Soil Dynamics
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Module - 2
Vibration Theory
Lecture - 4
Problems on Tortional Motion

Let us start with today's lecture on soil dynamics. Let us look at the slide. So, we are continuing with our module 2 on vibration theory. Let us recap, what we have done in the previous lecture.

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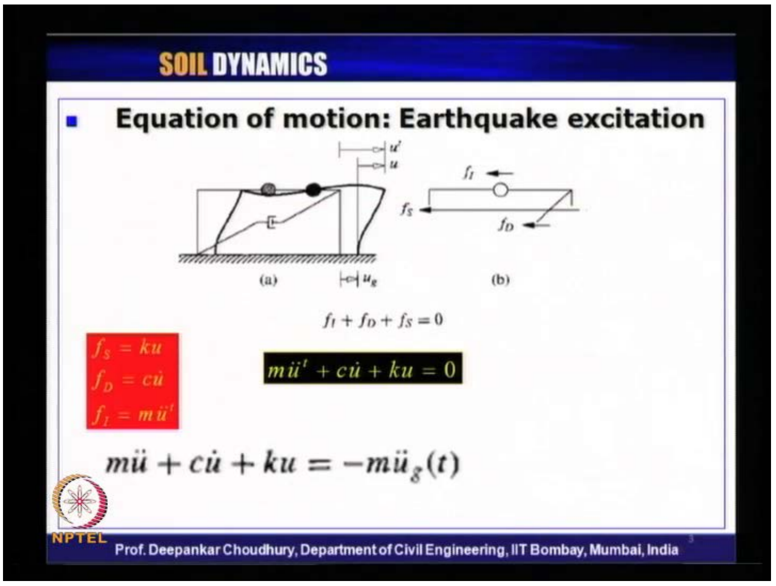

The major points were, we had discussed about the formulation of equation of motion when there is earthquake excitation and we had identified that it is a forced vibration case with earthquake ground acceleration u g double dot t at ground level. Then, the basic equation of motion for a single degree of freedom has been obtained as m u double dot plus c u dot plus k u equals to minus m u g double dot t, where this u is the difference between the ground displacement and the total displacement observed at mass level. So, this is the relative displacement, what we call it and this u g double dot t is the ground acceleration due to earthquake shaking. So, that creates the force vibration for this type of problem.

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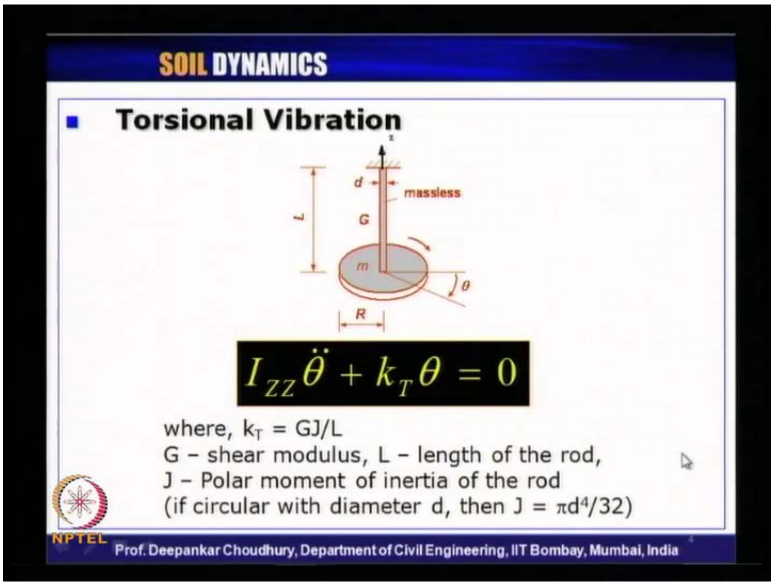

We have also seen the equation of motion for the case of Torsional vibration. Suppose this circular disc of radius R with mass m, if it is vibrating with a small rotational vibration or Torsional vibration with angle theta and this is the rigid rod connected to the disc with circular cross section and diameter of d and length L. We have seen the equation of motion is I z z theta double dot, where I z z is mass moment of inertia for this circular disc above this axis Z, plus K T theta, where K T is the Torsional spring constant equals to 0 and theta is the rotational displacement. How to compute the expression or value of K T, and that is to compute the Torsional spring constant? The expression is G J by L, where G is the shear modulus, L is the length of the rod and J is the polar moment of inertia of the rod.

If we consider the circular cross section of the rod, in that case, with diameter d for the circular rod, we can compute the polar moment of inertia J as pi d to the power 4 by 32. So, with this back ground of yesterday's lecture, let us continue to solve our problems what we had been solving.

