Soil Dynamics
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Module - 2
Vibration Theory
Lecture - 7
Decay of Motion

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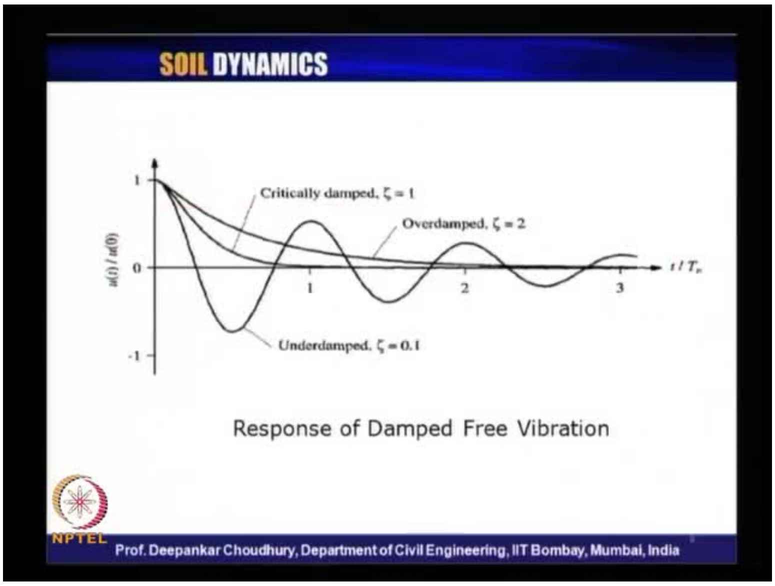

Let us start today's lecture on soil dynamics. We are continuing our lecture, so let us look at the slides. In module 2 vibration theory we are continuing with another sub topic, which is known as logarithmic decrement or decay of motion.

So, what is decay of motion? It is the case only for the under damped free vibration. In the case of under damped free vibration the response of the system, that is the displacement profile with respect to time we have seen it like this. A harmonic function with decaying magnitude of the these amplitudes that keep on decaying with respect to time or with cycles, because of the presences of the exponential decay function in the solution.

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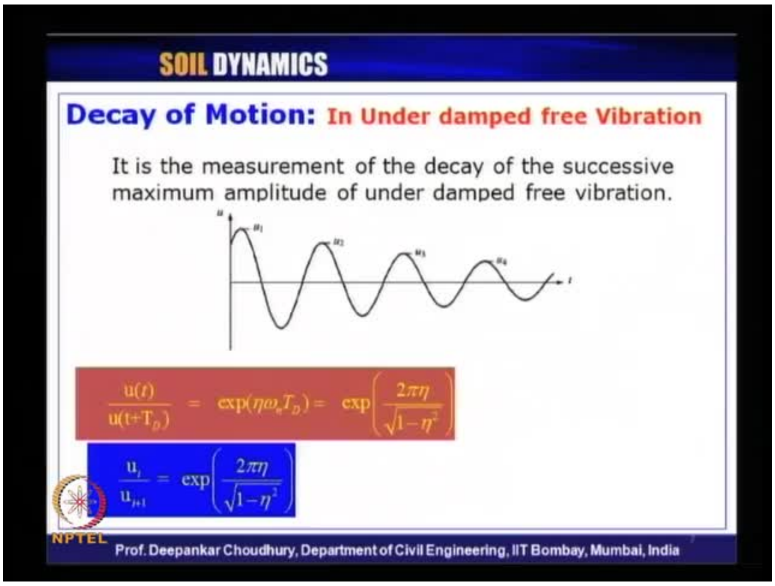

So, decay of motion, it is the measurement of the decay of the successive maximum amplitude of under damped free vibration. So, what does it mean? If u 1 is the maximum amplitude here, and u 2 is the maximum amplitude in the second cycle, u 3 is the maximum amplitude at the third cycle, u 4 is the maximum amplitude at the fourth cycle then the decay or the logarithmic decrement.

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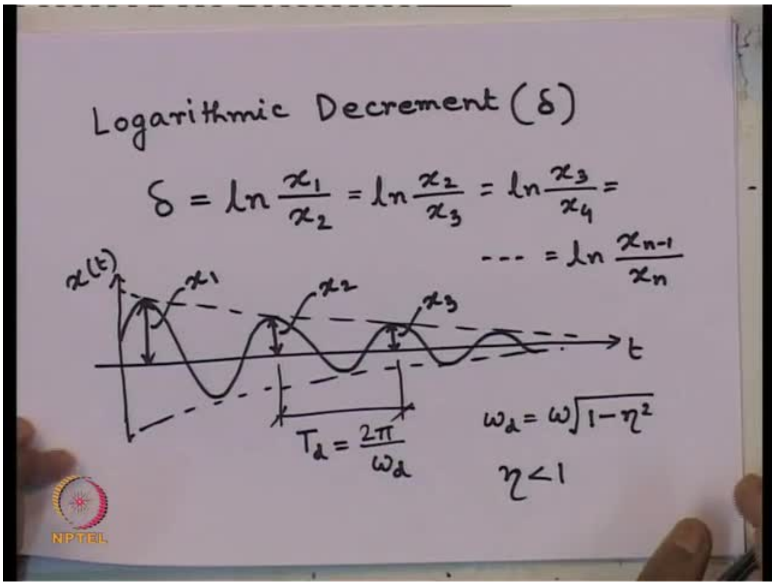

So let me write it down here, it is called logarithmic decrement. It is denoted generally by the term delta, Greek letter delta. So, how it is defined? Delta is defined as natural log l n x 1 by x 2, when we have the response of x of t with decay function like this envelope of the decay function. So, this is say x 1, this one is x 2, this one x 3. So, as the definition says, the decay of motion or the logarithmic decrement is the measure of the decay of successive maximum amplitude successive, means one after another. So, it can be natural log of x 1 by x 2 or natural log of x 2 by x 3 or natural log of x 3 by x 4, it can be written as natural log of n minus one th to nth, in the cycle. That is how the logarithmic decrement is expressed as. And how much is this time period? That is nothing but damped period.

Under damped period means damped period as I have already mentioned in the previous lecture this is 2 pi by omega d, where we have this omega d equals to omega root over 1 minus eta square because this occurs only for eta less than one case, that is the under damped condition. Now, if we want to simplify and find out what is the expression for delta in terms of other parameters. So, let us do that.

