Soil Dynamics
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Module - 2
Vibration Theory
Lecture - 5
Damped Free Vibrations

So, we are starting today's lecture. We are continuing with our module 2 on vibration theory. Now, we will move to our next topic on damped free vibration. So, for damped free vibration, let us now look at the slide.

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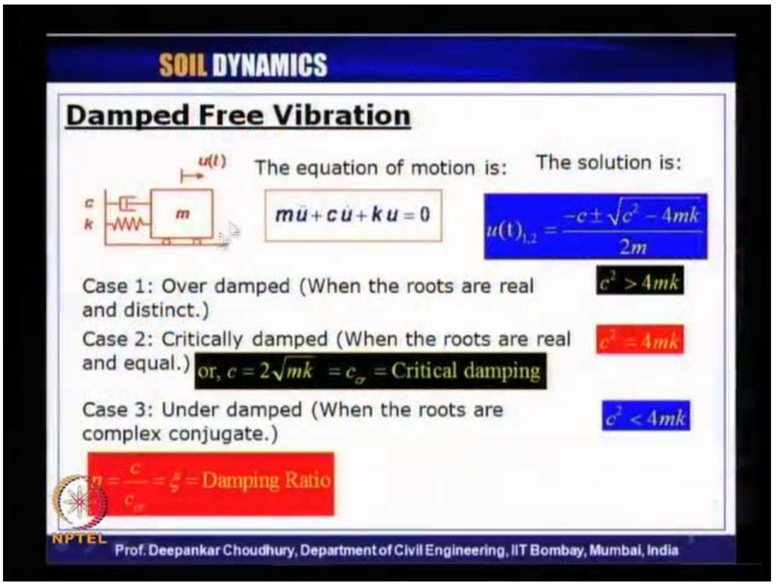

For damped free vibration, the basic model of single degree of freedom system will look like this; that is the damper now present, because it is a damped case we are considering. So, this c is no longer a 0 value, but it is a non-zero value. The spring is present, mass is present and the single degree of freedom is nothing but the displacement u of t, but still we are considering the case of free vibration that is why there is no externally applied dynamic load. So, f of t or p of t whatever you say that is 0.

So, the equation of motion can be written as for a linear model, for damped free vibration case as m u double dot; inertia force plus c u dot; damper force plus k u spring force equals to 0. There is no externally applied dynamic load. So, this is the governing equation of motion. Now, let us look at the solution of this equation of motion.

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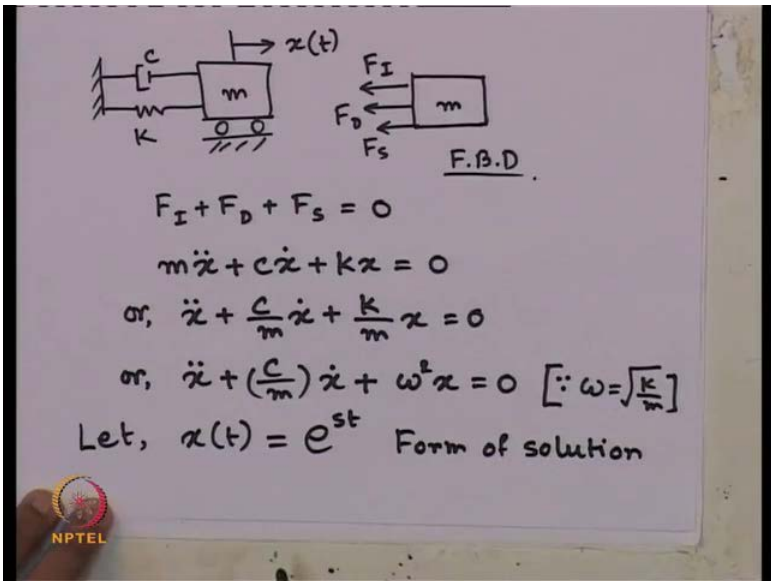

Let me draw it once again, the basic single degree of freedom system, what we are discussing just now. With the free body diagram c, k, x of t is a single degree of freedom. So, the mass we have F I, F D and F S, that is our free body diagram and using D'alembert's principle F I plus F D plus F S equals to 0, and then using the linear model we got the governing equation of motion m x double dot plus c x dot plus k x equals to 0. Now, we want to solve this second dotted differential equation for x, so x of t, now we are going to find out.

Let us divide both the sides of the expression by m, which we can write it like this or x double dot plus c by m x dot plus omega square x equals to 0 because we know omega equals to root over k by m is the natural frequency. So, that is why k by m we have written as omega square. Now, for this type of second order differential equation as we start with assuming a form of a solution, so let x of t equals to e to the power s t is the form of a solution. So, this is the form of solution for this differential equation. What we will get?

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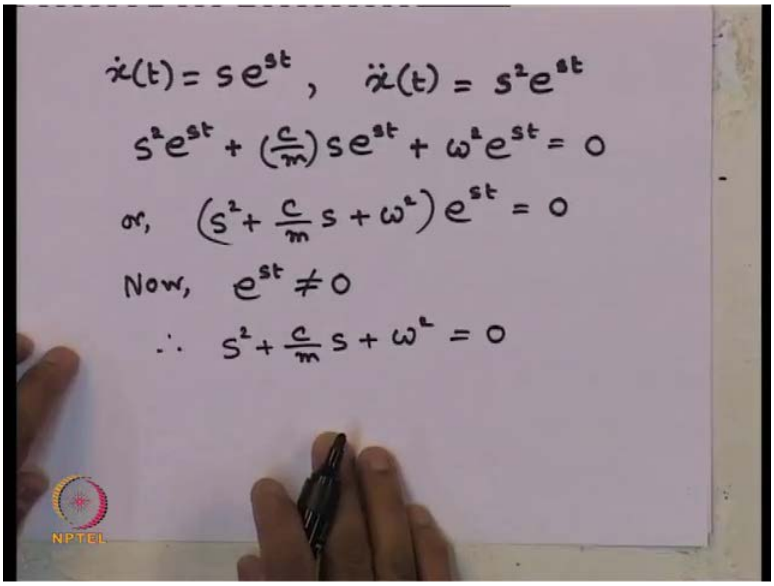

Now, x dot t will be s e to the power s of t and x double dot t, that is velocity, we have differentiated once and acceleration we are differentiating twice. The displacement function we get these expressions. Let us put it back to our governing differential equation we will get s square e to the power s of t plus c by m s e to the power s of t plus omega square e to the power s of t equals to 0. Or s square plus c by m s plus omega square times e to the power s of t equals to 0. Now, e to the power s of t cannot be 0 because otherwise it will give a trivial solution, so this component has to be 0. Therefore, s square plus c by m s plus omega square has to be 0. So solving this equation we will get two roots of this.