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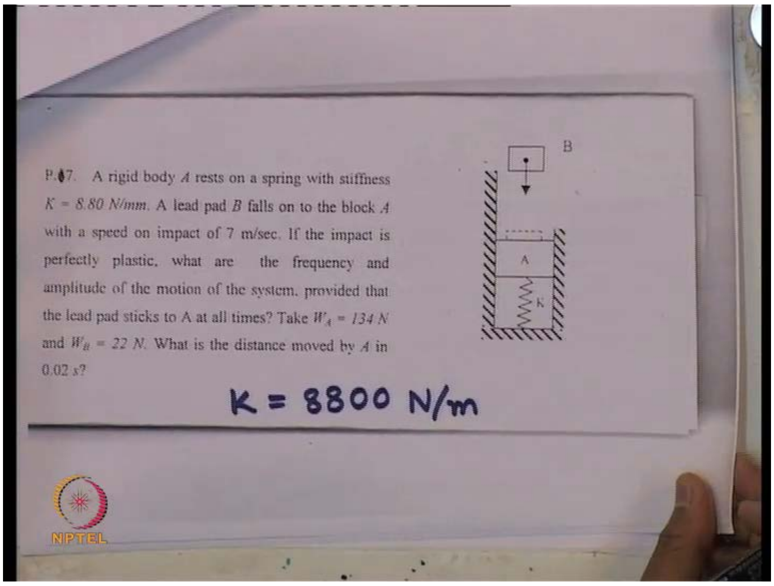

Now, for the next problem number 7, here is the picture. It shows a rigid body A. So, this is the rigid body A rests on a spring. It rests on a spring here with stiffness K value has 8.8 Newton per millimeter. A lead pad B, this is the lead pad B, which falls on to the block A with a speed on impact of 7 meter per second. So, this lead pad is falling on this mass or rigid body A with an impact velocity or speed of 7 meter per second.

If the impact is perfectly plastic, what are the frequency and amplitude of motion of the system, provided that the lead pad sticks to A at all times. The weight of rigid body is given as 134 Newton and weight of lead pad B is given as 22 Newton. What is the distance moved by the rigid body A after 0.02 second of the impact. So, that also has been asked in this problem.

So, what does it mean? The impact is perfectly plastic; that means, after the impact, the lead pad B does not bounce back. It sticks to rigid body A; that means that impact is perfectly plastic. There is no elastic component and there is nothing comes out or it did not bounce back. That is the meaning of perfectly plastic.

In this problem, look, the unit of K is not in SI unit. It is given 8.8 Newton per millimeter, which is in SI unit is L is 8800 Newton per meter. We have to convert this as we have mentioned all SI units we considered. Now, if we consider the equilibrium of momentum at the point of impact, what it should give us? The mass of lead pad B times the velocity or the speed with which the lead pad is striking with the impact to A, should be equals to the momentum after the impact when the combined mass is mass A plus mass B times the velocity.

So, that velocity which is unknown to us right now has to be computed because, it will be the initial velocity for the vibration. Because, as we know, for solving a vibration problem, to find out the constant, we should know the initial condition completely. So, if there is no initial displacement, the initial velocity has to be known. So, let us see if there is initial displacement or initial velocity or not.

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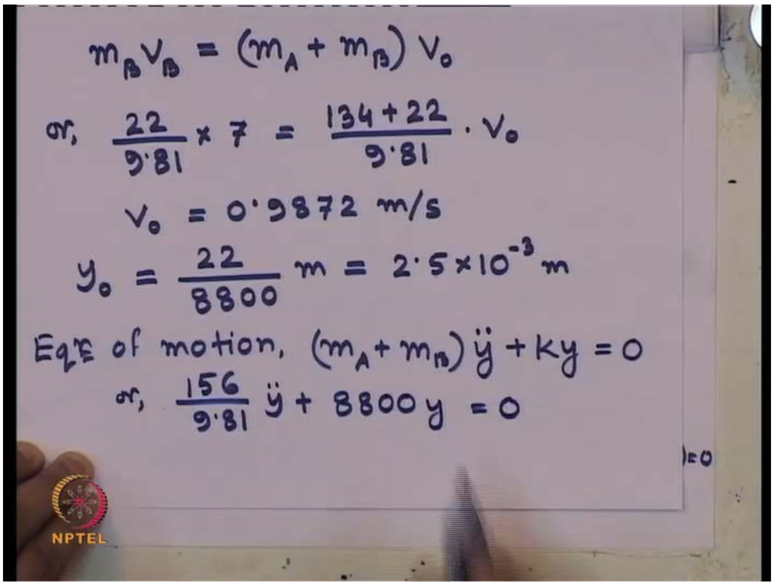

So, what we can write from moment equilibrium condition equilibrium of the momentum, that mass of B times the velocity with which it is striking should be equals to mass of A plus mass of B combined together because, after the impact, they are together, times the velocity, let us say V naught, this is the initial velocity for the vibrating system. So, in this all parameters are known accept V naught.

