes  $\frac{1}{\sqrt{1 - \lambda^2/\omega^2 + 2\eta\lambda/\omega}}$  whole square.

Now, let us define one parameter  $r$ . What is  $r$ ?  $r$  is the ratio of the external frequency to the natural frequency of the system, which is known as frequency ratio. So,  $r$  is called frequency ratio, it is defined as the ratio of the applied external frequency to the natural frequency of the system. And what is let us look at this term  $\frac{P}{m\omega^2}$ . What we can simplify for this term it is  $\frac{P}{m}$  times what is  $\frac{1}{\omega^2}$   $\frac{K}{m}$ . So, it is  $\frac{K}{m}$  by  $m$ ,  $m$  gets cancelled I get  $\frac{P}{K}$ . Now, what is  $\frac{P}{K}$ ?  $P$  was the amplitude of the dynamic load and  $K$  is the stiffness means spring constant.

So, load by stiffness is nothing but the static displacement. So, this we are denoting as say capital  $X_s(t)$ ,  $s(t)$  means static. So, this  $\frac{P}{K}$  is nothing but our static displacement. What does it mean? Instead of the dynamic load if I apply a static load on the system with magnitude  $P$  the deflection of the system will be  $X_s(t)$ . So, let us look back again this expression what we are getting.



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The image shows a whiteboard with handwritten mathematical expressions. The first equation is  $x_s(t) = \frac{X_{st} \cos(\lambda t - \theta)}{\sqrt{(1-r^2)^2 + (2\eta r)^2}}$ . The second equation is  $\frac{x_s(t)}{X_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\eta r)^2}} \cdot \cos(\lambda t - \theta)$ . Below these, it defines DMF as the Dynamic Magnification Factor, which is the amplitude of the ratio  $\frac{x_s(t)}{X_{st}}$ . A small logo for NPTEL is visible in the bottom left corner of the whiteboard.

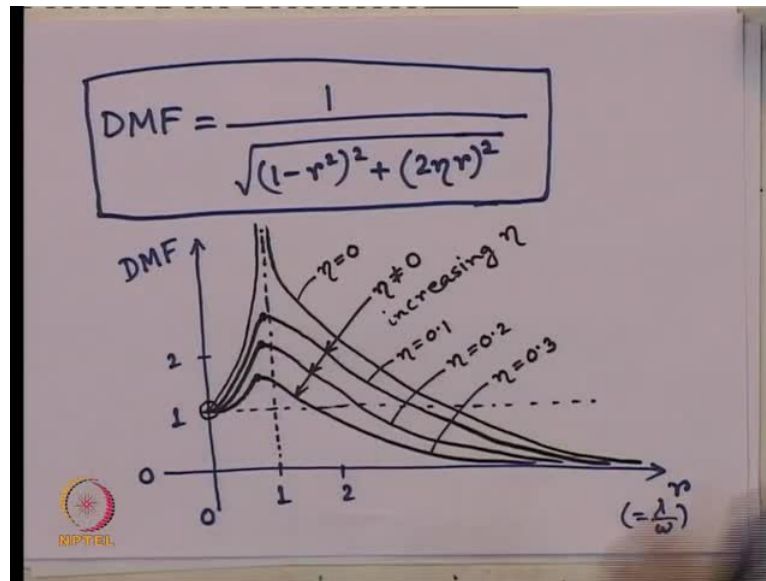
$$x_s(t) = \frac{X_{st} \cos(\lambda t - \theta)}{\sqrt{(1-r^2)^2 + (2\eta r)^2}}$$
$$\frac{x_s(t)}{X_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\eta r)^2}} \cdot \cos(\lambda t - \theta)$$

DMF = Dynamic Magnification Factor  
→ Amplitude of  $\frac{x_s(t)}{X_{st}}$

So,  $x_{st}$  is the steady state displacement, the steady state displacement we are writing now this  $P$  by  $M \omega^2$  let me put this expression here. So, it will be easy for us to follow. This is nothing but the static displacement times cosine of  $\lambda t - \theta$  by root over what I am getting from here  $1 - r^2$  whole square plus  $2\eta r$  whole square where  $r$  is the frequency ratio  $\lambda$  by  $\omega$ . So, we can simplify again this expression like this. The steady state displacement by the static displacement it is equals to  $1$  by root over  $1 - r^2$  whole square plus  $2\eta r$  whole square times cosine of  $\lambda t - \theta$ .