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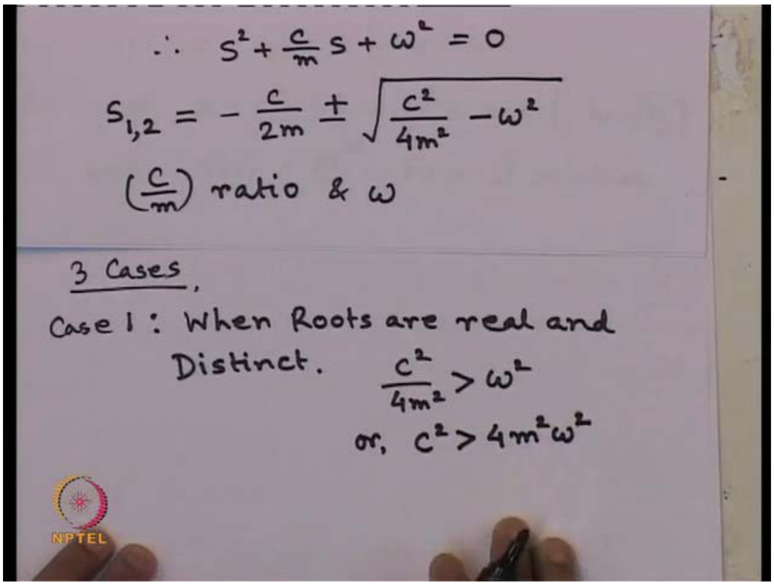

Root one and root two of s can be written as it is in the form of that, minus b by 2 plus, minus root over b square minus 4 a c by 2, a that a x square plus b plus c equals to 0, that form. So, c square by 4 m square minus omega square. So one root of this solution is minus c by 2 m plus root over c square by 4 m square minus omega square, and the other root s two will be minus c by 2 m minus root over c square by 4 m square minus omega square. These are the 2 roots of this second dotted differential equation. Now, one thing these roots s one or s two they depend on the value of the ratio c by m and omega.

So, these roots does not depend on the individual value of damping coefficient or the mass, but they are depend on the ratio of the damping coefficient and the mass. So, when for any design purpose we are going to select that, how the system is going to behave; we have to emphasize on the ratio of the chosen c by m, rather than just using one variable c or one variable m changing. So, suppose two systems are having same ratio of c by m, but mass and damping constants are different, does not matter their responses will remain same, as long as the ratio remains same.

Now, look at this roots, there can be three possibilities from this solution, we can get, what are those three possibilities? Let us look at it. So, three cases can arise; the first case, case one, let us say that will be when roots are real and distinct, when that condition will occur, when the roots are real and distinct, when this under root term is positive. In that case we will get real value roots and they will be distinct, two roots one is minus c by 2 m plus this value, and another is minus c by 2 m minus this value. So, the condition for this is c square by 4 m square has to greater than omega square, then we will get this case one or in other words we can write, c square should be greater than 4 m square omega square or c square should be greater than 4 m k because omega square is k by m.

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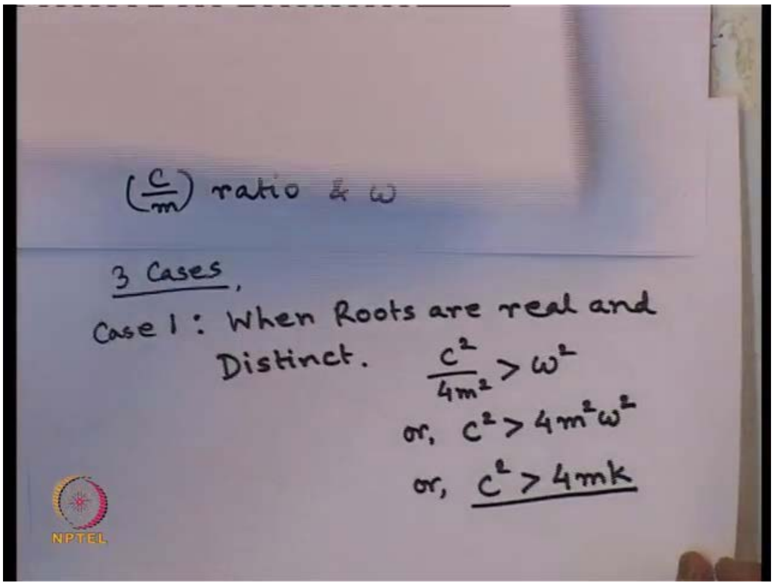

So, if we put k by m, m gets cancelled it should be the condition for roots to be real and distinct. Let us see what is the second case which can arise?

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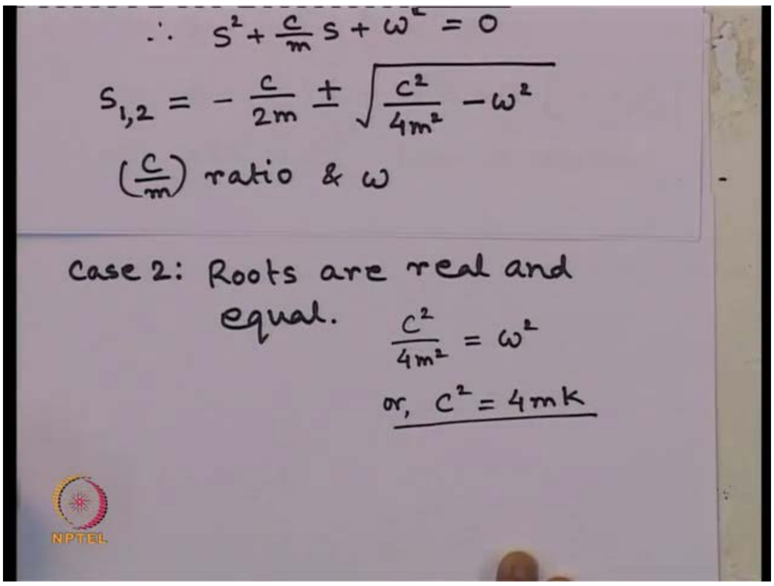

So, case two can be, roots are real and they are equal. When this condition will arise? When this under root term is 0, in that case what are the roots? Roots one will be minus c by 2 m and the root two s two will be the same value minus c by 2 m. So, roots are real values, this is the real value and they are equal. So, the condition for this is c square by 4 m square equals to omega square or c square equals to 4 m k, this is the condition to have roots real and equal.

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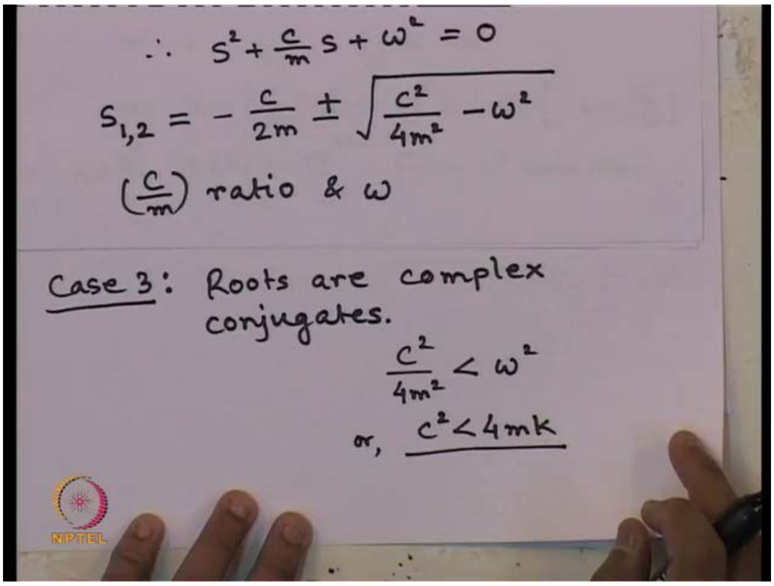

Now, let us see what is the third case. Case three can be, roots are complex conjugates. When it will occur? When this under root term is negative, then we will get a complex solution, complex value; minus c by 2 m plus of that complex number and the other root will be minus c by 2 m minus of that complex number, so that will form complex conjugate. The condition for this will be, this under root term c square by 4 m square is less than omega square or in other words, c square should be less than 4 m k. These are the three possibilities of solutions, which we can get for this differential equation from this governing equation of motion. Now, let us look at each of these three cases, what we can obtain. Let us look at this slide.

