Soil Dynamics
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Module - 2
Vibration Theory
Lecture - 6
Damped Free Vibrations, Definition of
Critical Damping and Problems

Let us start today's lecture on soil dynamics. So, we are continuing from previous lecture. Let us look at the slide. In module 2 vibration theory, we are continuing with what we have learnt till the previous class; a small recap of the previous lecture on the damped free vibration. This is the basic equation of motion and the solution has two roots like this, which can give us three different cases like over damped, critically damped and under damped depending on whether roots are real and distinct, roots are real and equal and roots are complex conjugate.

We have defined two new terminology. One is called critical damping, which is defined like this and another parameter, which is called damping ratio, is the ratio of the damping constant to the critical damping.

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Now, let me come to the definition of that critical damping once again how it is defined. The critical damping is the smallest value of damping, which is necessary to prevent the oscillation of the system when it is undergoing damped free vibration. So, for the case of damped free vibration, to prevent the oscillation, so if we do not want any oscillation to occur, the minimum value of the damping constant required to be provided is called the critical damping. For different materials, what are the typical ranges of the damping ratio? That is C by C c r value, which is commonly used in our engineering design. These are just typical ranges means, exact value can be obtained experimentally in the laboratory. I will tell you and describe you, how to obtain these values in the laboratory. So, these ranges generally lies between for concrete structures, the value ranging between 5 percent to 7 percent of the damping ratio generally. So, that is why the damping ratio value will be 0.05 to 0.07. Many times, you will see that we express the damping ratio in terms of percentage also. So, that is why I said 5 percent to 7 percent.

For steel structures, the typical ranges of damping ratio is 1 percent to 2 percent that is 0.01 to 0.02. For aluminium structures, the typical value of damping ratio ranging between 0.1 percent to 1 percent. So, the values are 0.001 to 0.01. Whereas for soil, for any soil structures, the range of the damping ratio generally lies between 10 percent to 20 percent. So, that is why 0.1 to 0.2. So, you can easily see that which material will behave as a good damper when there is a dynamic load applied to a system. So, of course, among these four listed categories of materials, soil will provide us as the best damping material.

So, suppose if there is an earthquake. Due to the earthquake, whatever vibrations are going to get generated and propagated through the media, because of the soil, we will get the values to be reduced because of having high damping ratio. But, it is not the case for concrete or steel. So, you can imagine the same earthquake, if it is travelling through a soil structure and a steel structure, the problem will be more for steel structure rather than a soil structure, because soil is having comparatively high damping ratio. So, it will dampen the value of the dynamic load as it travels through that media. So, it is better to have a high damping ratio when we are considering the design of a system for which the vibration is a problem and against that we want make the design. Here, another information I have written. You can see this. Most of the engineering structures are under damped. What does it mean? Look at all these ranges. The values of damping

ratios are much less than 1; that means, all these structures are under damped. That is, we have seen in case of damped free vibration, three cases of solutions, that is over damped, critically damped and under damped. Among these three, the last case, that is case 3, under damped, where damping ratio is less than 1 is the most common for our engineering structure. Then, why do we study the other two cases? Is there any engineering application or practical application of the other two cases? Like, do we really use over damper or critically damper? Critically damper, we can say that it is a transition between over damper to under damped case. So, it can define the border between the two. So, we have to see whether we use practically any over damper or not, because I made a statement that most of the engineering structures are under damped. That is, the case 3 with damping ratio less than 1. Let us see at this slide again.

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So, these are the three different cases of damp free vibration, which we have discussed in detailed just now. That is, over damped, critically damped and under damped. For case 1, over damped, damping ratio must be greater than 1, eta greater than 1 and the result of responses, there is no oscillations. Case 2 for critically damped; damping ratio must be equals to 1 and the result is again no oscillation, because this case, we have defined the critical damping as the condition when this is the minimum value of damping when no oscillation occurs. Right? There is no oscillation in the system and it is transition between the first case and third case. What is third case? When the damping ratio is less than 1 and there we have oscillations, but with a decay function. As I had mentioned just

now, why we study the other two cases if there is no application? But it is not so. There are few applications in engineering system for over damped case. The examples of some of the over damped system we use, like shock absorber. Like, when we want to isolate any system, which is going through a large vibration to another system adjacent to it using a shock absorber or vibration isolation, in those cases, we want a damper which will not allow the dynamic motion to oscillate, but to die it down with respect to time soon without any oscillation. That is the objective of using of over damper. So, that is why shock absorber are good examples of over damper.