What it shows us, the ratio of the steady state displacement to the static displacement is a harmonic function with amplitude or the maximum value of this comes out to be this value. So, the maximum value of the ratio of the steady state displacement to the static displacement is known as dynamic magnification factor DMF dynamic magnification factor DMF. So, what is the definition of dynamic magnification factor? It is the amplitude of this ratio of steady state displacement to the static displacement that is defined as dynamic magnification factor. So, what is the expression for the dynamic magnification factor? This one, so let us write that.

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So, DMF is given by the expression  $\frac{1}{\sqrt{(1-r^2)^2 + (2\eta r)^2}}$ . This parameter is extremely important. Why? Because it gives us the idea that under the dynamic load how much the displacement will magnify compared to your static displacement. So, that is why this terminology DMF is extremely important for any engineering design. So, if it is very important let us see the response or variation of this DMF with respect to the variation of different parameters like  $r$  and  $\eta$ .

So, let us plot the variation of DMF with respect to  $r$  and see how it looks like. So, x axis I am using  $r$  which is nothing but our frequency ratio  $\lambda$  by  $\omega$  and the y axis I am plotting dynamic magnification factor DMF. If we look at this expression very carefully  $r$  can vary from 0 say 1, 2 and then keep on going here,  $r$  can increase here and DMF dynamic magnification factor let us start it from 0 say 1, 2 and other values. If we take an undamped condition that is  $\eta$  equals to 0, when the value of frequency ratio  $r$  equals to 0 what is the value of dynamic magnification factor? When  $r$  is 0  $\eta$  is 0, the dynamic magnification factor is 1, what does it mean when  $r$  is 0  $r = 0$  when it can happen?

$\omega$  cannot be 0 as I said the natural frequency of the system cannot be 0. The possibility of  $r$  equals to 0 means  $\lambda$  is 0 that means there is no externally applied dynamic load. So, for that case how much will be the displacement for the system? It should be nothing but the static displacement that is why we are getting the DMF equals to 1. So, that is the physical significance which we have understood. Let us draw certain

lines here it will be easy for us later on. So, I am just drawing few lines. So, the starting point when  $r$  equals to 0 and  $\eta$  equals to 0 the DMF is always 1 that is static displacement is equals to your dynamic displacement because we do not have any vibration, we do not have any dynamic load applied on the system.

So, of course, it has to be 1 when  $\eta$  is non-zero that is when we are considering damped material, damped system with that if we have  $r$  equals to 0 that is again the frequency ratio 0, what should be the value of dynamic magnification factor? Again 1, if we put  $r$  0 here  $\eta$  is having some value, but still we have dynamic magnification factor 1 which is again justified physically because without presence of any forced vibration or applied dynamic load on the system, the displacement should be equals to the the steady state displacement or longer term displacement should be equals to the static displacement nothing more than that. So, that is why it will always start from the 1.

Now, let us look when  $r$  equals to 1 and we have undamped system that is  $\eta$  equals to 0 with  $r$  equals to 1, what should be the value of DMF?  $r$  is 1 we are putting in this it becomes 0,  $\eta$  is also 0. So, what it becomes?  $1$  by  $0$  means infinity so at  $r$  equals to 1, it goes to infinity. So, how the variation should be? Variation should be something like this. Close to  $r$  equals to 1, it will be tangential to this line. It goes to infinity and let us look when  $r$  is very, very high and  $\eta$  is 0. What will happen to the value of DMF?  $r$  tends to infinity.

If I put  $r$  tends to infinity in this expression what I will get? This will be  $1$  by infinity means DMF will approach to 0 with very high value of  $r$ . So, the other side of the curve will be something like this, tangential here goes down here, approaches 0 at very high value of  $r$ . So, this curve we got for  $\eta$  equals to 0 that is undamped case. Now, if we take damped condition that is  $\eta$  is non-zero still the curve will start from this point 1 when  $r$  equals to 0. When  $r$  equals to 1 what happens?  $\eta$  is non-zero, so we have some value here say  $\eta$  is 0.1 so it is 0.2 times 1.