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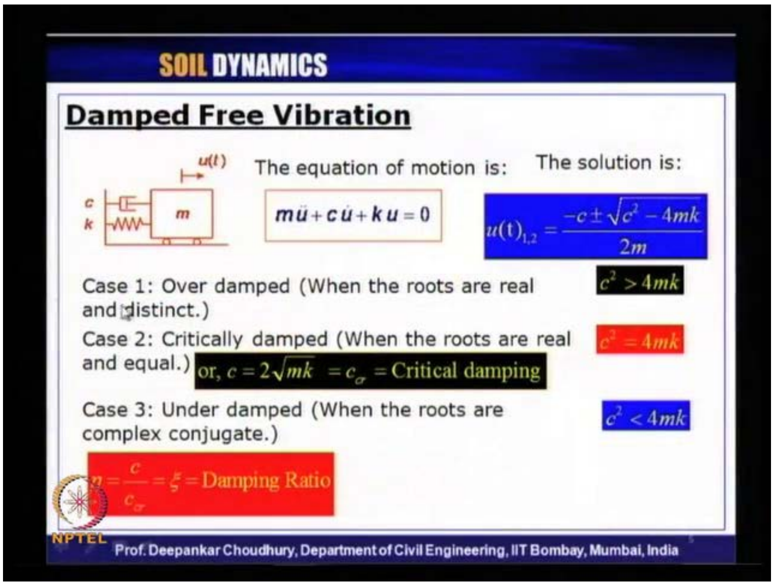

What I have told just now, case one is called, now the nomenclature I am giving you. The first case, that is when the roots are real and distinct, that is with the condition of c square greater than 4 m k is called over damped. So, if this type of vibration occurs we call it as over damped free vibration. The second case, when the roots are real and equal with the condition of c square equals to 4 m k that is called critically damped free vibration, that condition is called critically damped free vibration. And the third case, which is the condition with roots are complex conjugates arise due to c square less than 4 m k is called under damped free vibration. Now, let us look at each of these three cases, first let me start with case two.

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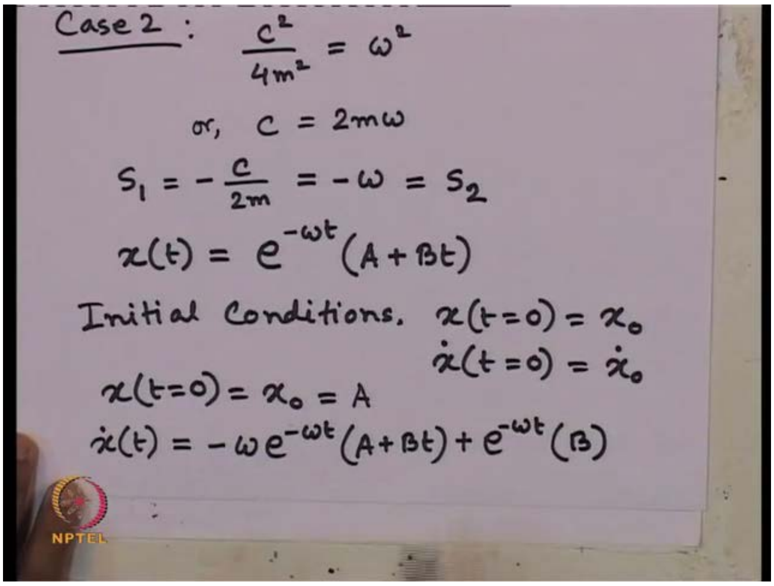

Let us start with case two first, then I will go to case one. So for case two, what we have, roots are real and equal and for that condition we have seen, c square by 4 m square has to be equals to omega square. So, that under root term is 0, which gives us c equals to 2 m omega and the roots s one equals to minus c by 2 m under root term is 0, which is equals to, if we use these relation, minus omega; this is equals to s 2, a real value root and they are equal, both the roots are same.

So, what will be the solution for this type of case, when roots are real and equal. We know the form will be what we had assumed the form of solution; that e to the power that particular equal root times t, then a polynomial function because of equal roots. So, e to the power, what is the root s minus omega, so minus omega t, e to the power s t was our general solution, so what is the value of s? Value of s is minus omega times A plus B t. So, from our knowledge of ordinary differential equation, we know that for equal roots, real roots we will have a solution like this, where A and B are the constants, which we have to determine from the initial conditions.

Now, initial conditions at x t equals to 0, let us say x naught and the velocity x dot at t equals to 0 is x naught dot, are given to us. If we put in this form of solution or the equation at t equals to 0, what we will get? x at t equals to 0, which is x naught equals to, if we put t 0, this becomes 1, A plus B times 0, so only A, so the constant A is nothing but x naught.

And now, we differentiate this expression, what we can write; the velocity expression will be, minus omega e to the power minus omega t times A plus B t, I have differentiated this exponential function first, then e to the power minus omega t times; I am now differentiating this polynomial function, times b. Now, let me put the other condition that is the condition of initial velocity in the solution. Let us look back here.

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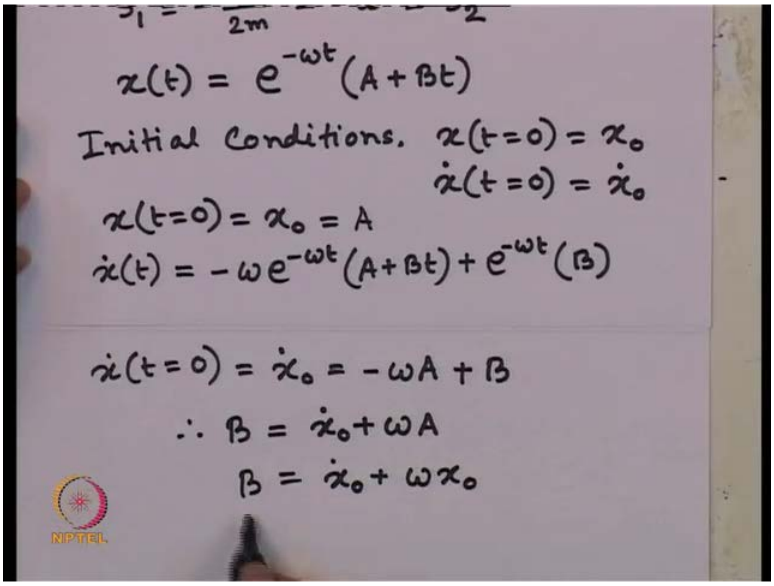

So, x dot at t equals to 0 is x naught dot, known to us. If we put in this expression, t equals to 0, what we are getting; minus omega times 1, A. So, minus omega A plus this gives us 1, this is B, so plus B. Therefore, the other constant B is nothing but x naught dot plus omega times A. Now, how much is our other constant A, which is also known now, x naught. So, the another constant B is x naught dot plus omega x naught. So, what is the complete solution now?