Then the door damper; in the doors we use the damper. Why? We do not want when we open and close the door, we do not want the door should oscillate like this. So, when we are opening the door with damper, after sometime we want the door to be closed in one cycle itself. It is not good that door will oscillate like this and then close after sometime. So, that is another good engineering example or applications of over damper, where we use, so that, it closes without any oscillation. So, good door dampers use a over damper system.

Another good example in engineering practice for over damper system are the measuring instruments in laboratory. Whatever a sophisticated instruments we use, measuring instruments, like for example, galvanometer or we can say displacement transducer and things like that, what we want? When we are doing any test, during that time we want the reading. But, if the needle inside the instrument is oscillating too much, it will be difficult for us to get a proper reading with fine accuracy. So, if the instrument is very sophisticated and very well designed, it has to give the correct data without any oscillation. The needle is not supposed to oscillate there. For the better instrument design, we have to use over damper, so that, there is no oscillation. It is another good example of over damping system.

But apart from that, let me repeat it once again. Most of the other engineering structures, whatever material we use for our civil engineering industry or any other engineering industry, all other engineering materials are of under damped nature. That is, with damping ratio less than 1, even much less than 1.

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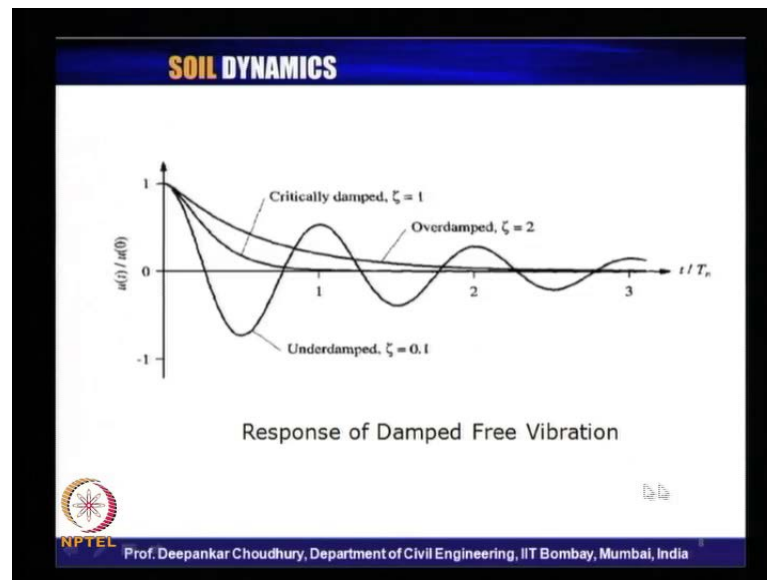
The response of all the three cases of damped free vibration is shown here. You can see. It is expressed here in different forms. The non-dimensional form, u of t, that is the x of t what I have found out the solution with respect to the initial displacement. So, that is why the curve starts at 0.1, because initially it will be 1. The ratio will be 1 and the x axis is represented by t time, for which we are plotting the response divided by t n, that is the natural period of vibration for the system.

So, it is just to represent the x and y axis in non-dimensional form. The response will remain same. Only difference is we are expressing them in non-dimensional form. That is why it starts from 1 and it oscillates like this. For under damped case, when we have damping ratio less than 1. This is the case when it is critically damped. That is, the damping ratio is equals to 1. This is the case for over damped, when damping ratio is more than 1. In these two cases, obviously, there is no oscillation. The response dies down without any oscillation. Whereas, in the third case of under damped, there are oscillations, but finally also, with time it will die down after few cycles or after few oscillations. So, with this knowledge, now let us move to the remaining two problems, which were available in our given problem sheet.