So, let us put the values here. In B is how much? 22 by 9.81 V B speed of impact is 7 equals to 134 plus 22 by 9.81 times V naught, which gives us V naught. As we solve this, we will get 0.9872 meter per second. Now, when the led pad is striking rigid body A, this is the initial velocity. Let us see if there is any initial displacement or not. What should be the initial displacement? If we denote it as y naught, that will be nothing but, the weight which is coming down and the stiffness of the spring from which we can find out the initial displacement.

So, 22 divided by, that is w by k. So much of meter. 2.5 into 10 to the power minus 3 meter is the initial deflection. So now, the initially condition of vibration is known. This is the weight of the lead pad B, which is causing the vibration, which is coming. Why the weight of A is not considered because, that was already with equilibrium with the spring. So, the initial condition starts because of the impact of the lead pad B coming on A.

So, that creates the extra deflection, which is nothing but our initial condition of displacement. So, the equation of motion will be mass of A plus mass of B. They are now vibrating together at acceleration; let us say y double dot plus k spring constant times y equals to 0. There are no other forces acting on it. Now, from this we can write 156 by 9.81 y double dot plus 8800 y equals to 0. This is our equation of motion. From this, we can find out what is the natural frequency of the system.

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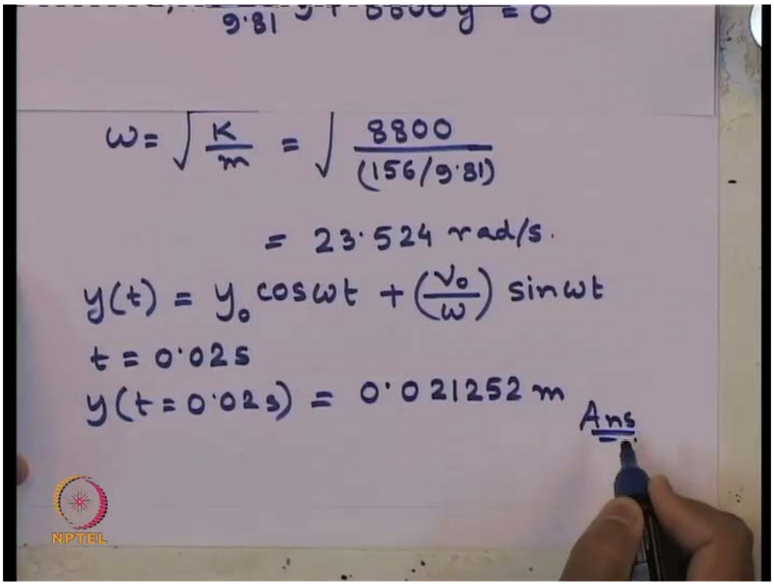

So, natural frequency omega is root over k by m. In this case, k is 8800 and m is 156 by 9.81. So, if we solve this, it comes out to be about 23.524 radian per second. For such undamped free vibration, this is nothing but an equation of motion for undamped free vibration. There is no damper and there is no externally applied dynamic load.

So, for such system, what is the solution? We know y of t equals to the initial displacement y naught cosine omega t plus initial velocity by omega sine of omega t. That is the solution we have see in the previous lecture. So now, y naught is known to us, initial displacement, and initial velocity V naught is also known to us. Omega, we have computed just now. What it has asked? After t equals to 0.02 second, how much is the displacement or the deflection of the system.

So, y at t equals to 0.021 second. If we put in this expression, we will get the value. It comes up to be about 0.021252. So much of meter. So, that is the answer of the problem. Now, before we move to the next problem, that is problem 68, we need to know the expression to compute for Torsional vibration. I have though discussed about the Torsional vibration in one of the previous lectures, let me show in the slide here.