So, 0.2 whole square, so we have 0.04 and here it is 1 so it becomes 0, but we have some finite value which we can obtain for  $r$  equals to 1 at non-zero value of the  $\eta$  and let us look what happens when  $r$  tends to infinity for non-zero value of  $\eta$ . When  $r$  tends to infinity whatever be your value of  $\eta$  this DMF goes to 0 or approaches to 0. So, the type of graph which we should get is something like this. Let me draw it first. So, it will

approach to 0 here. Then let me draw again another graph with this. Then let me draw another graph with this so these are the, for different values of  $\eta$ .

So,  $\eta$  non-zero, these are the graphs for  $\eta$  non-zero because here we will get some finite value. Now, what is this shows? This is in the decreasing values of  $\eta$  or increasing values of  $\eta$ ? If the value of  $\eta$  is increasing, what we will get? The DMF value will keep on decrease so this is in the increasing trend of increasing  $\eta$ . So, these three arrows what I have shown here that shows the increasing values of  $\eta$ . Say for example, this is for  $\eta$  equals to 0.1, this is for  $\eta$  equals to 0.2, this is for  $\eta$  equals to 0.3 like that just typical examples I am giving, the exact figure may be different, but trend will remain same.

Now, if you look at this variation properly what we know when the  $r$  equals to 1 means your applied external frequency matches with the natural frequency of the system. So, when  $r$  equals to 1 that means  $\lambda$  equals to  $\omega$ , means externally applied frequency equals to the natural frequency of the system. So, as a layman what we know? When the applied frequency, external frequency matches with the natural frequency of the system the resonance occurs.

So, that is why when  $\eta$  equals to 0 we got DMF as maximum or it goes to infinity actually, but when  $\eta$  is non-zero there also we got some very high value of DMF with  $r$  equals to 1, but those are not infinity. Infinity becomes only when it is a undamped case. In case damped case it will be some high value of dynamic magnification factor. Now, at  $r$  equals to 1 really those are the maximum value, maximum point of DMF or something else. Let us look. Look at the curve I have drawn the peak, knowingly little before  $r$  equals to 1. So, these are my peaks, not at  $r$  equals to 1. Why? Let us see now. We will derive it. So, where the dynamic magnification factor is maximum we want to know.

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The image shows a whiteboard with handwritten mathematical work. At the top, it asks for the maximum DMF and the corresponding value of r. The DMF is given as  $DMF = \frac{1}{\sqrt{(1-r^2)^2 + (2\eta r)^2}}$ . The derivative  $\frac{d(DMF)}{dr} = 0$  is set up, leading to the equation  $2(1-r^2)(-2r) + 4\eta r(2\eta) = 0$ . Since  $r \neq 0$ , this simplifies to  $2\eta^2 = 1 - r^2$ , which is boxed to give the final result  $r = \sqrt{1 - 2\eta^2}$ . An NPTEL logo is visible in the bottom left corner of the whiteboard.

$$(DMF)_{\max} = ? \text{ for } r = ?$$
$$DMF = \frac{1}{\sqrt{(1-r^2)^2 + (2\eta r)^2}}$$
$$\frac{d(DMF)}{dr} = 0,$$
$$2(1-r^2)(-2r) + 4\eta r(2\eta) = 0$$
$$\therefore r \neq 0$$
$$\Rightarrow \boxed{r = \sqrt{1 - 2\eta^2}}$$

So, we want to find out DMF max. When it occurs? That is for what value of r that maximum DMF occurs. What is the expression for DMF? The expression is 1 by root over 1 minus r square whole square plus 2 eta r whole square. So, if we want to find out the maximum dynamic magnification factor what we need to do? We need to find out differentiate it with respect to r and equate it to 0, then let us do that operation what we will get from this denominator by expanding it I can write it like this 2 1 minus r square times minus 2 r plus 4 eta r into 2 eta equals to 0, by differentiating with respect to r.

Now, r is non-zero. So, from which we can simplify and write it like this 2 eta square equals to 1 minus r square which will give me the value of r is 1 minus 2 eta square. So, that is the value of r at which DMF becomes maximum. So, now let us put the value of r in this expression of DMF. So, that will give us the maximum value of DMF. Let us do that.