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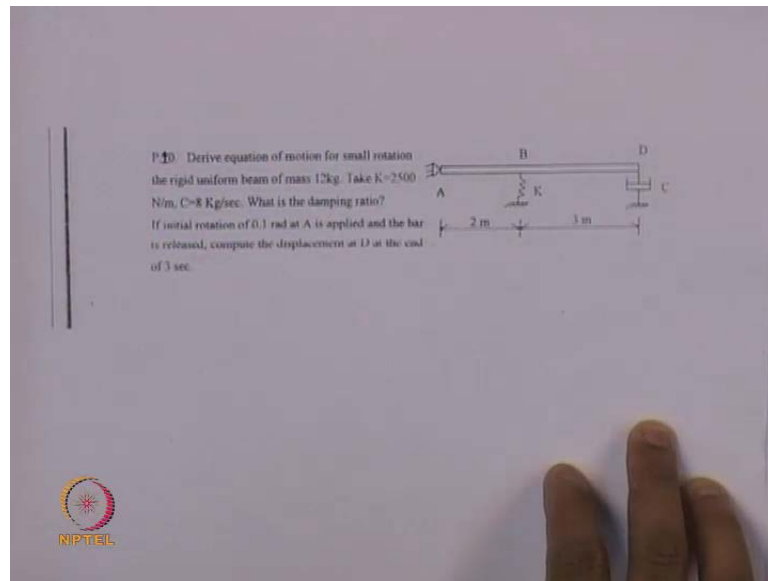
So, problem number 10, it says derive equation of motion for small rotation. The rigid uniform beam of mass 12 kg; this beam A B D is having a mass of 12 kg. There is a spring connected at a distance of 2 meter from this hinge support. That spring constant is given as 2500 Newton per meter and 1 damper is connected at this point D, and that is 5 meter from this hinge support and the damping constant is given as 8 kg per second. So, what is the damping ratio? This is the first question. Then, it is asked, if the initial rotation of 0.1 radian at A is applied and the bar is released, compute the displacement at this point D at the end of 3 second.

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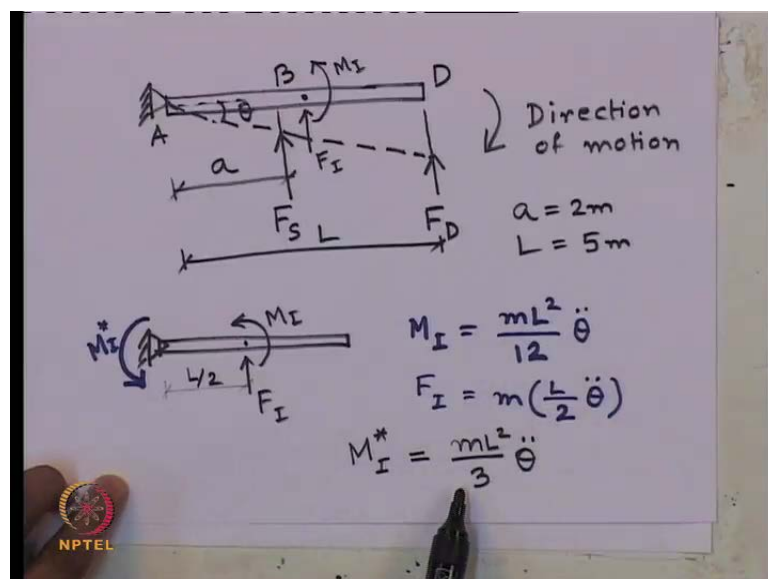
So, let us see how we can solve this problem. First, we need to draw the free body diagram. So, we have a hinge support here for the beam. Now, let us consider a small rotation. So, this is our direction of motion for the beam with a small rotation theta. At this location the spring was connected. So, the spring force will try to push it back to its original position. So, the spring force here. Here one damper was connected. So, one damper force will try to push it back to its original position F of D and what are the distances? This is, say this distance is A and let us say this distance is L. This point is A, this point is B and this point is D. Now, beam is having a mass. So, A is how much? A is 2 meter and L is 5 meter. So, C g of the beam is somewhere here, 2.5. Now, what will happen to the inertia of the beam? It has displaced vertically down as well as it has rotated. So, there will be some inertia force here and some inertia force here. So, F I and M I at the C g.