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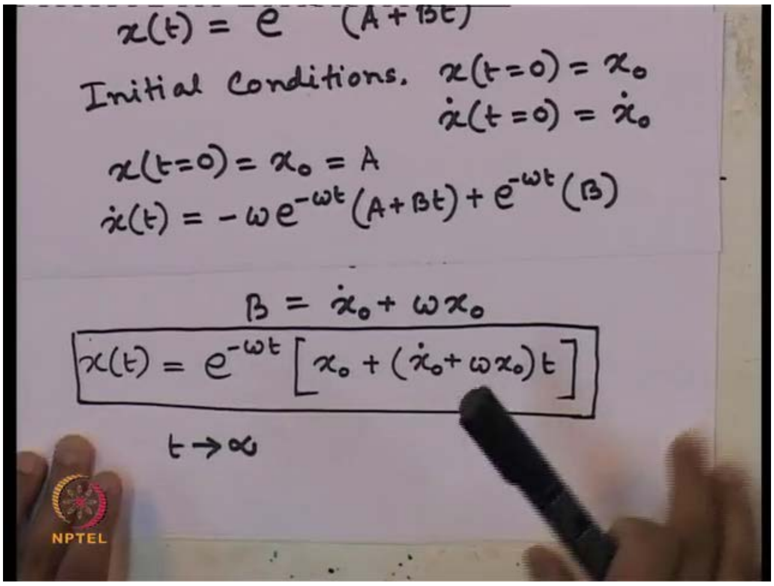

We can write the complete solution x of t will be e to the power minus omega t times A, means x naught plus B, means x naught dot plus omega x naught times t. So, this is the final solution using the initial conditions for the case of, roots are real and equal. Now, how this response we can plot? Because finally, we want to look at the behavior of the response, that is how x of t varies with respect to time. If we look at this expression, let us see what are the components, it has... It is having one exponential component and a polynomial component. Now, if t tends to infinity, what will happen to this function x of t? This exponential is a with a power of negative power, which calls as decay.

So, this is an exponential decay function and this is the polynomial with increasing with t function. Now one function is increasing and another function is decreasing. Now, what will be the combined response of these two? From our knowledge of mathematics, what we know? So, let us look at the solution of this case, where roots are equal and real. x of t is a combination of two function; one exponential function another polynomial function. Exponential function is a decay function, which decreases with increase in time because of this minus omega factor, and this polynomial function is an increasing function, that is with increase in time it increases.

So, the combination of them, will it be increasing or decreasing? Now, from our knowledge of mathematics, what we know? There are several functions among them one dominates over the other one. Like in the case of integration also in our high school mathematics, we know while integrating which function dominates over which one, accordingly we take during integration, which performance has to be done first.

Similarly, when we take the behavior of combined effects of several functions, which function dominates over the other one? Always exponential dominates over polynomial and then the last one in the rank was trigonometrical function, right? From our knowledge of mathematics, what we know? Exponential always dominates, what does it mean? This decay or decreasing function dominates over this increasing function of polynomial, what does it mean? If we want to plot the behavior of this combined effect of two functions; exponential decay and polynomial increased function, the combined effect will be a decay function because of the higher order or higher dominancy of the exponential function over the polynomial.

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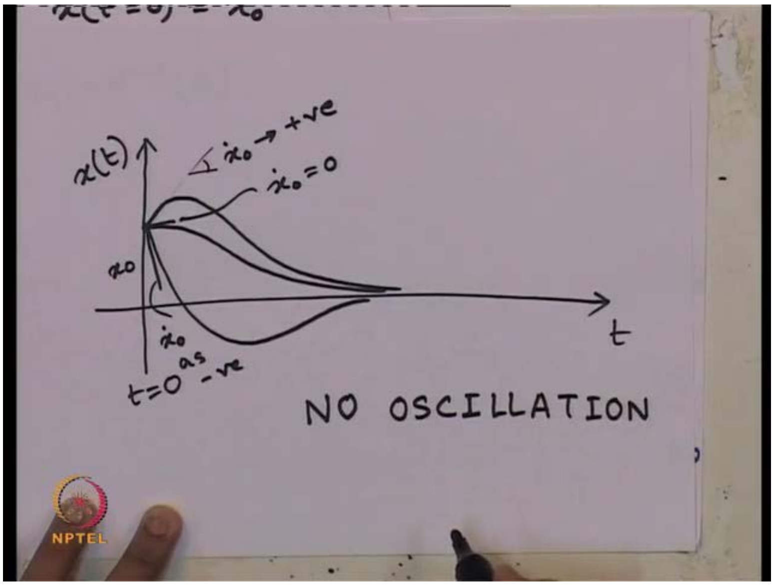

So, the response will look like this, with increase in t, if we want to plot x of t. It will start from some point here at t equals to 0 depending on its initial displacement x naught. It can be either like this or it can be like this or it can be like this. What are the three cases I have drawn? This one is for x naught dot, being positive; this one for x naught dot as negative and this one with x naught dot equals to 0. As the exponential decay function dominates with increase in time, it will tend to 0 or it will diminishes like this, without any (( )) oscillation. So, the final result of this vibration is no oscillation because of the presence of this decay function like this. Now, let us come to the first case, that is the case when roots are real and distinct.

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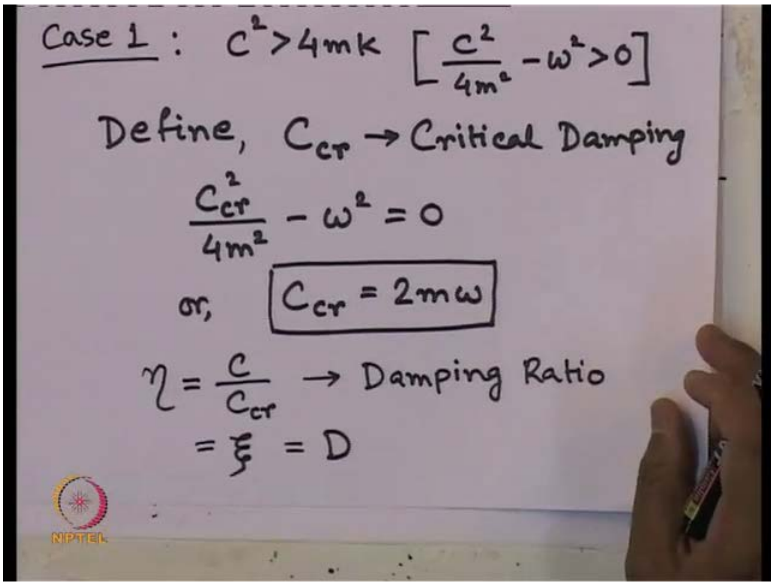

So, case one now we are doing, with the condition that c square is greater than 4 m k, which we got from c square by 4 m square minus omega square is greater than 0, that is another root term is positive. In this case, let us define one new parameter. Let us define a new parameter C c r, which is called critical damping, critical damping constant. So, what, how we are defining this critical damping constant, in such a way that C c r square by 4 m square minus omega square equals to 0 or in other words, C c r equals to 2 m omega. That is the definition of critical damping constant equals to 2 m omega. Now, if we compare this with our previous solution, for case two were roots were real and equal, this denotes to that case. So, we have just now solved a case of critically damped conditions, so C c r equals to 2 m omega.

Now, let us define another parameter, eta as ratio of the given damping constant to that critically damping constant. So, eta is known as damping ratio, so the damping ratio is defined as the ratio of damping constant to the critically damping constant. In some book you will find they use some other notation like this, so anyway it does not matter, some book has used the notation of capital D, these are the standard notations used for damping ratio in different text books. So, for our course I will try to stick to this Greek symbol eta, as the symbols for damping ratio. But I have mentioned that we will find a similar different symbols, but same for damping ratio in other text books. So, with this condition what are the roots now we are getting, for this case, when roots are real and distinct.