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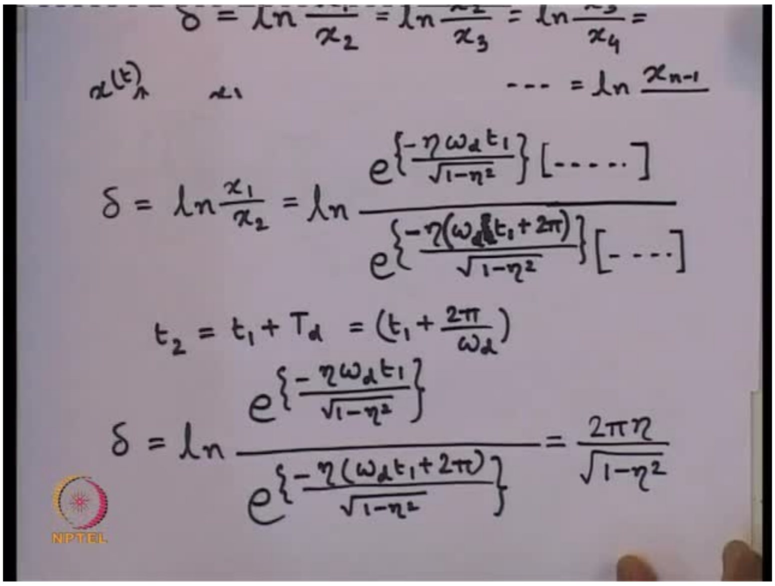

Delta is given as l n say x 1 by x 2, as I have mentioned, let me put the picture once again here. Suppose, this time where x 1 occurs is t 1, where x 2 occurs let us say the time is small t 2 where x 3 occurs this time is small t 3 in the time scale. So, delta will be l n. What is the value of our x 1? That is e exponential raised minus eta omega t, so in case of t we can put t 1. Times a big expression in terms of the initial displacement and initial velocity, which is a harmonic function. For x 2 the expression will be, e to the power minus eta omega e 2 times a big expression containing initial displacement and initial velocity with a harmonic function in terms of cosine and sin functions.

What does it mean? We can rewrite it in the simple form like this e to the power minus eta omega d t 1 by root over 1 minus eta square divided by, it was x 1 was e to the power minus eta omega t. t is 1 for x here and omega is nothing but we know this expression omega is nothing but omega d by root over eta over 1 minus eta square. So, just I have put this expression here then a function containing that harmonic function in terms of initial displacement and initial velocity. For x 2 it will be e to the power minus eta omega d t 1 plus 2 pi. So, instead of by root over 1 minus eta square times that harmonic function. Look at here t 2 will be nothing but t 1 plus t d t 1 plus t d, so 2 pi by omega d. So, it should be the bracket will not be here the bracket should be here. Now am I correct?

Minus eta omega d times t 1 plus 2 pi by omega, so it becomes omega d times t 1 plus 2 pi, correct? If it is not yet clear let me put it here t 2 is nothing but t 1 plus t d. Look at here t 2 is nothing but t 1 plus t d that means t 1 plus 2 pi by omega d. So, in case of x 2 it will be eta omega times t 2, in case of t 2 I put t 1 plus 2 pi by omega d multiplied with omega d. So, it becomes omega d times t 1, so this is not there mega d times t 1 plus 2 pi. Let me write it again more clearly, so delta l n e raise to minus eta omega d t 1 by root over 1 minus eta square by e to the power minus eta omega t d t 1 plus 2 pi by root over 1 minus eta square. Why I have not written this harmonic part because harmonic part they repeats after cycle. So, they gets nullified in both numerator and denominator, they cancels each other.

So, this will be our final expression, now from this on simplification what we can write? This will give us e raise to minus eta 2 pie by root over 1 minus eta square, will come out in the denominator with a minus sin. So, that means e raise to eta times 2 pi by root over 1 minus eta square, what does it mean? It is natural log and this is e. So finally, it will give us 2 pi eta by root over 1 minus eta square. So, this exponential and this exponential gets cancelled, from here I have separated it out. Exponential times a plus b. We can write e to the power a times, e to the power b so that is what I did. e to the power b is nothing but minus 2 pi eta by root over 1 minus eta square in the denominator which in the numerator.

If I take it the minus sign will vanish and this is exponential and this is natural logarithm. So of course, the coefficient will be the final result, so that is the expression for the logarithmic decrement and this expression is used in the laboratory to find out the damping ratio of any system subjected to free vibration, that is in this case the under damped system which is subjected to free vibration. How to obtain the damping ratio?

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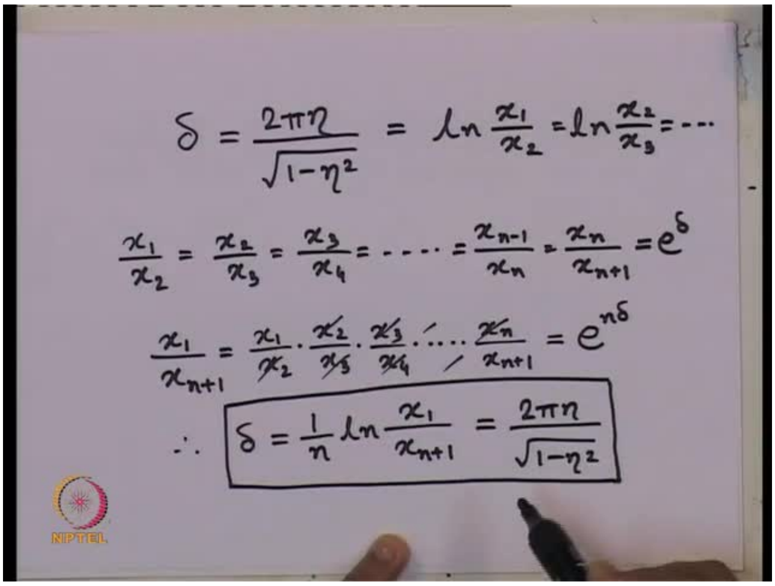

This is the way we compute 2 pi eta by root over 1 minus eta square, which is equals to l n x 1 by x 2 or l n x 2 by x 3 and so on. So, what we can do in the laboratory? For any system, single degree of freedom system like ours, one mass, one damper and one spring subjected to a single degree of freedom, if we let it vibrate under a free vibration condition and if the system is under damped, that is if damping ratio is less than one for that system when it is vibrating, we can measure it is peak displacement or peak amplitude or the maximum the amplitude at two successive cycles. That is suppose the first cycle and in the second cycle, so we can measure easily x 1 and x 2.