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$$\begin{aligned}
 (DMF)_{max} &= \frac{1}{\sqrt{\dots}} \\
 (DMF)_{max} &= \frac{1}{\sqrt{\{1 - (1 - 2\eta^2)\}^2 + 4\eta^2(1 - 2\eta^2)}} \\
 &= \frac{1}{\sqrt{1 - 1 + 4\eta^2 - 4\eta^4 + 4\eta^2 - 8\eta^4}} \\
 &= \frac{1}{\sqrt{8\eta^2 - 12\eta^4}}
 \end{aligned}$$

So, DMF max that is equals to now I am putting this value of r in that expression root over 1 minus r square. So, r square is now how much? 1 minus 2 eta square, I am just putting the value of r, let us put it here again. So, r is now root over 1 minus eta square. So, I put r square is 1 minus 2 eta square this whole square plus 4 eta square r square. r square is again 1 minus 2 eta square. See, if we simplify this, what we are getting? On simplification we get 1 1 gets cancelled, this becomes 4 eta to the power 4 with a minus sign and then let me do that, it will be better 1 minus 1 it will give me minus. So, minus minus plus 2 into 2 4 eta square then I am getting plus so it becomes minus 4 eta to the power 4 plus this gives me 4 eta square minus 8 eta to the power 4. That is what we get, right?

So, on simplification we are getting it like this, 8 eta to the power 4 4 eta square. So, it gives us 8 eta square minus 12 eta to the power 4. I made some mistake somewhere, this is 1 minus 1 gets cancelled this is... I did this whole square, sorry let me put it once again. I make the whole square of the bracket term, but that has to be simplified and then...

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The image shows a whiteboard with handwritten mathematical steps for finding the maximum Dynamic Magnification Factor (DMF). The steps are as follows:

$$\begin{aligned} (\text{DMF})_{\max} &= \frac{1}{\sqrt{\{r - (r - 2\eta^2)\}^2 + 4\eta^2(1 - 2\eta^2)}} \\ &= \frac{1}{\sqrt{4\eta^4 + 4\eta^2 - 8\eta^4}} \\ &= \frac{1}{\sqrt{4\eta^2 - 4\eta^4}} = \frac{1}{2\eta\sqrt{1 - \eta^2}} \\ (\text{DMF})_{\max} &= \frac{1}{2\eta\sqrt{1 - \eta^2}} \end{aligned}$$

A small logo for NIPTTEL is visible in the bottom left corner of the whiteboard.

So, DMF max is now 1 minus 1 minus 2 eta square whole square plus 4 eta square 1 minus 2 eta square. This, this gets cancelled, this becomes plus 2 eta square. So, that will give me 4 eta to the power 4 plus 4 eta square minus 8 eta to the power 4 that is it. 1 by root over 4 eta square minus 4 eta to the power 4 which is equals to I can take out 2 eta root over 1 minus eta square. Therefore, the expression for maximum dynamic magnification factor is 1 by 2 eta root over 1 minus eta square. This is another important expression which we use for our engineering solutions.

So, it is independent of the value of r because we know at which point of r it will occur. So, that point is r equals to root over 1 minus 2 eta square, and the maximum value of that depends on what is our value of eta. So, the maximum value of dynamic magnification factor for an undamped case, that is when eta is 0, it is infinity. So, that is get cross checked from this expression also, but when eta is having some value where the maximum occurs, let us look back again this expression of r where the maximum occurs.

Where the maximum occurs for undamped case that is eta equals to 0 at r equals to 1 which we have seen in our plot. That is, it becomes infinity at r equals to 1, but when eta is non-zero it is having some value, it is not at r equals to 1, but at slightly lower value of r, that is why in the curve when I draw, I draw the curve maximum point little ahead of r equals to 1 the maximum occurs for non-zero values of eta and as the value of eta increasing the maximum DMF point will keep on shifting away from r equals 1 towards r

equals to 0. It will shift slowly from  $r$  equals to 1 to  $r$  equals to 0. So, let me put the graph again here and compare with respect to this expression.

Now, it will be easily understandable for us at  $\eta$  equals to 0 maximum occurs at  $r$  equals to 1 with increase in value of the  $\eta$ , this maximum point shifts towards  $r$  equals to 0 little away from 1 just before 1, but this point keep on shifting. So, remember this profile or projectile or whatever you say this curve, it follows another curve the peak of the each point with different values of  $\eta$  and what is that magnitude of maximum value for different  $\eta$ ? That is what we have obtained just now. This is  $\frac{1}{2\eta} \sqrt{1 - \eta^2}$  that is the maximum value for different values of  $\eta$ . So, this is the discussion on dynamic magnification factor. Let us look at the slide now, here again the frequency ratio is defined as the externally applied frequency to the natural frequency of the system.