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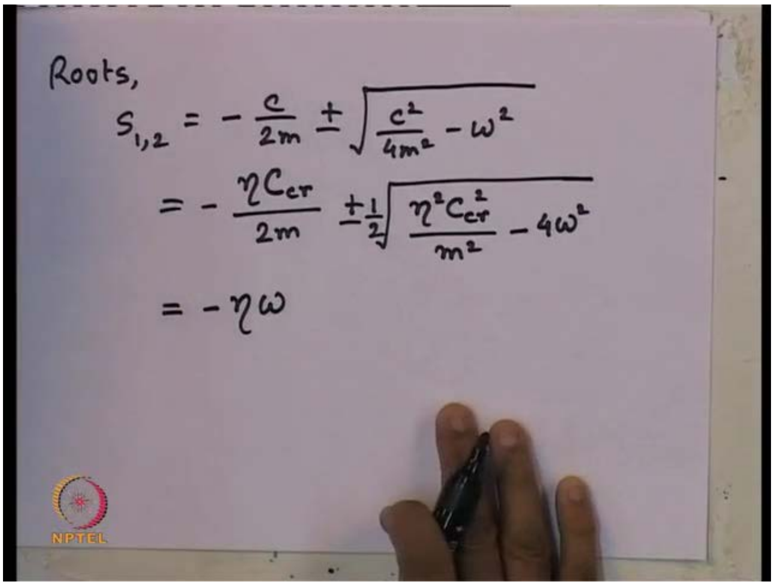

Two roots will be minus c by 2 m plus, minus we had c square by 4 m square minus omega square, that was the solution we got. Now, c is how much? The damping ratio times C c r .We have defined eta equals to c by C r, so we can write c as eta times C c r plus minus, let us take half out here outside the root eta square, C c r square by m square minus 4 omega square, fine. So, 4 we have taken outside the root, so half I have taken here, c is nothing but eta time c r, so c square is eta square C c r square by 4 m by m square minus, as 4 from the denominator we have taken out, here 4 omega square. Now, this will give us minus eta omega, why? Let us look at the expression earlier we had given C c r is 2 m omega. So, what is omega?

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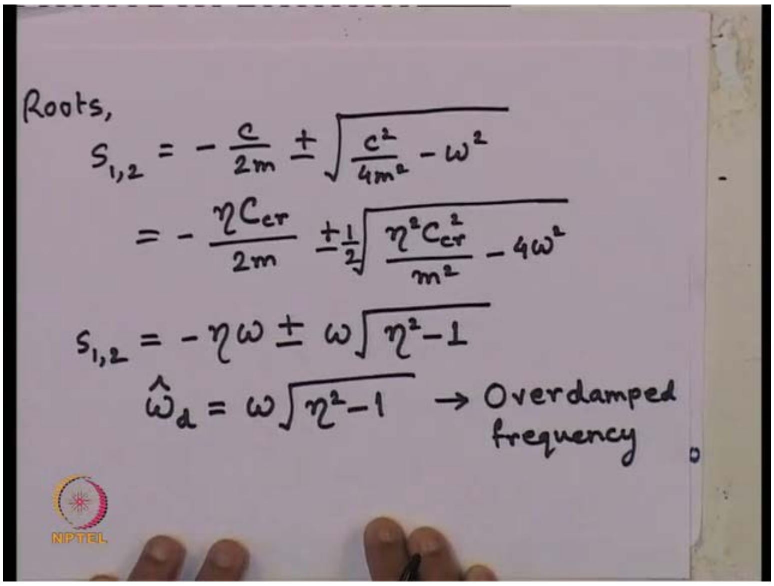

C c r by 2 m, so the same relation we have used here, C c r by 2 m is omega, so minus eta omega plus minus omega root over eta square minus 1. Here, also if we take this 2 inside, what we will get? This term will give us omega square and 4 was not there, so omega square if we take out, it will come as omega, so eta square minus 1, fine. So, this is the solution for two roots, one root is minus eta omega plus omega root over eta square minus 1, and the second root is minus eta omega minus omega times root of eta square minus 1. Let us define another parameter omega d cap, which is expressed as omega times eta square minus 1, which is known as over damped frequency. So, over damped frequency omega d cap is defined as, omega times root over eta square minus 1. And for this condition what we had, c square is greater than 4 m k. What does it mean?

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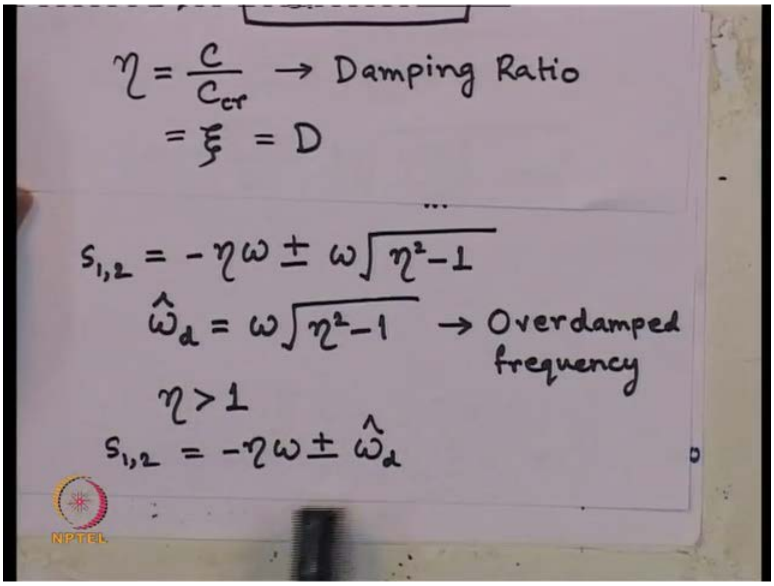

We have the value of eta is greater than 1, so with value of eta greater than 1 and omega d cap written like this we have two roots now; eta omega plus minus omega d cap. Now, if these are the two roots, these roots are real value and they are distinct. What will be the form of solution?

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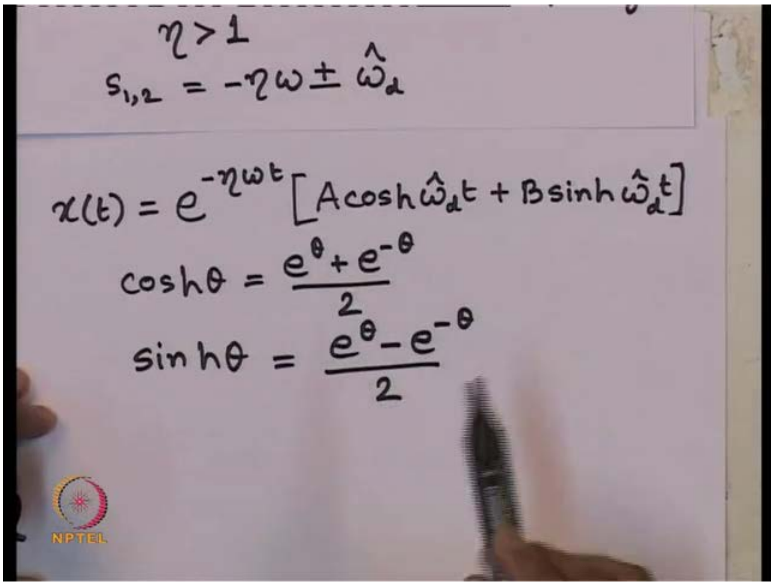

The form of solution will be x of t equals to e to the power s t that is what we have taken. So, e to the power this one minus eta omega t times A cos hyperbolic omega d cap t plus B sine hyperbolic omega d cap t. So, that is the complete solution with A and B are two constants, which has to be determined from the initial conditions. And what are the hyperbolic functions? We know, suppose cos of hyperbolic theta is written as e to the power theta plus e to the power minus theta by 2 and sine hyperbolic theta is written as e to the power theta minus e to the power minus theta by 2. So, with this now using initial condition we will get the two constants A and B.