Put the values in this expression and this side only parameter unknown is eta. So, we can solve for eta and get the, what is the damping ratio of the system. So, this is the conventional way to find out the damping ratio of any system in the laboratory, using free vibration technique. And it can be any other two successive displacement also, second cycle, third cycle it can be third cycle, forth cycle. But now the problem can arise in this case is like when the system is undergoing free vibration, it may be difficult for us when we are noting down the displacement at every successive cycle, while recording the reading because of manual error and things like that we may miss in between some cycles.

So, if we try to take successive displacement of or amplitude in successive cycles, that is first or second or second or third, it may be possible that we may miss some of the cycle. So, rather than taking that strain, what we can do? We can measure suppose first cycle what is the displacement and say n th cycle what is the displacement, that is after n number of cycles, how much the displacement occurs? Then chance of recording that maximum displacement that error will get minimized. So, if we use that technique whether still we can use the expression to obtain the damping ratio of the system? If yes, how? Let us see. So, now we have written the ratio like this we are measuring this successive displacement or amplitude x 1 by x 2, x 2 by x 3, x 3 by x 4 in that way we can write x n minus 1 x n, x n by x n plus 1, which can be written as equals to how much? e to the power delta.

So, all these ratios if we write down from the definition of the logarithmic decrement, we can write this is all are equals to e to the power delta. So, if we now want to measure this first amplitude and n plus one th amplitude, what we will get? x 1 by x n plus 1 that should be equals to x 1 by x 2 times x 2 by x 3 times x 3 by x 4, x n by x n plus 1. So, this this gets cancelled, this this gets cancelled, here also it gets cancelled here also it get cancelled. Finally, it remains like this, what does it mean? From 1 to n, I have taken n number of cycles, so e to the power delta repeating for n times. Therefore, therefore if we simplify this what we are getting? Delta is given by 1 by n l n x 1 by x n plus 1 equals to 2 pi eta by root over 1 minus eta square.

So, this is most important thing which we have to use, to compute the damping ratio of any system. So, in the laboratory you measured say first cycle, displacement or amplitude and say n plus one th cycle or displacement and amplitude after n cycle, using this expression you know your left hand side completely on the right hand side only unknown is eta, you can find out easily the value of eta. So, with this knowledge of logarithmic decrement, it is having a very good practical use. Let us move to the slides here. Here what I have derived just now is given here in the form of expressions in terms of u the same thing.

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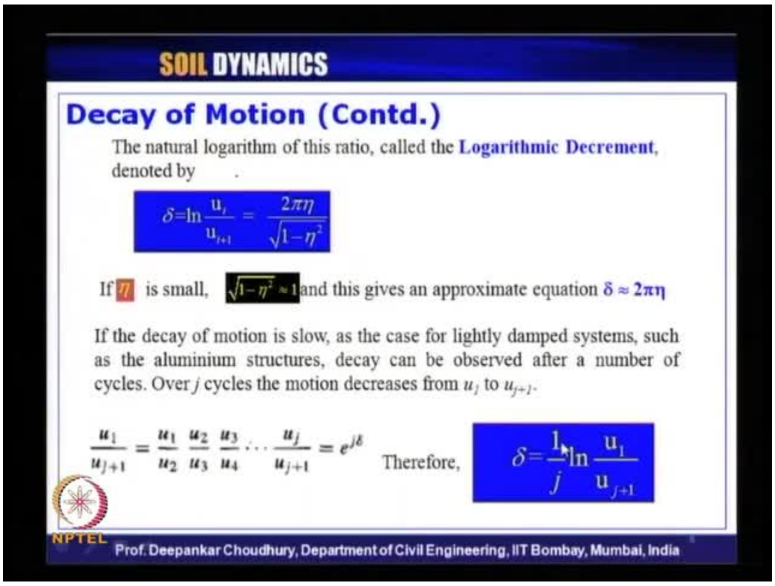

And logarithmic decrement delta is 1 by after how many cycles and the displacement in the first cycle and the j plus oneth cycle. So, after jth cycle the motion has been decreased, so this is due to the decay of motion this logarithmic decrement has occurred, which is given by this expression 2 pi eta by root over 1 minus eta square. And if the value of eta is very small, suppose for aluminum structures like we have discussed that has a typical range of about 0.1 percent to 1 percent the damping ratio. For those type of problems this will be almost equals to 1, so we can approximate also that the value of delta is nothing but approximately equal to 2 pi eta when eta is pretty small.

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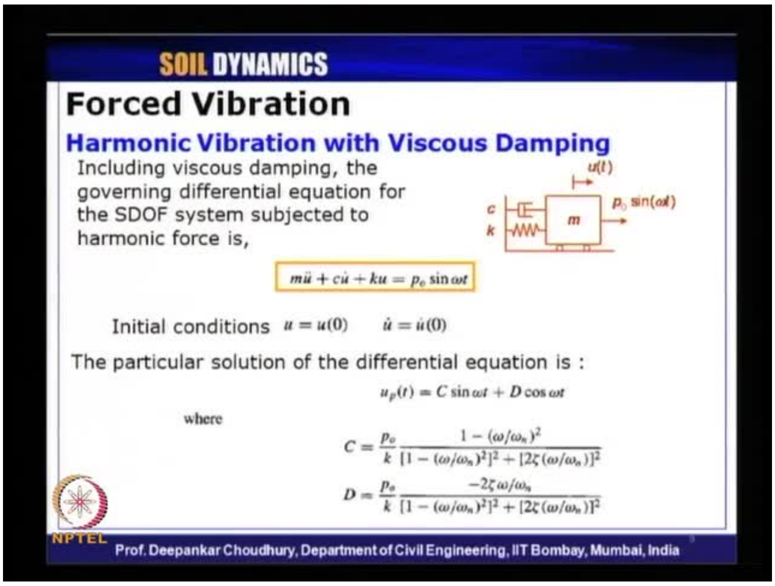

Let me come to the next topic of our discussion, forced vibration. So, in the free vibration we have solved both the cases of undamped free vibration as well as damped free vibration and within damped free vibration we got, three cases; over damped free vibration, critically damped free vibration and under damped free vibration Now, we are moving to the another major type of vibration which is forced vibration. So, in forced vibration what will be the basic model for the single degree of freedom system? We have the mass, we have the damper, we have the spring and the single degree of freedom is u of t and the dynamic load; that is the load which is varying with respect to time is applied to it.