Now, if it so, earlier I had shown you, let me repeat it once again. For any beam or any rod, if at C g we have two inertia forces M I and linear inertia force F I, then instead of taking them at the C g of the beam, what we can do? We can transfer them at the end of a beam, say M I star. For beam, how much was the expression for M I at the center of the beam? M L square by 12. That was the mass moment of inertia about centre. How much was the inertia force at the center? F I will be; how much is the displacement? L by 2 theta. So, acceleration is this much. Mass into acceleration is F I and M I is mass moment of inertia into acceleration.

Instead of considering this M I and F I, what I said? We can as well consider a single inertia force M I star at the end. In that case, it will be how much? M L square by 3 theta double dot. Why because, we are taking moment about this point. Just transfer these two forces from C g of the beam to this point. So, M L square by 12 theta double dot plus M L by 2 theta double dot times L by 2. This is L by 2. So, M L square by 4 theta double dot. So, M L square by 12 plus M L square by 4 will give us M L square by 3 theta double dot.

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So, for our problem, we are either taking those two inertia forces at the C g of the beam or at the end, it does not matter. It gives us the same result. Finally, what we are doing is, we are taking moment equilibrium above that hinge support m. So, sum of M a is equals to 0 will give us M I star. Let us look back the free body diagram. We have M I star coming from this one. Instead of these two, I am considering M I star, that is anti clock wise. Now, this F s spring force and damper force is also producing anti clock wise moment about point A. So, F D times L plus F s times a equals to 0. That is the equation of motion.

Now, let us put the expressions known to us. M L square by 3 theta double dot plus; F D is how much? The damper force is nothing but, let us look back here. The damper force F D will be the damping constant C times the velocity at this point. How much is the displacement at this point L theta. So, velocity is L theta dot. How much is the spring force? At this point, F s is spring constant times the displacement at this point. How much is the displacement? a theta. So, we get F D is C L theta dot times L plus F s is k a theta times a equals to 0 or M L square by 3 theta double dot plus C L square theta dot plus k a square theta equals to 0.

We can put the values of the given properties and we can express them as now our canonical form like this. So, M star theta double dot plus C star theta dot plus k star theta equals to 0, where M star is M L square by 3, C star is C L square, k star is k a square.

So, the natural frequency will be root over k star by M star. If we put the expression of this k star and M star and the values of the given properties we will get omega as about 10 radian per second. Now, this is a damped system. So, how much will be the value of damping ratio? Damping ratio value is not given to us.

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So, we need to find out the value of the damping ratio. How to find it out? We know C equals to given as 2 eta omega M. Do you agree with me? C damping constant is given as 2 times damping ratio times omega times M, because C c r, the critical damping was given as 2 M omega. So, that is why eta is C by C r. So, we get this expression. So now, this is the generalized equation. In our case, this should be M star and C star, because it comes from the canonical form. So, do not fog it. It is not the actual values of a mod C, but it is coming from the basic equation of canonical form of equation what we have. So, it is not actual C and actual M. It is actually derived from this basic form of equation.

So, if you put the expression for C star, which is C L square and expression for M star, which is M L square by 3. Omega, already we have computed just now. So, everything is known. Only unknown is damping ratio. We can solve for damping ratio and it comes out to be about 0.1. From that, we get the under damped frequency, which is given as, this is less than 1. So, of course, we are considering the case of under damped system or under damped free vibration. Omega D comes out to be 9.95 radian per second. So, that

is what it was asked in the first question, what is the damping ratio. This is the answer of first part, damping ratio is 0.1.