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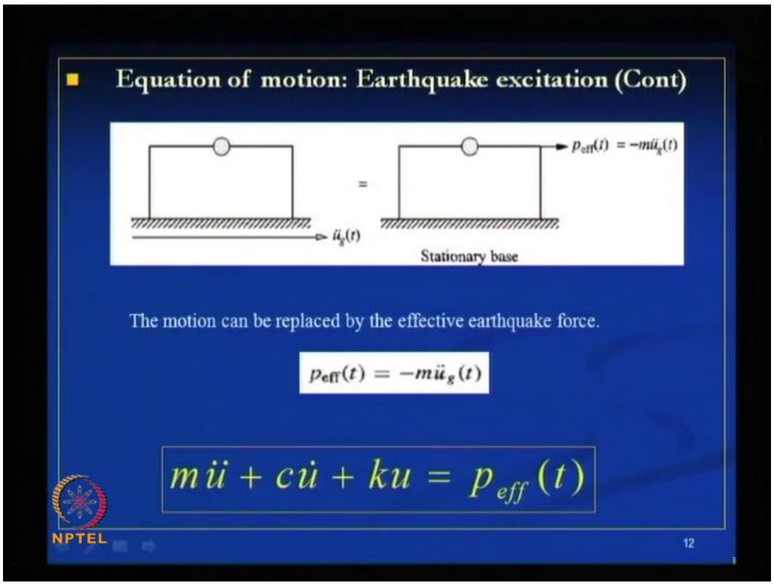

If there is a Torsional vibration for a plate like or disc like this, circular disc connected with a rigid rod, we are assuming that rod is mass less and the disc is having a mass of m and a radius of R and the diameter for this rod, which is connecting the disc to a fixed end is small d and length of the rod is capital L. In this type of rotational vibration or Torsional vibration, for a small rotation of theta, the equation of motion that we have seen earlier also, it can be expressed as I z z theta double dot plus K T theta equals to 0, where I z z is moment of inertia above this z axis for the disc. How to compute the K T? That I have not mentioned earlier. This K T can be computed as G J by L.

So, expression for K T is G J by L. What is G? G is nothing but, the share modulus. L is length of the rod and J is polar moment of inertia of the rod. So, if we have a circular rod with diameter d, in that case the expression for J will be pi d to the power 4 divided by 32. Fine.

So, using this expression, the Torsional spring constant can be computed for a Torsional vibration like this. So, after knowing this, now we can proceed to our next problem, problem number 8. Now, let us go to the problem number 8.

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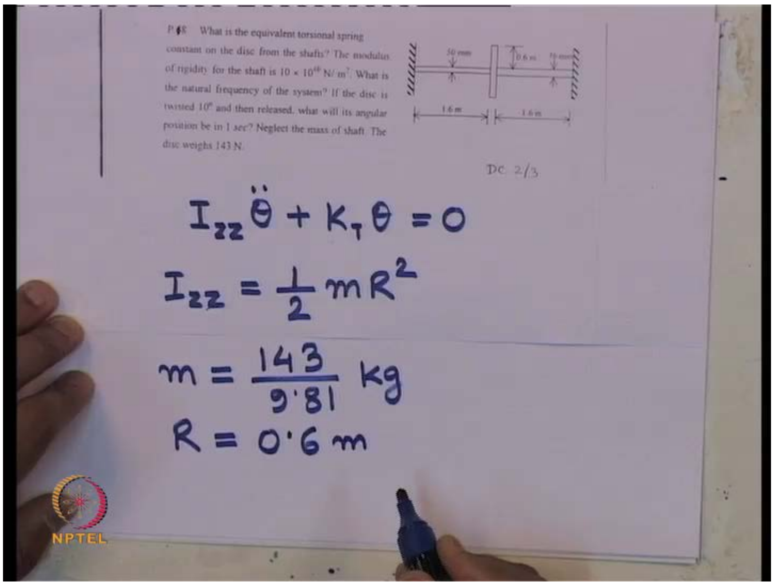

So, in this figure, what it says, what is the equivalent Torsional spring constant on the disc from the shafts. So, the middle portion, this section, this is nothing but a circular disc connected with two shafts of varying diameter. One is 50 mm diameter and another is 76 mm diameter. The modulus of rigidity for the shaft is given as 10 to the power 11 Newton per meter square. What is the natural frequency of the system if the disk is twisted 10 degree and then released, what will be its angular position after one second of vibration? What is mentioned, that is neglect the mass of the shaft.

So, mass of these two shafts can be neglected and disk weighs 143 Newton. So, weight of this circular disk is 143 Newton. So, how we can solve this problem for undamped free vibration case? So, in this case, the equation of motion as we have learned just now, I z z theta double dot plus K T theta equals to 0. This is the equation of motion. Now, how much is I z z for a circular disc? That is half m R square. That is the mass moment of inertia of a circular disc. Now, how much is the mass of the circular disc? Weight is 143 Newton. So, 143 by 9.81 kg. How much is the value of R? That is radius of the circular disc, which is given as 0.6 meter. So, with this information, now let us find out the K T due to the presence of the two circular shaft.