So, with these two roots, we finally obtain the solution can be expressed as x of t equals to e to the power minus eta omega t because minus eta omega was one root, and another root was omega d cap. So, A cos hyperbolic omega d t plus B sine hyperbolic omega d cap t, were cos hyperbolic and sine hyperbolic functions we know how these are written as. To continue with this solution, now let us find out the expression for the velocity that is x dot t from this solution form, so that we can put the initial conditions here.

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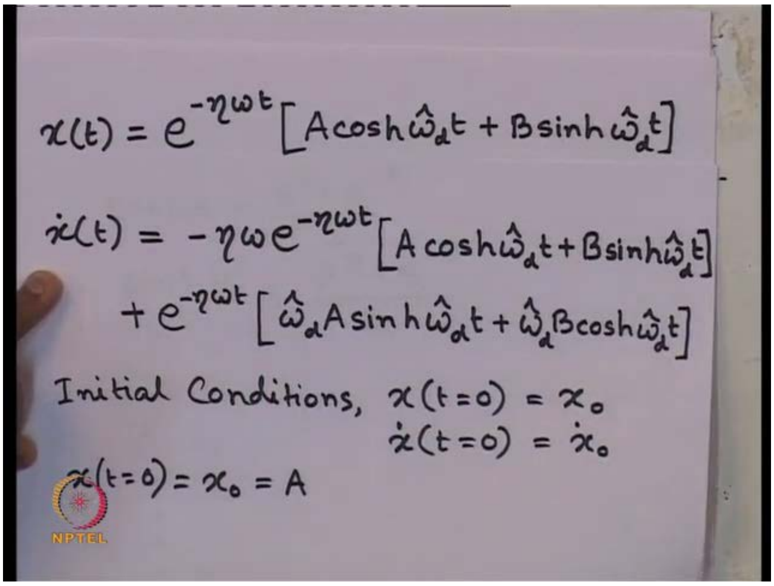

So, x dot t can be written as, minus eta omega e to the power minus eta omega t. Let me put the function here, so that it will be easy for us to follow. So, first I am differentiating this function. So, minus eta omega e to the power minus eta omega t times this remains as it is, so A cosine of hyperbolic omega d cap t plus B sine hyperbolic omega d cap t plus, now we need to differentiate this part. So, this remains as it is, e raise to minus eta omega t. Now, this factor comes out, so omega d cap A cos becomes sine, sine hyperbolic omega d cap t, and from this we get plus omega d cap B sine become cos, cosine hyperbolic omega d cap t.

So, for hyperbolic function we know the differentiation comes out like this. Now, what we do? We apply the initial conditions to obtain the unknown two constants, A and B. So, initial conditions that is x at t equals to 0 is x naught and velocity x dot at t equals to 0 is x naught dot. So, from the first expression what we can write, that x at t equals to 0, which is x naught. If we put t equals to 0 here, this becomes 1, this also becomes 1. So A remains, this vanishes, so this is only A. And in the second expression, now we will put t equals to 0, let us see what we will get.

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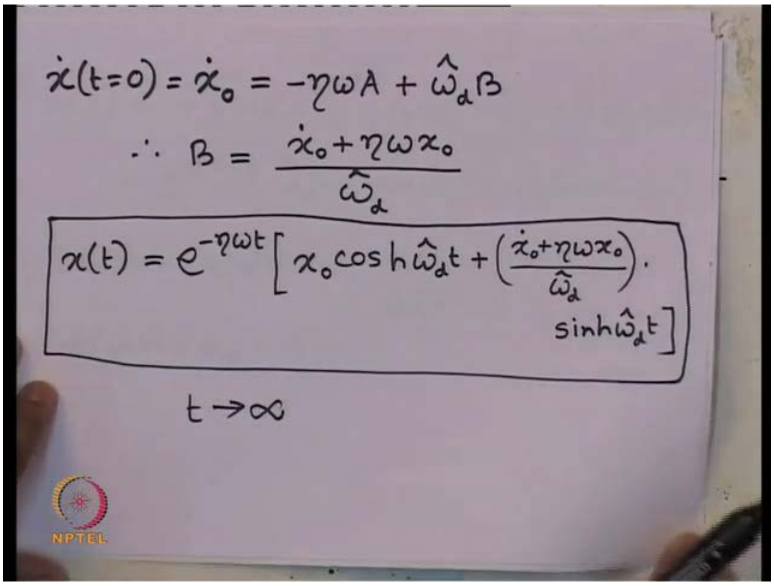

Now, we are putting x dot t equals to 0, which is x naught dot. This will give us, in this expression if we put t equals to 0, it will give us minus eta omega times A and from this expression we will get; this term vanishes, this remains, so plus this is 1 omega d cap B. Therefore, the constant B becomes x naught dot plus eta omega A, we have already got it, is x naught divided by omega d cap. So, the final solution takes the shape of x of t equals to e to the power minus eta omega t times, A is x naught cosine hyperbolic omega d cap t plus B is this term. So, x naught dot plus eta omega x naught by omega d cap times times sine hyperbolic omega d cap t.

So, that is the complete solution using the initial conditions for the first case of over damped. Now, if we want to see the response of this function, that is when t tends to infinity how this function x of t behaves, with respect to t. We can find out that again this is exponential decay and this is cos hyperbolic function, within which also we have exponential function because we have seen what are the cos hyperbolic and sine hyperbolic function. So finally, it will also decrease without any oscillations, so there also will be no oscillations.

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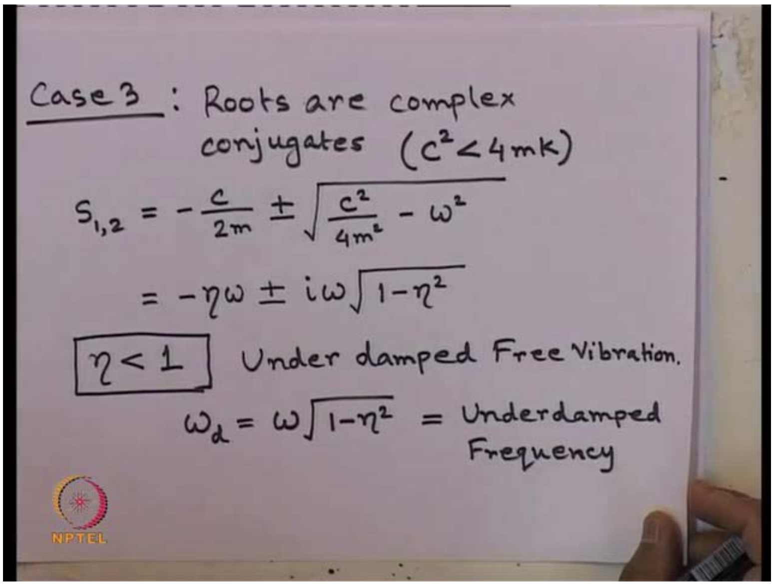

Now, let us move to the last case, that is case three when roots are complex conjugate. For roots to be complex conjugate, we have seen roots are complex conjugates, the condition we have; c square should be less than 4 m k. Now, what are the roots we had, is one and two. That was minus c by 2 m plus minus root over c square by 4 m square minus omega square, which we can now express using our damping ratio eta. So, this is minus eta omega as we have done for the previous case also plus minus. Now, this is a negative term, so one i will come out, i omega root over 1 minus eta square, same like the previous case, only difference is we have one imaginary function i here. And for this case, as c square is less than 4 m k, what we will get? Eta is less than 1, so this is the condition for this case three to give us roots as complex conjugates. So, this case is called under damped free vibration.