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**SOIL DYNAMICS**

**DMF (Dynamic Magnification Factor)**

$$DMF = \frac{u_p(t)}{(u_{st})_0} = \frac{1}{\sqrt{[1-r^2]^2 + [2\xi r]^2}} \sin(\omega t - \phi)$$

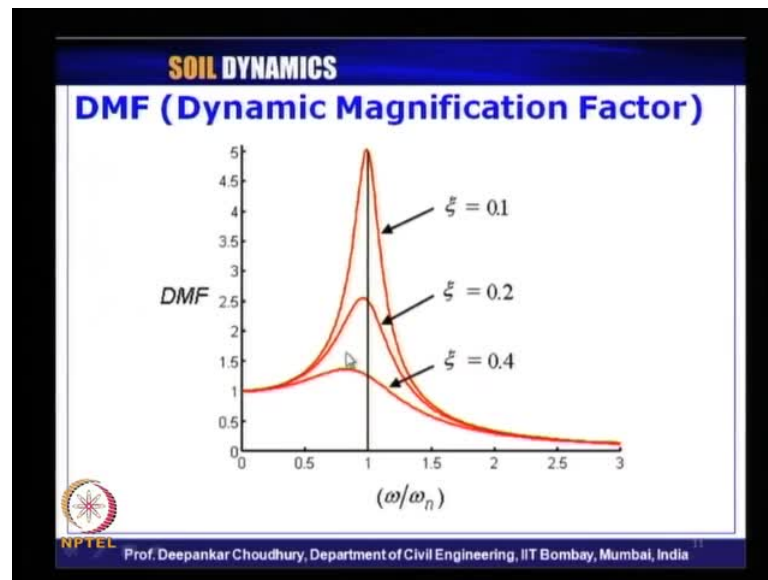
DMF can be plotted as function of the ratio  $\omega / \omega_n$  for different values of the damping coefficient  $\xi$ .

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And DMF is given by this expression.



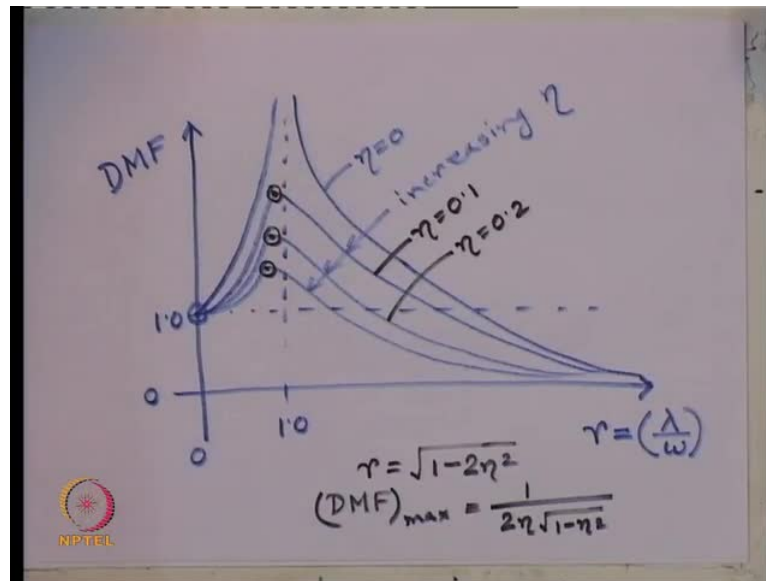
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And the plot is like this. So, peak is somewhere here little before  $r$  equals to 1 then with increase in the damping ratio it keep on shifting slowly towards left side. Now, the peak is here, now the peak is here. So, with this distribution of dynamic magnification factor we will stop here and we will continue further with the discussion of dynamic magnification factor more elaborately from the next lecture.

A small recap what we have done, the dynamic magnification factor how it varies with respect to frequency ratio that is the ratio of the applied frequency externally to the natural frequency of the system. How it varies for different damping ratio and where the peak occurs and what is the maximum dynamic magnification factor, all this things we have seen. So, continuing with this background let us further discuss on this dynamic magnification factor.