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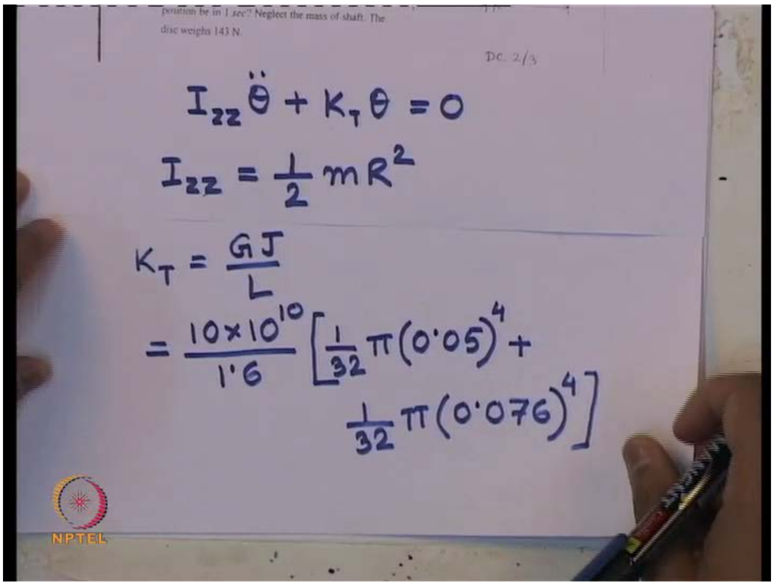

So, K T, expression for K T, we have seen that is G J by L. So, from two shafts, it will be, how much is G? G is constant for both the shafts. It is 10 into 10 to the power 10. Length of both the shafts are equal. So, that also we can take common. 1.6 meter, but the radius or diameter of the two shafts are different.

So, J is how much? 1 by 32 pi d to the power 4. For one shaft, the d is 0.05 meter to the power 4 and for the other shaft, 0.076 meter is the diameter to the power 4. So, that gives us the value of the Torsional spring constant for this problem. So, what next? We can put these values of I z z and the value of K T in our governing equation of motion and let us see how it looks like.

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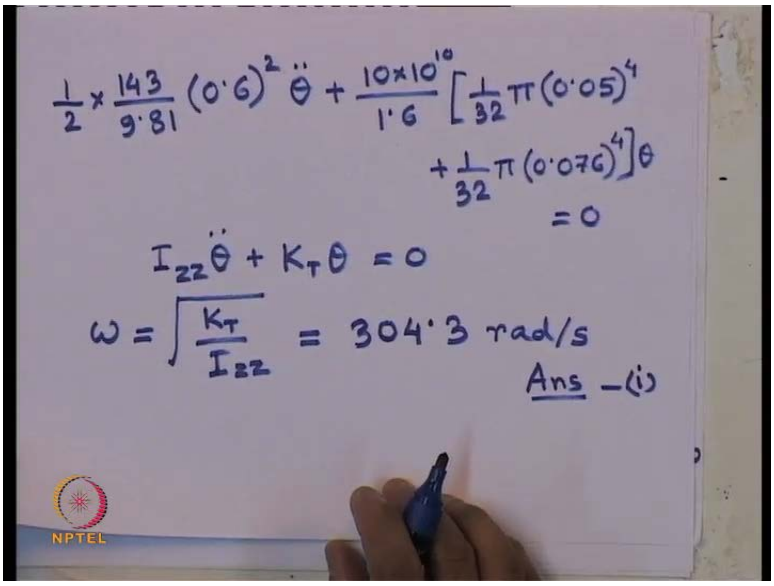

So, half into 143 by 9.81, that is m R square, half m R square is the I z z theta double dot plus 10 into 10 to the power 10 by 1.6 into 1 by 32 pi 0.05 to the power 4 plus 1 by 32 pi 0.076 to the power 4 theta equals to 0. This is the equation of motion, which can be written in our known form, and that is from which have actually written this expression K T theta equals to 0. Now, how to compute the natural frequency of the system? That is given by root over K T by I z z.

So, expression for K T is known to us and expression for I z z is known to us. If we put these values in this equation, we can find out what is the natural frequency of the system. After computation, it comes out to be 304.3 radian per second. So, that is what it was asked. Wwhat is the natural frequency of the system? This is the natural frequency of the system. This is answer of part 1 of the problem. Now, in the second part, it says that if the disk is twisted by 10 degree initially and then released, so, in this case of free vibration and initial rotation has been provided.