We had started solving the problem which was given to us in the problem sheet. So, let us continue with that. This was the problem statement with the picture given to us, which we wanted to solve. For this problem, we had already drawn the free body diagram with direction of motion like this and a spring force, damper force and the inertia force, how they are acting and what are their magnitude; that we have computed. Based on that, we have written the equilibrium equation of moment about this hinge support A. From which, the governing equation of motion takes the form of our conventional canonical form like this, where M star is M L square by 3, C star is C L square and k star is k a square.

Now, after knowing all the given values, we computed what is the natural frequency of the system. It has been computed like 10 radian per second. Based on that, the damping can be expressed by this expression and critical damping ratio is expressed by this expression and remember this was actually star. So, eta is nothing but, C star by C c r star, which we have computed as 0.1, which is less than 1. So, it is a case of under damped free vibration.

Now, in the next step, we have computed damped frequency of the system, which is given by the natural frequency of the system times root over 1 minus eta square, where eta is less than 1 and the damped natural frequency comes out to be about 9.95 radian per second. So, we had stopped up to this part and what was the next part of the question of the problem? It is asked that, if initial rotation, that is theta naught, that is theta at time t equals to 0 at point A, that is at the hinge, it is given as 0.1 radian. So, it says, if initial rotation of 0.1 radian at A is applied and the bar is released, what does it mean? The initial velocity was 0, because only the rotational displacement has been applied and then, the bar has released, so that, it can vibrate under free vibration condition. It is asked that at the end of 3 seconds, so, when t equals to 3 seconds, how much will be the displacement at the end D. So, we need to find out at the end of t equals to 3 seconds, how much will be the rotation at point A and if we multiply rotation with respect to the length of the beam A D, we will get the displacement at D. Pretty simple. Now, referring to the solution what we know for a case of under damped free vibration, what was the solution available to us?

Let us write it in the form of theta. So, theta t will be equals to e to the power minus eta omega t times theta naught cosine of omega D t plus theta naught dot plus eta omega theta naught by omega D times sine of omega t. That was the complete solution using the initial conditions of theta naught and theta naught dot. In our case, we know eta equals to 0.1, we have already computed. t is seconds and omega we have computed as 10 radian per second. Omega D also we have computed as 9.95 radian per second. Theta naught is given as 0.1 radian and theta naught dot is 0. So, using all these values in this expression, we can compute easily theta at t equals to 3 seconds will be how much. If we put all these values here, we get theta at the end of t equals to 3 seconds comes out to be minus 4.76 into 10 to the power minus 4 radian. So, what does it mean this minus sign? That the rotation after 3 seconds, occurs in the opposite direction of the initial rotation given, because initial rotation theta naught that equals to 0.1 radian, we have considered as positive.

Suppose, whatever was the direction of that rotation, if it is clock wise, then at the end of 3 seconds, this magnitude of rotation is going to occur at point A in the anti clock wise direction or vice versa. That is what the minus sign signifies here. So, once we know this, so, displacement at point D, at the end of D equals to 3 seconds will be how much? This minus 4.76 into 10 to the power of minus 4 radian times the length of the beam, which is 5 meter. Right? So much of meter will be the displacement; just L theta. So, this

is the answer of the second part of the problem. Now, let us move to the next problem number 11.