So, this is very much present, this externally applied, externally applied dynamic load is present. So, the governing equation of motion becomes like if we say u as the degree of freedom then m u double dot plus c u dot plus k u equals to that externally applied load. So, let me formulate the solution for this problem.

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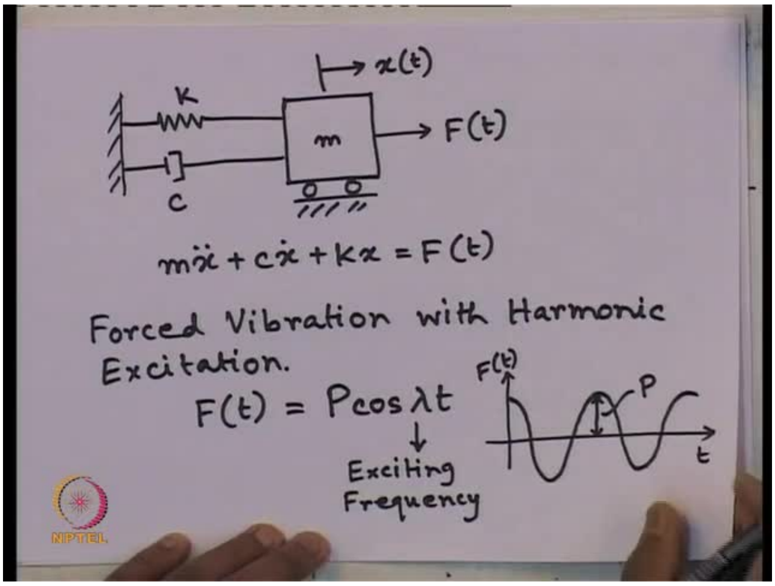

Let us look at the picture, let me draw it once again for you. The single degree of freedom system I am drawing, mass spring damper. I have degrees of freedom say x of t, what we were using earlier also and there is one dynamic load applied to it f of t. So, we have equation of motion now m x double dot plus c x dot plus k x equals to this externally applied dynamic load F of t. Now, the first case, let us do the solution for forced vibration with harmonic excitation. So, this case of forced vibration with harmonic excitation, that means this externally applied dynamic load is harmonic in nature. Let us solve for that case first, then we will see the all other different cases.

Let us take F of t as P cosine lambda t, this is nothing but a harmonic load and dynamic load function of t. So, if we want to draw the profile of F of t with respect to t, it should look like this. So, that is the profile of the externally applied dynamic load with this amplitude is P. So, this dynamic load is having amplitude P and what is lambda? Lambda is called exciting frequency; exciting frequency that is the frequency with which the externally applied load or external excitation has been given or has been applied to the system. So, with this now the governing equation of motion for this particular case of forced vibration subjected to harmonic excitation.

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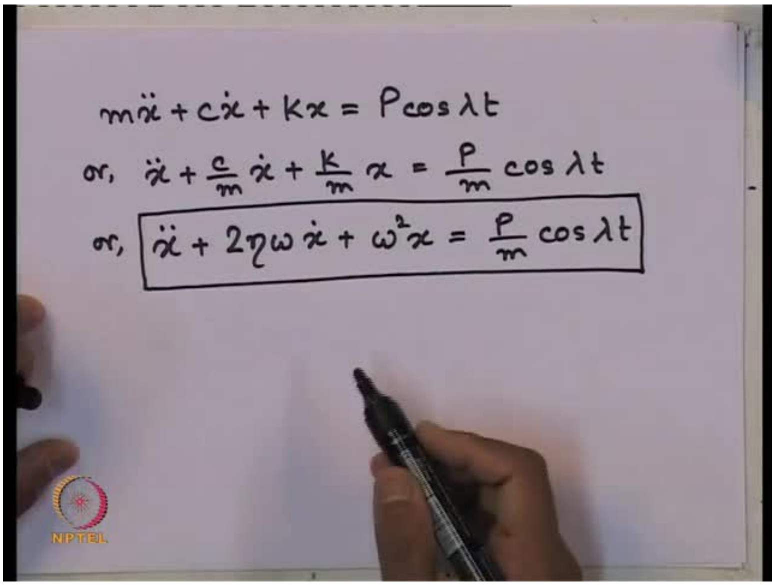

We can write like this, m x double dot plus c x dot plus k x equals to P cosine lambda t or x double dot plus c by m x dot plus k by m x equals to P by m cosine lambda t. I divided both the sides with m or x double dot plus c by m, we can write down as 2 eta omega. Earlier we had seen and x dot and k by m is nothing but our omega square, where omega is the natural frequency of the system omega square x equals to P by m cosine of lambda t. Now, this differential equations second order differential equation, we need to solve to get the solution of what is the response of x of t, varying with the respect to time.

Just a small recap, what we had learnt. The decay of motion for the case of under damped free vibration, which is also known as logarithmic decrement. The measure of the decay of successive maximum amplitude in the case of under damped free vibration expressed in natural log is the measurement of the logarithmic decrement and that logarithmic decrement is denoted by this parameter delta, which is expressed as on simplification 2 pi eta by root over 1 minus eta square.

So, eta is a function of only the damping ratio of the system and in the laboratory we can use this method to determine the damping ratio of any system by measuring the maximum amplitude of say first cycle. And then after jth cycle, j plus one th cycle. If we measure, then the delta can be expressed as 1 by that number of cycles after which we are measuring the maximum amplitude and natural log of that u 1 by u j plus 1, this we can equate with respect to this expression and this parameter is known to us we have measured this values and number of cycles, so only unknown eta we can compute. Then we had started with our next topic, next sub topic on forced vibration and in the case of forced vibration, that is the basic single degree of freedom model that is with mass spring dash pot with single degree of freedom u of t and externally applied dynamic load is acting on the system.

Now, the governing equation of motion for a linear model is expressed by m u double dot plus c u dot plus k u equals to that applied dynamic load. Now, we will continue with harmonic excitation, that is if in the case of forced vibration the applied dynamic load is harmonic in nature. We tried to see, now let us look at this sheet here. So, this is our basic single degree of freedom model with linear model, this is the basic equation. We are considering the case of forced vibration with harmonic excitation. So, F of t varies with respect to t harmonically and the function we have assumed as P cosine of lambda t where P is called the amplitude of the externally applied load and lambda is nothing but the exciting frequency.