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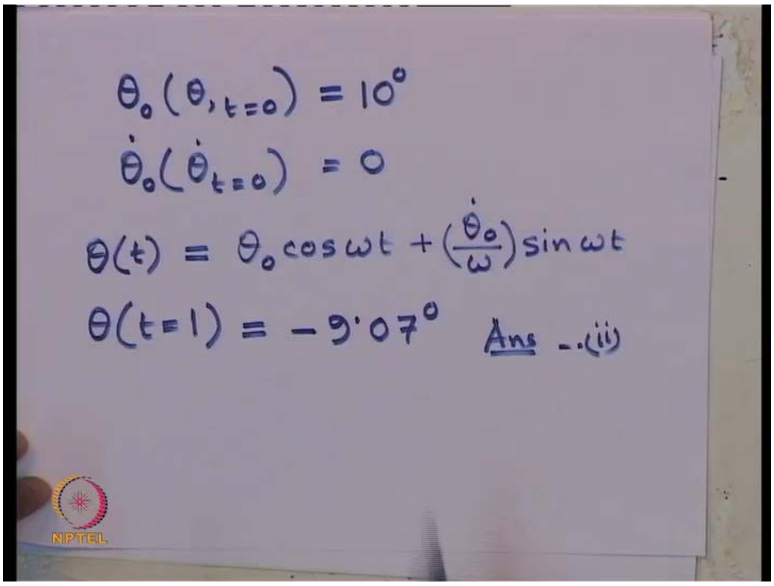

So, theta naught, that is theta at t equals to 0 is given to us as 10 degree and nothing is mentioned about the velocity. So, we can consider theta dot naught that is the velocity at time t equals to 0 as 0. Now, what is the solution from the governing equation of motion? The solution for un damped free vibration is theta t equals to theta naught cosine omega t plus theta naught dot by omega sine of omega t. That is the complete solution using the initial conditions of theta naught and theta naught dot. That derivation we have seen earlier.

So, what is asked in this problem? That theta at the end of t equals to 1 second is how much. So, we have to just put in this expression the value of t as 1 second, and value of omega is known to us now. Just now we have computed. It is 304.3 radian per second and theta naught is given to us as 10 degree and this one is 0. So, if you put the values and get the solution, it will be coming as minus 9.07 degree. So, that is the final answer, because it asked what is the angular position at the end of 1 second. This is the angular position after 1 second. What this minus sign indicates? That after 1 second, the rotation of the disk will be in the opposite direction of its initial rotation. Suppose the initial rotation was given in clock wise direction and that has been considered as be positive. So, that initial 10 degree was considered as positive. In that case, after 1 second, its rotation will be in the anti clock wise direction of the magnitude of 9.07 degree. That is what the minus sign indicates. As we now, the variation of theta of t with respect to time will be cyclic. So, it will go from positive to negative and things like that. That is what the solution also indicates. In this case, now let us move to the next problem, problem number 9. So, let us look at the problem.

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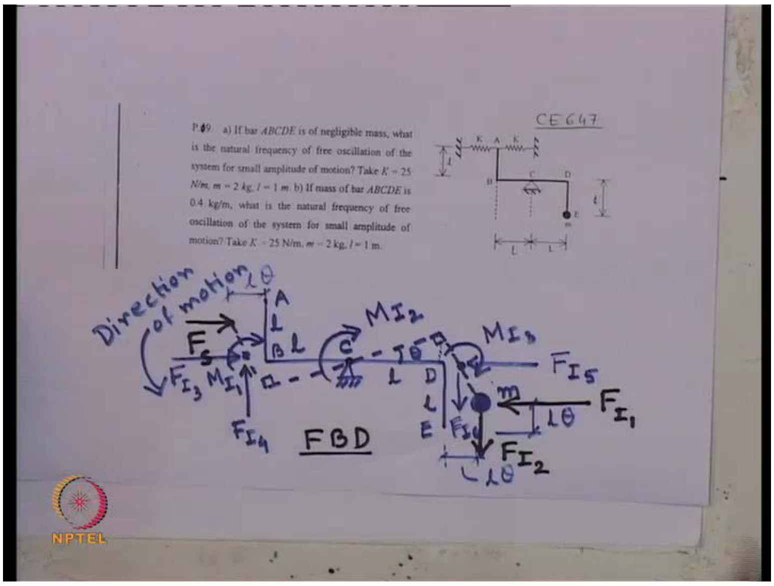

So, this is the picture. It says that, if this bar ABCDE is of negligible mass, that is no need to consider the mass of this bar, what is the natural frequency of free oscillation of the system for small amplitude of motion? It is mentioned that take the value of K as 25 Newton per meter. Mass of this bob, that is concentrated mass at this end E is 2 kg and length of each of this arm for this bent rod is given as 1 meter.