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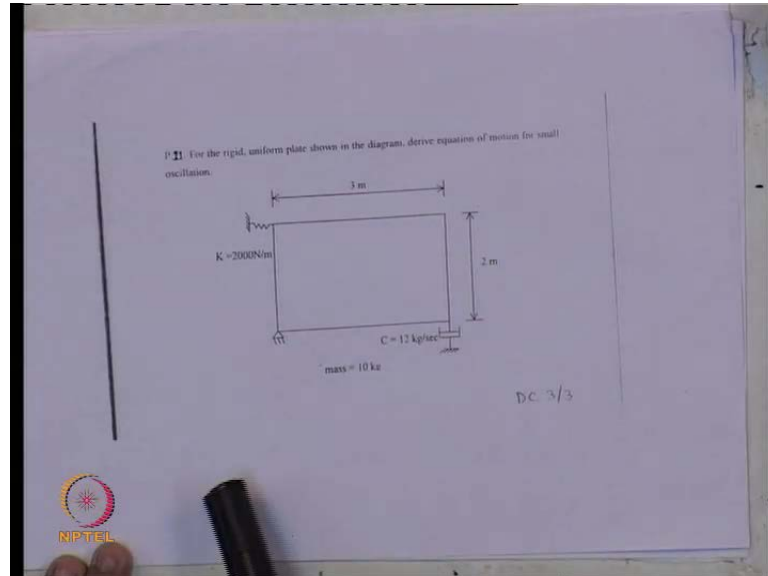
So, in problem number 11, what it says is, for the rigid uniform plate, so, this is the rigid plate, uniform plate shown in this diagram, derive the equation of motion for small oscillation. So, we have to just write down the equation of motion for this plate for small rotation. So, rotation can occur about which point? About this hinge support. So, about this support, we can provide a small rotation to the plate and then, we have to write down what is the equation of motion for that. There is a spring connected at this end of the plate and a damper is connected at the other corner of the plate.

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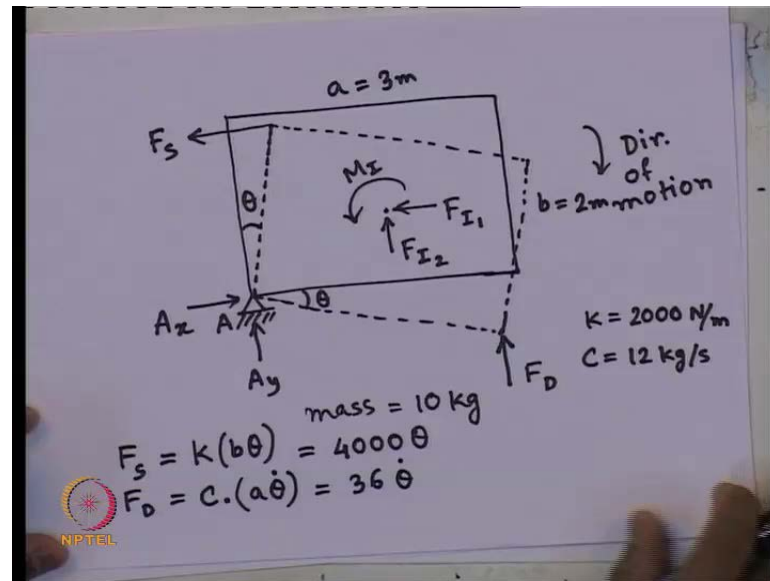
So, let us draw the free body diagram of this problem. So, we have a rigid uniform plate. This is the hinge support; this corner of the plate and let us apply a small rotation. So, we have applied a small rotation theta about this hinge and this is our direction of motion. Here will be some support reaction at this hinge, say this point is A. So, A y and A x are the support reactions. We have a damper here. So, this point has been moved. So, the damper force will come here and it will act in this direction F of D and linear damper. So, in this direction it will work and there is a linear spring attached to it, which has moved here. So, at this location, we have spring force F s. F s will act in this direction to bring it to its normal position. F D will act vertically upward to bring it back to its normal position.