So, with this the basic governing equation of motion we had simplified in this form and that is our final second order differential equation, which we now need to solve for x of t to get the response of the system with respect to time. So, let us do that. Now, for this type of solutions, say second order differential equation, what we know the x of t the complete solution, will compose of two parts. What are those?

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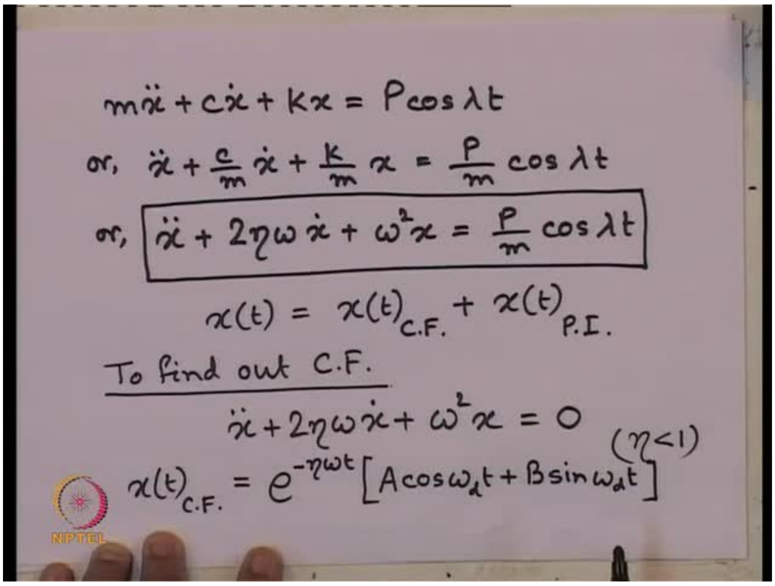

x of t complementary function plus x of t particular integral because if we have a second order differential equation with on the right hand side some parameter like this, then the total solution can be represented by two parts, one is called complementary function and the other is called particular integral, and how to find out these two different parts? Let us see, to find out complementary function what we need to do? This governing differential equation we have to equate it with respect to 0; that is we have to write down the equation in the form of this equals to 0 and then if we solve this equation we will get the solution which is nothing but x t complementary function, right?

So, if it so already we have solved this equation in the case of under damped free vibration. That is the third case as I have mentioned, so let us considered our system is under damped eta is less than 1, I am considering. So, when we are writing down the solution x of t the complementary function is known to us. What was the solution? Solution was e to the power minus eta omega t times A cosine omega d t plus B sin omega d t. That was the solution. One point I want to highlight here in this solution, when we are writing the constants A and B, we need to put here as A and B only, not that x not c cos omega d t plus that function of x not and x not dot of sin omega d t.

Why because the initial conditions if we used in the case under damped free vibration, that is the complete solution, but in this case also we have initial condition, but we cannot use that initial condition at this moment. Why? Because it is not the complete solution, it is only a part of the solution. So, there is another part attached to the complete solution, so the determination of the constants should not be done at this moment, but it has to be done at the end of getting the complete solution. So, remember this point it is very important, do not write the solution of final solution of under damped free vibration, but write the basic solution with the unknown constants A and B like this. So, now let us find out the second part and then we will come to this constant determination of the constants.

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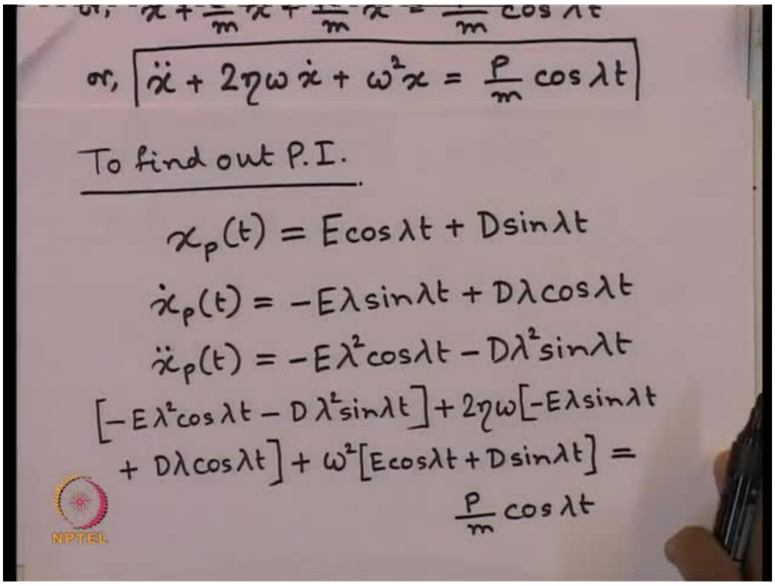

So, to obtain the second part that is to find out particular integral, what we can do? Let me write the particular integral x P of t, let us assume the particular integral as a solution which will fit to the response. So, this response what we already got, we can assume the particular integral in the form of some harmonic function like this because our applied load is also, let us look here our applied external load is also harmonic in nature. So, the solution of particular integral that depends on this parameter on the right hand side of the second order differential equation whatever is there based on that we make a suitable selection of the particular integral.

So, let us assume that form of the solution of the particular integral is a harmonic function like this, e cosine of lambda t plus d sin of lambda t, it will be a function of this given parameter on the right hand side to determine the particular integral. So, I have assumed a harmonic function as a solution because of this reason as I mentioned with lambda t as the parameter, but combinations of cosine and sin with unknown constants E and D. So, these unknown constants now we need to determine, which will satisfy this basic differential equation. So, let us see how it will satisfy this basic differential equation, step by step.

If I differentiate this with respect to t, I will get the velocity, so that will be minus E lambda sin of lambda t plus d lambda cosine of lambda t, that is the velocity of the particular integral part. What is the acceleration? Again lets us differentiate the velocity with respect to time, we will get minus e lambda square cosine of lambda t minus D lambda square sin of lambda t. Now, this expressions of acceleration velocity and displacement must satisfy, the basic governing equation this one. So, what we need to do? We need to put these expressions of acceleration, expression of velocity and expression of chosen displacement in this equation and equate it with respect to the given right hand side parameter, that is the way we obtain the particular integral. So, let us put these three expressions in this equation.