Now, let us define another parameter omega d, which is expressed as omega times root over 1 minus eta square, which is called under damped frequency. So, earlier we have seen omega d cap, which is called over damped frequency. Now, we are defining another parameter, which is omega d only given by omega times root over 1 minus eta square is called under damped frequency, and the difference between these two in terms of the expression, what we have seen, for omega d cap. Let us look here once again. So, that it will be clear to us the difference between the two cases, omega d cap was omega times eta square minus 1 because eta was always greater than 1, in that case, that is over damped frequency. Whereas, in third case eta is always less than 1, so omega d is omega times root over 1 minus eta square, which is called under damped frequency.

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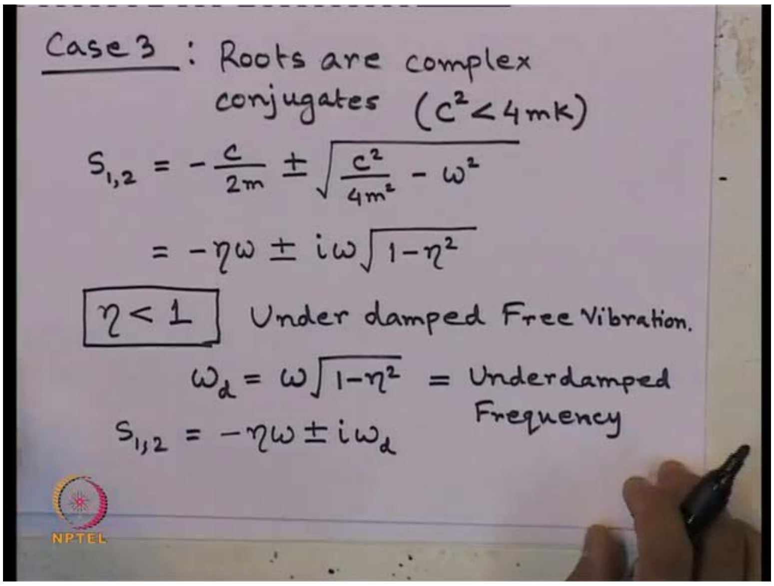

So, the two roots of the solution s one and s two will be minus eta omega plus minus imaginary function i omega d, these are the two roots. So, if we have two roots like this, what should be the solution?

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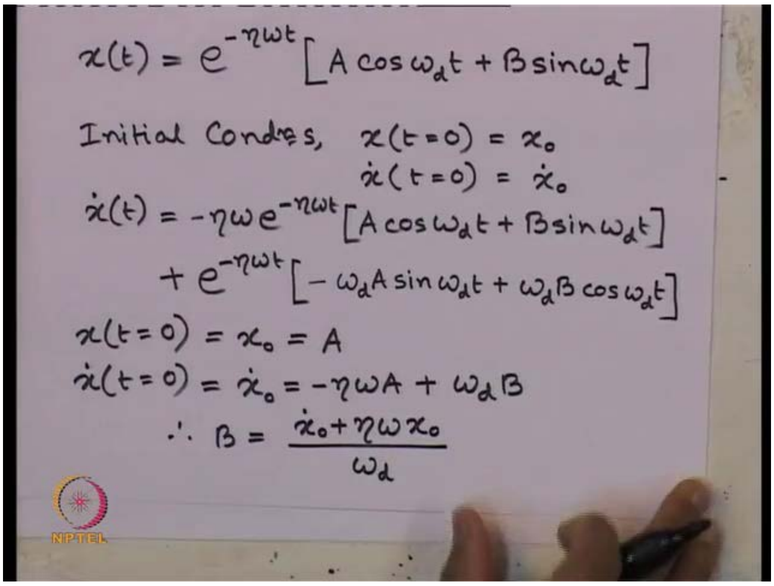

Let us see, so the solution will be x of t equals to e to the power minus eta omega t, let me show the roots once again here so that it will be easy for us to follow, so this function s one or s two, e to the power minus eta omega t times. We have imaginary function, so it is in the form of complex conjugate roots. So, we can express them in terms of trigonometrical functions A cosine omega d t plus B sine omega d t. This we have see in the case of undamped free vibration also, that is when roots are complex conjugate that time we can express them in terms of trigonometric function. So, that is how the final solution we have expressed in this form where A and B are constant, which we need to find out form the given initial conditions.

Now, let us apply the initial conditions given to us, that is x at t equals to 0 is x naught and the velocity x dot at t equals to 0 is x naught dot. So, if we want to write down x dot t for this expression, what we will get? First let me differentiate this function, so minus eta omega e to the power minus eta omega t times, this remains as it is. A cosine omega d t plus B sine omega d t plus, now let me differentiate this function, keeping this as constant. So, e to the power minus eta omega t times cosine becomes sine minus omega d A sine omega d t plus sine becomes cosine omega d B cosine omega d t.

Now, let me put that initial condition given to us x at t equals to 0 is x naught, if we put this in this expression this becomes 1, this is 1 because of t is 0. So, only we get A, this vanishes because sign function with 0 vanishes. So, A equals to x naught and in the velocity expression, if we put x dot at t equals to 0 as x naught dot, what we will get? This becomes 1, so minus eta omega. This A remains here, this vanishes plus, this becomes 1, this vanishes, this remains, so we get omega d B, which means the constant B will be x naught dot plus eta omega, A is x not divided by omega d.

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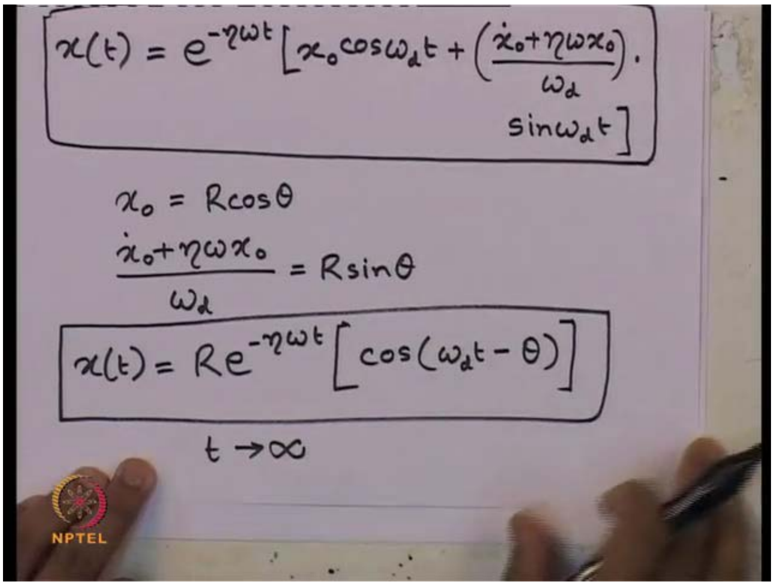

So, the complete solution using this initial conditions can take the form of x of t equals to e to the power minus eta omega t times, A is x naught cosine omega d t plus B is x not dot plus eta omega x naught by omega d times sine omega d t. So, that is the complete solution using the initial conditions for under damped free vibration, when damping ratio is less than 1, and we have the condition that c square is less than 4 m k.