What else will act? At the centre of the plate, there will be the vertical inertia force, horizontal inertia force and a rotational inertia force at the center of the plate. Why these three inertia forces? Because when the plate is rotated about its support at one corner of the plate, the c g of the plate has been displaced both horizontally as well as vertically and also the plate has been twisted or rotated. So, that is why the inertia force in all the three directions will act. This vertical inertia force will try to bring it back to its original vertical position, the c g. The horizontal inertia force will try to bring the plate back to its original position of c g and the rotational inertia force will try to bring it back to its original position. Mass of the plate is given to us. Mass of the plate is 10 kg in the problem it is mentioned and the dimensions are this side. Let us say this a, which is 3

meter and this dimension, let us say b, which is given as 2 meter. What next we need to do? We need to write the expressions for these forces. If a spring force will be how much? That will be this length times theta times k. So, k into b theta will be the spring force. k is given as 2000 Newton per meter and C is given as 12 kg per second. So, if we put the values of k and b, we will get, it comes out to be 4000 theta and the damper force F D, that comes out to be the damping constant C times. How much is the displacement at this point? a times theta. So, velocity will be a theta dot. So, a theta dot is the velocity. So, it comes out to be 12 into 336 theta dot. Next, we have to find out the inertia forces.

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So, let us write down. The horizontal inertia force F I 1; to do that, let us keep the figure with us. So, it will be easy for us to follow the problem. So, F I 1, the horizontal inertia force will be mass times the acceleration at this point. So, horizontally how much it has moved? The center has moved horizontally by the amount b by 2 theta. So, acceleration at the centre will b by 2 theta double dot. So, the inertia force in horizontal direction will be mass times b by 2 times theta double dot. Now, we can put mass equals to 10 kg given to us and b is 2 meter. So, it is 10 theta double dot. How much will be the vertical inertia force F I 2? That is mass times the acceleration in the vertical direction. So, vertically how much distance it has moved? Vertically it has moved by a distance of a by 2 theta. The center has moved by the distance a by 2 theta. So, acceleration will be a by 2 theta double dot. So, if we put the value, it comes out to be 15 theta double dot. M I, how much will be the M I? The rotational inertia force for a plate, the expression is M by 12 a

square plus b square theta double dot. So, this is the mass moment of inertia of a plate with sides a and b.

So, now, we can put the values 10 by 12 3 square plus 2 square theta double dot. That is the value of M I. Aafter computing all these forces, what we need to do is, we need to write down the equilibrium equation. So, when we are taking equilibrium for this problem, we have to take equilibrium about the hinge support, that is the corner of the plate about point a. Moment equilibrium about point a should be equals to 0. So, let us do that. Sum of moment about point a should be 0 will give us, let us look at the picture once again here, M I is anti-clock wise about a. Now, F I 1, this horizontal inertia force is creating a moment about point A is how much? This length by 2. So, b by 2 plus, this is also anti clock wise; so all plus, F I 2, that is creating moment about point A, F I 2 times a by 2 plus this F D is producing again anti clock wise moment. So, F D times a plus F s is producing again anti clock wise moment about a, and F s times b should be equals to 0. So, that is the moment equilibrium about point a.

So now, if we put all the values of the forces already we have written, we will get this comes out to be 9 plus 4 13 130 by 12 theta double dot. So, 130 by 12 theta double dot plus, F I 1 is 10 theta double dot times b by 2 and b is 2 meters. We will get 10 theta double dot plus, F I 2 is 15 theta double dot and a is 3 meters. So, 1.5 15 into 1.5 times theta double dot plus, F D we have calculated 36 theta dot and a is 3 meters. So, 36 into 3 theta dot plus, F s we have calculated 4000 theta, so, F s is 4000, b is 2 meter into 2 into theta and that equals to 0.

So, if we take all of them together, 130 by 12 plus 10 plus 15 into 1.5, this theta double dot plus 36 into 3 theta dot plus 8000 theta equals to 0, which is nothing but our equation of motion. That M theta double dot, I should write M star theta double dot plus C star theta dot plus k star theta equals to 0. So, this is the equation of motion. That is what it is asked. Formulate the equation of motion, where M star is this expression, C star is 36 into 3 and k star is 8000. From which, if somebody asked find out the natural frequency and damping ratio, everything we can compute from our know expressions. But, we have to remember always that this is the basic equation or canonical form of equation. So, not the individual value of M and k and C we should use, but we should use this common form or the generalized form of the expressions of M star C star and k star. So, we will stop here today. We will continue our lecture in the next class with this problem.