Let us see how it looks like? So, it will be minus E lambda square cosine of lambda t minus D lambda square sin of lambda t, that is our acceleration plus 2 eta omega times velocity minus e lambda sin of lambda t plus D lambda cosine of lambda t plus. The next term is omega square times x. x is E cosine of lambda t plus D sin of lambda t, which will be equated with respect to this given term that is P by m cosine of P by m cosine of lambda t. Now, when we are representing the equation like this, how to find out the unknown constants E and D from this expression? We know from the basic mathematics that if there is a relation like say A cos theta plus B sin theta equals to say C cos theta plus D sin theta, then the individual terms of cosine and sin the coefficients of them must be equal to hold this relation of equality.

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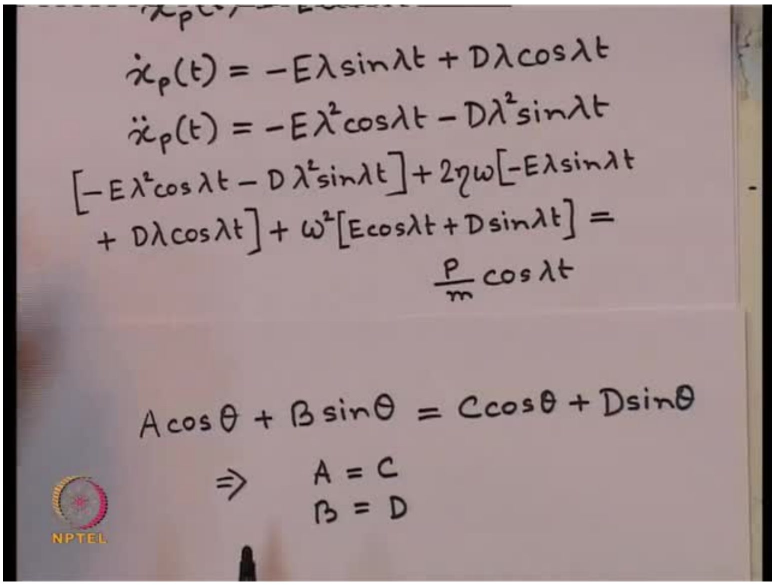

So, from this we know that A must be equal to C and B must be equals to D because this is the coefficient of cos theta, this is the coefficient of sin theta they are equating, so that is from our basic mathematics we know, this equating sin terms and cosine terms we get this relation. So, using this basic mathematics, what we can find out for our equation? Let us look at here, in all these components are combinations of cosine function and sin function. Now, let us equate all the cosine functions both the sides and equate all the sin functions on both the sides. On the right hand side we have only cosine functions and no sin functions, so what we can write from this expression? Let us note it down here.

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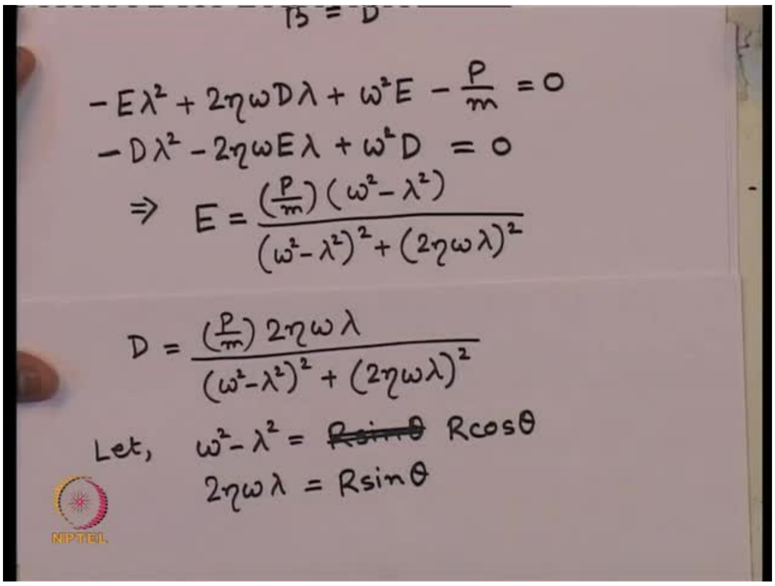

So, it will give us minus e lambda square, I am taking out the cosines terms first, minus so E lambda square then here we are getting 2 eta omega D lambda, so plus 2 eta omega D lambda and this is another cosine term plus omega square E and this the right hand side term is P by minus P by m equals to 0. So, this we got by equating the terms involved with cosine functions. Now, let us take the terms involved with sin functions what we get minus D lambda square then minus 2 eta omega E lambda and plus omega square D this is equal to 0 because on right hand side we do not have any sin functions.

Now, from these two equations, look now we got two equations and we have two unknown E and D, so easily we can find out the two unknown from the two equations. If we resolve these two equations, what we will get? The final expression I am giving you E comes out to be P by m omega square minus lambda square by omega square minus lambda square whole square plus 2 eta omega lambda whole square. And what will be the expression for D? The expression for D will be, D equals to P by m 2 eta omega lambda divided by omega square minus lambda square whole square 2 eta omega lambda whole square.

So, that is the solution for the constant E and constant D, so by knowing these two, we know what is the particular integral solution. Now, we know x P of t. Now, let us simplify this term little bit. So, let us assume that omega square minus lambda square is R sin theta and this term 2 eta omega lambda, I am sorry let us say omega square minus lambda square is R cosine theta, we are assuming omega square minus lambda square as R cosine theta and this term 2 eta omega lambda as R sin theta, so what we get?

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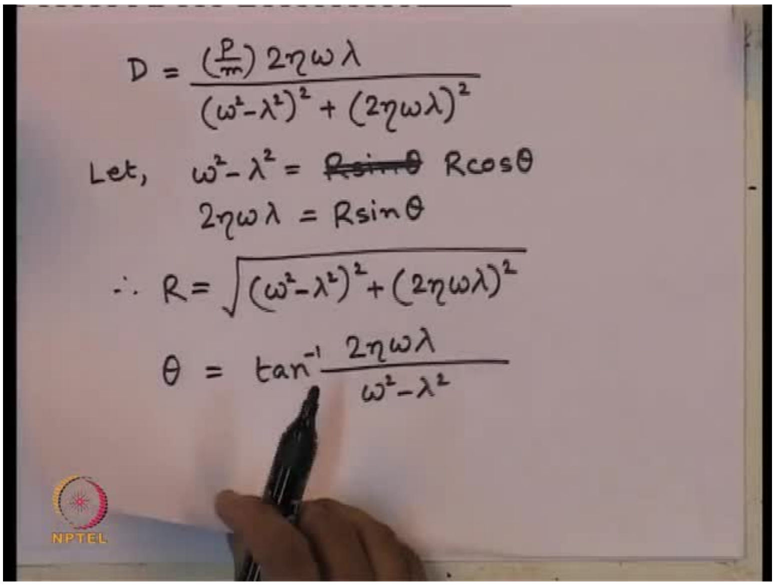

What is R? Therefore, R is root over omega square minus lambda square whole square plus 2 eta omega lambda whole square, that is our R. And theta is nothing but tan inverse this 2 eta omega lambda by omega square minus lambda square. That is how we have chosen these two parameters in terms of R and theta. So, if we simplify from the solution of x P of t, what we will get? Let me write down this first in terms of D and E parameter. So, our final solution is like this.