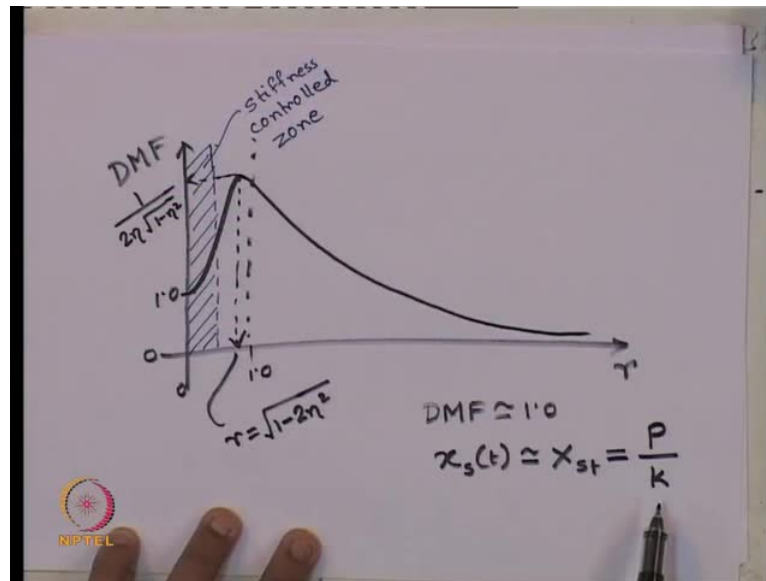
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Let me draw it once again to understand the physical significance of different parameters involved in the dynamic magnification factor because that will finally, help us in the design process to decide on which parameter we have to control or which parameter we have to properly consider for a good design of any system which is subjected to this kind of harmonic excitation or forced vibration with harmonic excitation. So, this is our x axis frequency ratio and the y axis dynamic magnification factor.

We have seen that it starts from always dynamic magnification factor 1 and it goes to infinity and here also it comes infinity to goes to 0 for the undamped case eta equals to 0, but for damped conditions the variations are like this with increasing eta. So, that is what we had seen in the previous lecture. So, suppose as I say eta 0.1 say eta 0.2 like that and the maximum occurs just before r equals to 1 and the value of r where DMF is maximum is r equals to root over 1 minus 2 eta square for DMF max and that value of the DMF max is 1 by 2 eta root over 1 minus eta square. That is the maximum value of dynamic magnification factor. Now, with this if I want to plot just for any damped case, for any damped case the DMF with respect to r.

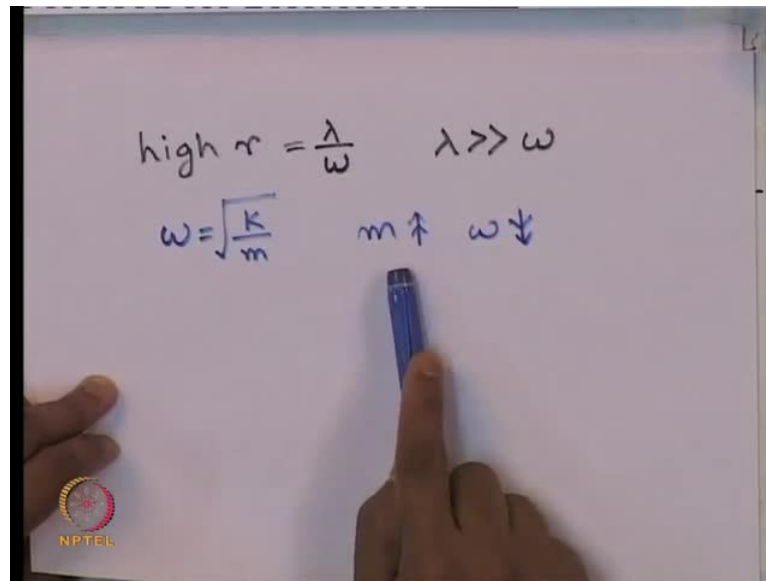
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Simply, I can plot it like it starts from 1, it attains a peak here, then comes back, then approaches 0 as  $r$  tends to infinity. So, this value as I have mentioned is  $r = \frac{1}{\sqrt{1-2\eta^2}}$  and this maximum value of DMF is  $\frac{1}{2\eta\sqrt{1-\eta^2}}$ . Now, let us see the different zone in this graph. DMF equals to 1 or very close to 1, what does it mean? That means the steady state displacement is almost equal to our static displacement. Dynamic magnification 1 means steady state displacement is equal to the static displacement and very close to 1 means they are equally almost equal.