So, that is the first part of the problem. So, let us see how we can solve this one. To do that, first we need to draw the free body diagram. Let us draw the free body diagram of this problem for a small angular rotation. That is what it is mentioned. For a small angular rotation, we have to derive the equation of motion. Let us say that this is the small rotation we are providing about this hinge point and about this point C.

So, this will be the displaced position after providing a small rotation. So, this is our direction of motion we have given to the system. This point D has moved up to this place, and point E has moved to this plus, where point mass was attached to m and there was a spring connected to it. So, what are the forces acting? Let us denote this armed length was L, this is also L, this length is also L and this length is also L. This point was point B and this point was point A. For a small rotation, say theta, as this is the rigid bar, it will try to maintain 90 degree at this junctions also. Now, this mass has displaced by both horizontally as well as vertically. How much it has been displaced? This mass has been displaced by, this is its vertical displacement and this much it has been moved up.

So, how much is the magnitude? This will be L theta for small theta. How much it has moved horizontally? It has moved horizontally by again L theta, because these dimensions are L, so, this angle will be L theta and this angle also will provide us the distance of vertical as L theta, this horizontal distance as L theta. At this point, the horizontal movement of at spring, that is also l theta. Now, these are linear spring. So, they will be subjected to spring force only because of this linear displacement along the direction of the spring.

So, what are the forces coming into picture? This rod ABCDE is not having any mass. So, no question of arising of mass moment of inertia for that rod. But, this concentrated mass M will be subjected to inertia forces, because it has been displaced. So, it will have acceleration both in vertical direction as well as horizontal direction, because it has two directions of movement. So, there are two inertia forces. The horizontal inertia force, say F I 1, this will act in the direction from right to left. Why because, the bob will try to go back to its original position. So, the inertia force will try to bring it back to its original position. So, that is why the direction of the horizontal inertia force is in this direction.

The vertical inertia force will act vertically downward direction. Let us say that is F I 2. For similar reason, that vertical inertia will try to bring the mass back to its original position. So, that is why it will act in this direction. The spring force, spring force F S will act in the direction from left to right, because it will try to bring it back to its original position. So, this is the complete free body diagram of this problem. Now, let us compute all these values and then, for equilibrium, we have take moment about this hinge point C. So, moment about point C should be 0.

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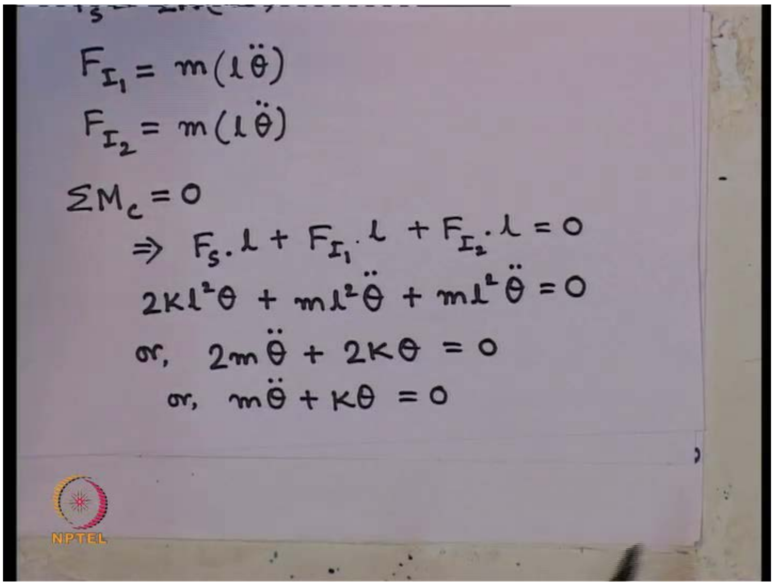

So, the spring force F S is how much? Spring force will be, let us look back to the problem. The spring force will be 2 K times l theta. F I 1, that will be mass of that point, and mass given to us times acceleration. Acceleration in the direction of, F I 1 we have taken horizontally. So, in that direction of horizontal direction, whatever acceleration it is experiencing. So, how much is the horizontal displacement is l theta.