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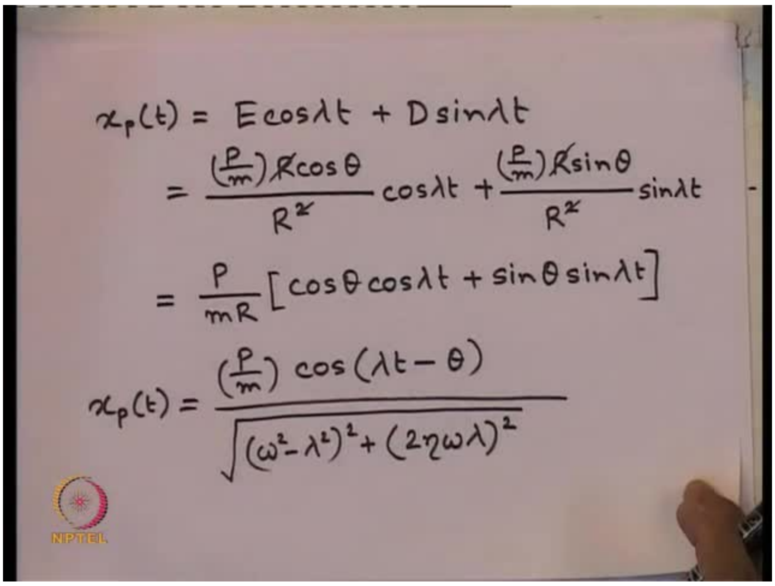

x P t, we had assumed E cosine of lambda t plus D sin of lambda t. Now, E we got the expression is P by m, let me put the expression of E, so it will be easy for us to follow. So, E was P by m omega square minus lambda square. What is that we have assumed? That is R cos theta, then in denominator we have this parameter which is nothing but our R square times cosine of lambda t. What is D? Let us take the expression of D, it will be easy for us to follow. So, from the expression of D, we can write it is P by m this 2 eta omega lambda we have considered as R sin theta and this denominator is nothing but R square.

Then, we have sin of lambda t, so what we get as R is not 0, we can cancel these. We are getting P by m R times cosine theta cosine of lambda t plus sin theta sin of lambda t, that is our x P t, from which we can write down now the solution of x P t, the particular integral part will be P by m. This we can write as cosine of lambda t minus theta divided by R and what is R? Root over omega square minus lambda square whole square plus 2 eta omega lambda whole square. So, this the solution for particular integral, and we got the solution for complimentary function also. Now, we have to add these two components to get the total solution. So, the total solution will be.

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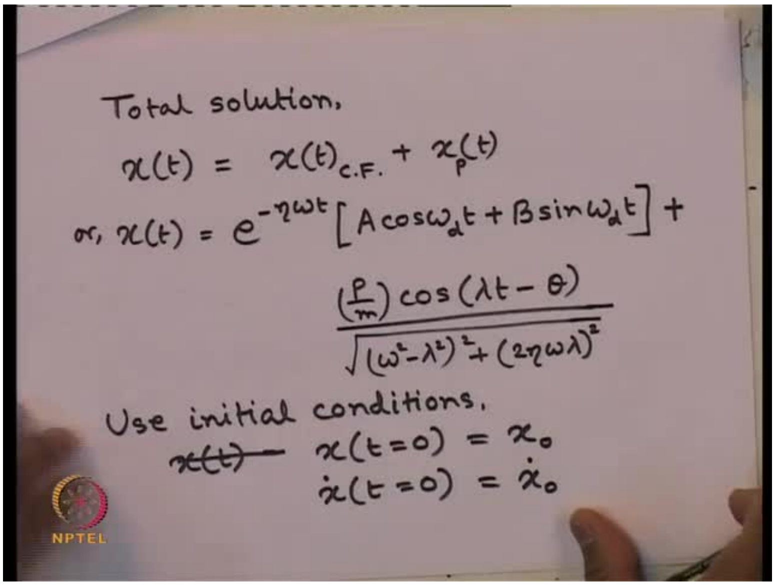

Let me write down, so the total solution or the complete solution x of t which is equals to x of t complimentary function plus x of t particular integral x P t. It can be written as E to the power minus eta omega t times A cosine omega D t dammed frequency plus B sin of omega D t, that is the solution of complementary function plus this particular integral part, that is P by m cosine of lambda t minus theta by root over omega square minus lambda square whole square plus 2 eta omega lambda whole square. So, this is our complete solution.

Now, we have to use initials conditions. So, this is the place where you should use initial conditions not before that to determine A and B. Now, we will be using x at t equals to sorry x at t equals to 0 is given to us as x 0 and the velocity x dot at t equals to 0 is given to us as x not dot. These two initial conditions now we will use for this expression to get that coefficients A and B. Let us do that, so before doing that what we need to do? Let us differentiate this x of t first with respect to t to write down the expression for that velocity.