Now, let us see, if we want to represent this expression or this form of solution in terms of polar coordinate system. As similar way we have done for un damped free vibration. So, here also we can do that, by expressing x naught as R cosine theta and this x naught dot plus eta omega x naught by omega d, this function has R sine theta. Then the solution will be given as R e to the power minus eta omega t times cosine omega d t minus theta, so that will be the form of solution in polar coordinate system, with R as amplitude of motion, and theta as phase. So, if we look at the response of this function with t tends to infinity, that is variation of this displacement function x of t with respect to time t, if we want to plot, let us see how it looks like.

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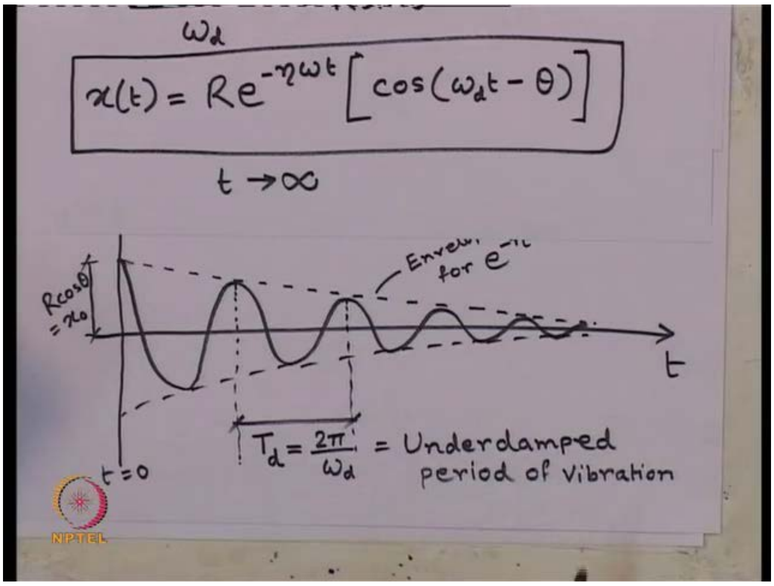

Now, let me put the solution also here so that we can compare the expression and the variation and how to draw this easily; x of t I am plotting with respect to t, so at t equals to 0, we have some value of x naught here. That x naught is nothing but R cosine theta and with some initial velocity it will start. And look at the response, it is a harmonic function, cosine function with a exponential decay function combined to it. So, the variation will be like the first case, that is undamped case. It will be a harmonic variation, but the amplitude will not remain same, but it will keep on decreasing because of the presence of the exponential decay function.

So, how the combined effect will give us a harmonic function with an exponential decay of the amplitude? So, easy way to draw this one, if we can first draw the envelop for the exponential decay. Suppose if this is the envelop for exponential decay, so this is envelop for e to the power minus eta omega t.

So, this is the decay function, exponential decay function I have drawn, what will happen? This will start oscillating between these two and finally, with increase in time it will die down because of the presence of this exponential decay function. So, it is a harmonic function with decay in the amplitude, as the time is increasing. So, that will be the variation for… This is the variation for un damped free vibration and from this point to this point, what is this called? This is T d, T d is nothing but 2 pi by omega d, which is called under damped, T d is under damped period of vibration.

One more thing I want to inform you, in several books you will find, for several problems in many cases it is mentioned damped period of vibration, so it is a common terminology. Most of the case we do not use the complete word under damped, but damped period of vibration for the case of free vibration means under damped case only.

Generally, the over damped case we do not consider because there is no oscillation. So, the getting the value of T d cap does not arise, so that is why the damped period of vibration if it is asked, that atomically means it has to be under damped period of vibration.

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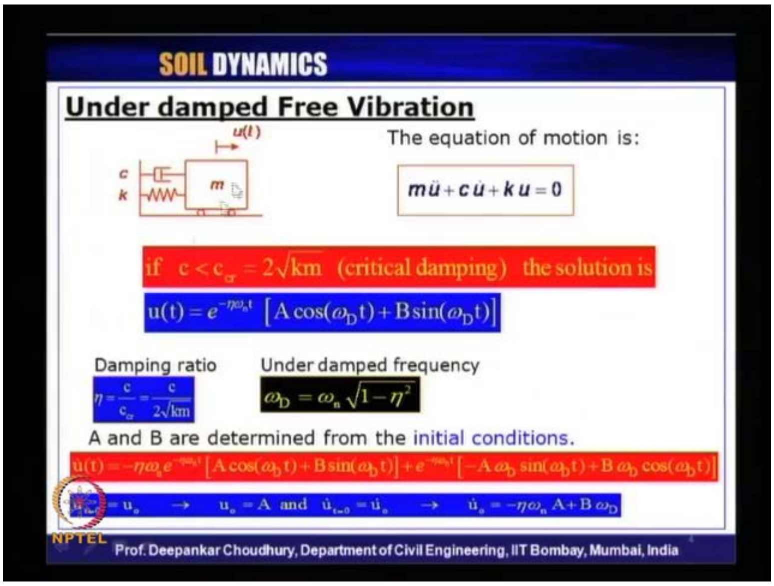

So, now let us look at the slides with all these results what I have obtained just now. Let us look back at the slide, the last solution only is shown here, the under damped free vibration case, with the condition of c will be less than the critical damping constant C c r. The solution is of this form and under damped frequency is defined like this and the displacement function, and after differentiating the velocity function.

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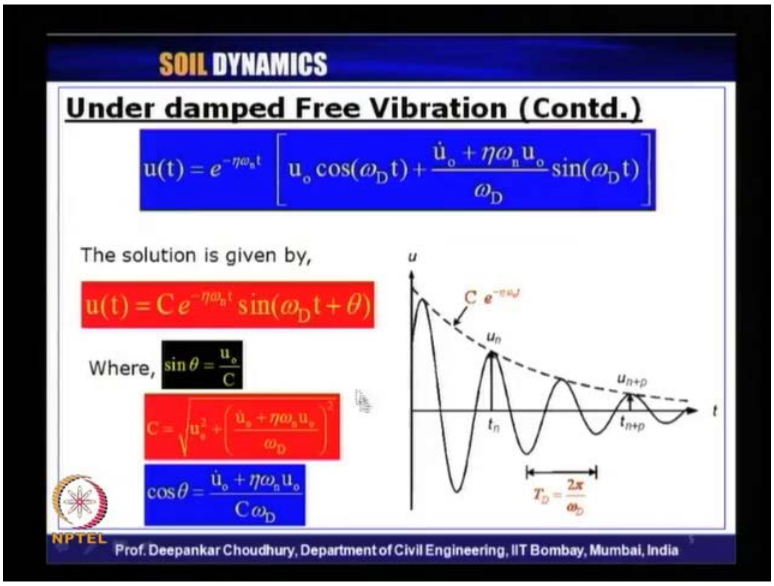

Then using the initial conditions we get the combined solution in this form, which can vary like this. We have seen how the envelop varies, this is the exponential decay and the harmonic function. So, we will continue our lecture in the next class, we will stop for today.