Now, how much is our static displacement; that is nothing but the amplitude of the load  $P$  by the stiffness  $K$ . So, what does it mean? This value of DMF very close to 1 is dominated or affected by the stiffness. So, if the DMF is affected or dominated by stiffness what can we do? We can mark suppose a particular zone very close to DMF equals to 1. So, this zone we can shade and we can say this is a stiffness controlled zone or stiffness dominated zone because in this zone if you change the value of the stiffness of the system you will have a direct effect on the DMF. Because the DMF in this region is directly related to the stiffness parameter, hence, it is called the stiffness controlled zone. Now, let us see what about the other two zones or other two parameters rather than. Now, let us look at the condition when the frequency ratio is very high.

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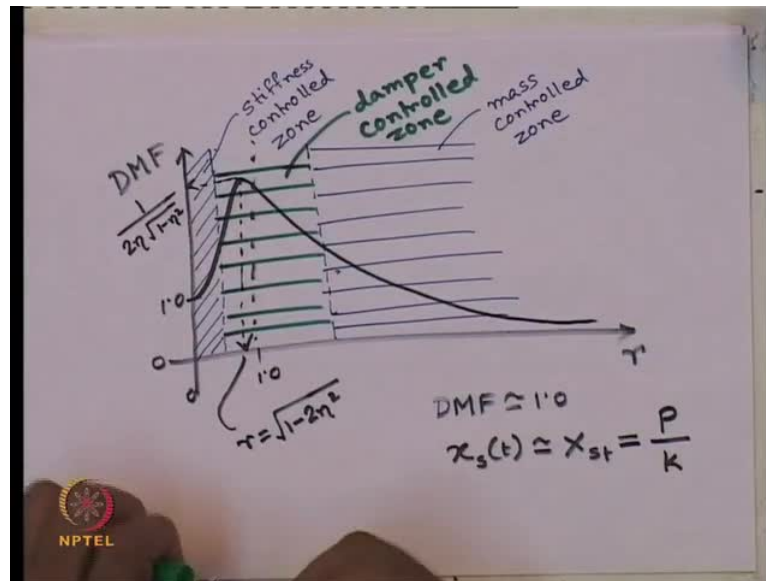


So, for very high value of  $r$ ,  $r$  means  $\lambda$  by  $\omega$ . What does it mean? That means  $\lambda$  is very high compared to the natural frequency, that is applied external frequency of the system is very high compared to the natural frequency of the system which will give us a very high value of  $r$ . Now, how it can happen? We hardly have any control on the applied external frequency. As a civil engineering designer or a civil engineer when we are going to design a machine foundation, we hardly have any control on this  $\lambda$  because it is given by the machine producer or the manufacturer of the machine.

So, they give us the input parameter that is at which frequency the machine will operate. So, it is given by them, it is just an input parameter for us, civil engineers or the machine foundation designers. What we can control is the natural frequency of the system. So,  $\lambda$  if we want to make that very, very high compared to the natural frequency of the system, what we should do? See, natural frequency again root over  $K$  by  $m$ . If we increase the mass if we increase the mass of the system, mass of the foundation very high automatically our natural frequency will decrease. That will give us finally, this condition.

So, we have now identified that the very high value of  $r$  means it can come from by increasing the mass of the system. So, that means this region of very high value of  $r$  is dominated by the mass of the system very much. Now, let us go back to our previous picture here.

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So, what we can do? We can now find out another location. So, say this zone where value of  $r$  is very high this can be classified as mass controlled zone. So, if you want to keep your  $r$  value for the design very high you have to control on mass. If you want your  $r$  value to be very low you have to control on the stiffness of the system and what about this central area? Obviously, the third remaining parameter in the system what we have the damper. So, this region, the central region is known as damper controlled zone or damper dominated zone which is obvious from the previous variation.