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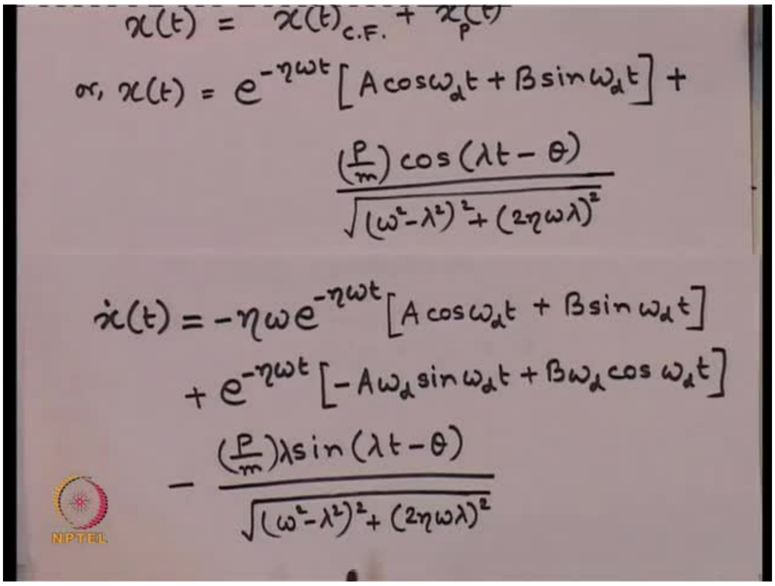

See if we differentiate this, so first time differentiating this component minus eta omega E to the power minus eta omega t times. This remains as it is A cosine omega D t plus B sin omega D t. Then I am differentiating this component, E to the power minus eta omega t times minus A omega D sin of omega D t plus B omega D cosine of omega D t. And then I need to differentiate this with respect to t, cosine to it will become sin, so minus P by m sin lambda t minus theta. Now, with respect to t there is a coefficient lambda, so I am bringing that lambda here, fine? And the dominator is independent of t so we can write down omega square minus lambda square whole square plus 2 eta omega lambda whole square. So, that is the expression for velocity and this the expression for the displacement, complete solution. Now, we are putting the initial conditions in these expressions.

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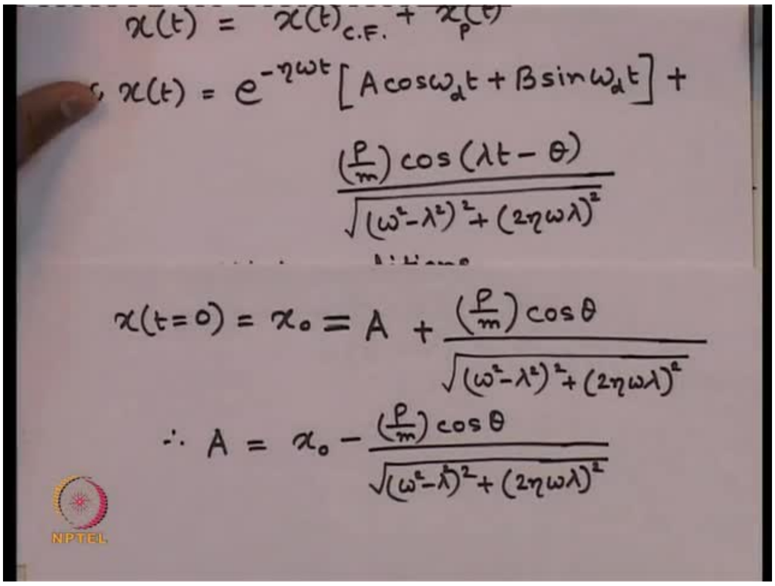

First let us put the displacement, that is x at t equals to 0 is x naught. If we put t equals to 0 in this expression for the displacement, what we will get? This becomes 1, this A it vanishes and here what remains, this term remains. So, let us see this will be equals to from the first term we are getting A and from the second one we are getting P by m, t we are putting 0, so cosine of minus theta is cos theta. So, cos theta by root over omega square minus lambda square whole square plus 2 eta omega lambda whole square, therefore the coefficient A is equals to x not minus P by m cos theta by root over omega square minus lambda square whole square plus 2 eta omega lambda whole square.

So, that is the value of A. See if we look back to our previous case of under dammed free vibration, the coefficient A was obtained as x naught only. So, here it is not the case, here it is having some other component, so that is why I told that for this case of forced vibration we need not to use the initial conditions at the beginning with complimentary function only, we should use it at the end of getting total solution, remember this. Now, to obtain the other coefficient B we need to use the expression for the velocity. Let us bring back the expression for velocity.

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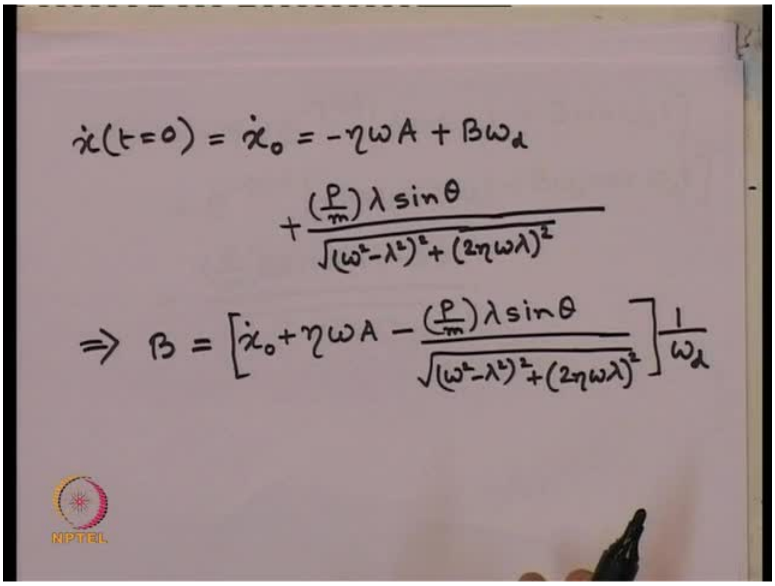

Now, I am using x dot at t equals to 0 is equals to x not dot, I am putting t equals to 0 in this expression. The first term will give me minus eta omega times A, this vanishes. The second term will give me plus, this becomes one this vanishes because of sin here cos, so it will be B omega D and from the third term what I will get? Minus P by m, lambda this becomes sin minus theta, sin minus theta is minus sin theta. So, let me take it out here so it will become plus. Sin theta divided by the denominator remains as it is omega square minus lambda square whole square plus 2 eta omega lambda whole square.

So, from which we can get now easily, let me close this. Therefore, b is nothing but how much? x not dot plus eta omega A minus P by m lambda sin theta by root over omega square minus lambda square whole square plus 2 eta omega lambda whole square, this by 1 by omega D. So, that is the expression for the other coefficient B. Here also you can find it out that B is not only like earlier under damped case, the function of x naught dot and x naught but also the particular integral is involved in the solution. So, now you know the A you can put it here you will get the complete solution. You can write it later on what is the complete solution. So, let us stop today here, we will continue our discussion on this in the next lecture.