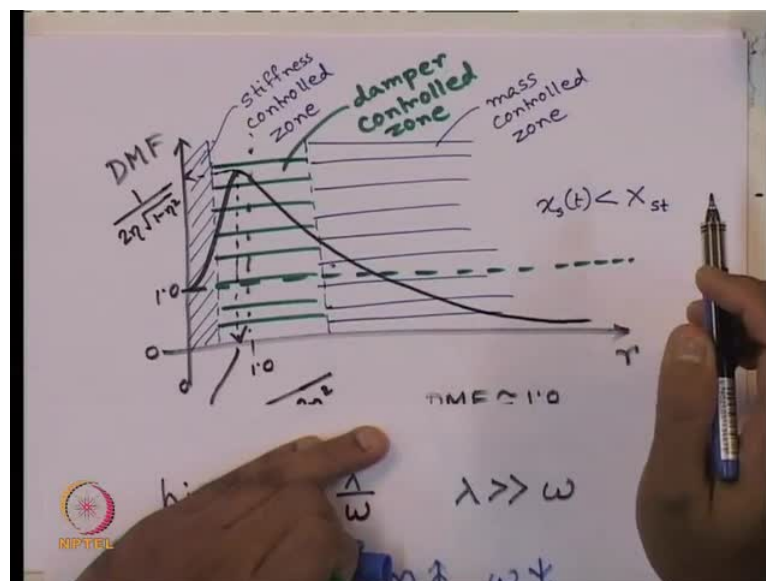
Now, let me place this figure here. See; if the value of the damping ratio changes this region the peak value keep on changing. So what does it mean? It is this region value of DMF is directly related or controlled or influenced more by the damper. So, what does it mean in the design of your foundation system suppose which is subjected to a harmonic excitation like this, whether you have DMF very close to 1 or whether you have  $r$  value very less than 1 or very high value, compared to that you can control by changing either the stiffness or the mass and if you control over the damper you will get the change in the peak value of the dynamic magnification factor.

That is why DMF max is a function of your damping ratio only. So, these are the basic three regions which are called stiffness controlled zone, damper controlled zone and mass controlled zone in the variation of DMF versus  $r$ . Later on we will see when we will take up the actual design of machine foundation, how by controlling this three we

can get a better response. Now, coming back to mass controlled zone let me give you the physical significance of the mass controlled zone. Let me keep this picture as well and this expression. So, for mass controlled zone as I said for high value of  $r$ ,  $\lambda$  should be very much higher than the  $\omega$  natural frequency and for getting very low value of  $\omega$  we have to increase the mass. So, what does it mean physically? Why it happens?

Very high value of mass means the system is pretty heavy. It is a heavy weight system which we are considering. Now, for this heavy weight system we have applied a very high value of external frequency. So,  $\lambda$  is pretty high, but the system is so heavy and so rigid though the frequency of excitation externally applied is very high, it is not able to respond properly. So, in other words the dynamic magnification factor for very high value of  $r$ , it will be even less than 1.

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Let us look back to this graph. Let me draw this line at DMF equals to 1. Look at here. So, at very high value of  $r$  DMF is less than 1 and DMF less than 1, what does it mean to us? DMF less than 1 means our steady state displacement is less than the static displacement. So, your dynamic displacement what you are getting is lesser than the static displacement. So, in other words what does it mean? As a designer if you have designed your system or the foundation to take care of static load only, it is sufficient enough to withstand the dynamic load. Correct?

Because the displacement it is going to experience maximum is the static displacement because its dynamic steady state displacement is much less than that, so for that kind of foundation if we have designed it properly to take care of static load that is good enough for taking care of the dynamic load. So, we need not to design it specially for the dynamic load. So, that is one added advantage we are getting that is without having a proper knowledge of dynamics even if we design it very well under any static load, it should be sufficient enough to take care of the dynamic load because of this reason of very high value of mass. Now, physically let me make you understand in a more better way, what it happens.

Externally applied frequencies pretty high and system is very rigid. So, why the dynamic displacement is less than the static displacement? What is the physical significance? Means you are applying a frequency, very high frequency to a rigid system. Its mass is so heavy that it is not getting enough time to respond for it. So, its dynamic displacement is less. Let me give you a kind of funny example actually. Suppose, if you slap somebody from this side and then wait, slap, it will go like this, then slap it from the other side it will go like this, but if you increase the frequency too much and the system is rigid, what will happen? You slap it continuously like this the system will not be able to identify that which direction it should move.

So, it will remain almost like a static. So, that is the meaning why the dynamic displacement for that case is so low or even lesser than the static displacement. This is the main reason that frequency externally applied is very high, mass is pretty rigid or very high. So, the system is not able to mean give a proper response to the subjected excitation. It is not getting a chance to respond to it. So, it is occurring so fast it is not able to give proper response. So, that is the good example or good way I thought that people will understand that.

If I apply a large frequency to a less mass system it can respond, but to a very high mass system it cannot respond that fast. So, that is why its dynamic displacement is pretty low. We will continue our lecture in the next class.