So, horizontal acceleration will be l theta double dot. The vertical inertia force F I 2, vertical inertia force it has been displaced by vertically with the amount l theta. So, vertical inertia force will be m times l theta double dot. Now, we are writing momentum equilibration equation about point C, which gives us the spring force times l plus F I 1. That is horizontal inertia force times l plus F I 2 times, this length l equals to 0 and all are producing clock wise movement about point C.

So which, now if we put the expression for each of them, we will get 2 K l square theta plus m l square theta double dot plus m l square theta double dot equals to 0. Or, we can take out l square and we can write it as 2 m theta double dot plus 2 k theta equals to 0 or m theta double dot plus k theta equals to 0.

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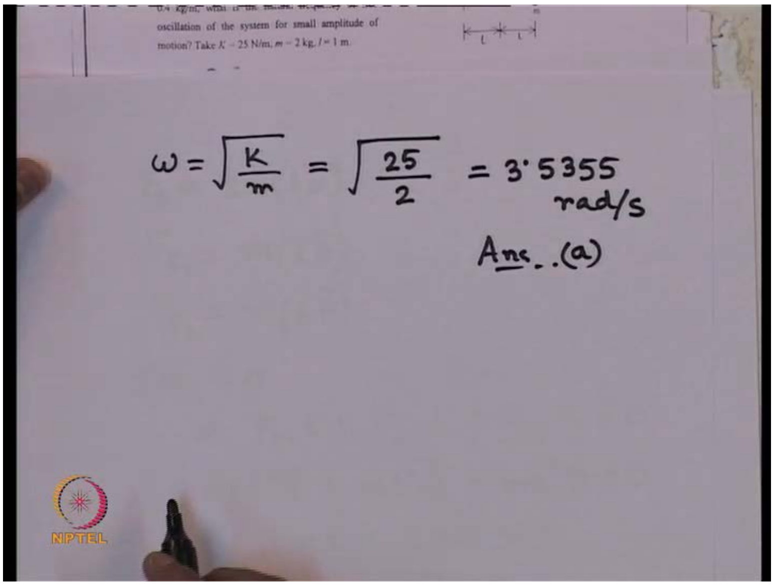

So now, to find out the natural frequency of the system, omega is root over k by m. Now, values of k and m both are given to us. k is 25 m is 2 kg and it will come as 3.5355. So much radian per second. So, that is the answer for part A. Now, let us us go to the next part of the problem, part B. Let us come back to this figure again. What it says in the part B? If mass of the bar ABCDE is 0.4 kg per meter, so, the uniform mass for the bar is given, that is per meter length. How much is the mass for that bar? What is the natural frequency of free oscillation of the system for small amplitude of motion? So, the same problem, what we have solved in part A, the only difference in part B is, now we have to consider the mass of this bent rod. What does it mean? The moment of inertia for the bent rod also we have to consider. So, in this free body diagram, what are the changes or the additions will be there? Let us see.

So, this portion of the rod, that is B C D, this segment, it has been rotated about its center point, because length are equal. See, it has only rotated. It has not been displaced horizontally or vertically. What does it mean? When we are taking mass moment of inertia for this rod, it will come only the Torsional inner force, which will try to bring it back to its original position. So, we will have inertia force for this segment B C D acting in this direction. Let us say, that is M I 2. 2, I am using for this segment. Now, for this segment A B, what has happened to this bar about its center? Its center has been rotated as well as it has been moved both horizontally as well as vertically. So, about its center point, what are the inertia forces will act? The Torsional inertia, let us us denote it as M I 1, which will try to bring it back to its original position. So, that is why the direction.

Now, we will have a vertical inertia force acting vertically upwards, because this point has moved down. So, it will try to bring it back to its original position. The horizontal inertia force in this direction, it will try to bring it back to its original position. So this, let us denote it as F I 3 and this as F I 4. So, for the segment A B, F I 3 is the horizontal inertia force, which will try to bring it back to its original position and all these are acting at the center of the bar A B. F I 4 is the vertical inertia force, which is acting vertically upward to bring it back to its original position and M I 1 is the Torsional inertia force, which will try to bring it back to its original position.

Now, looking at the segment D E, this also has been displaced about its center both horizontally and vertically as well as it has been twisted. So, for this, again we will have a Torsional inertia force M I 3 in this direction, so that, it brings back to its original position. The horizontal inertia force in this direction F I 5 and vertical inertia force in this direction F I 6, it will bring it back to its position. This will bring it back to its position and this will bring it back to its original position. Now, what are the expressions for this new forces coming because of the mass of that bent rod ABCDE? Let us write the expressions for each of them.

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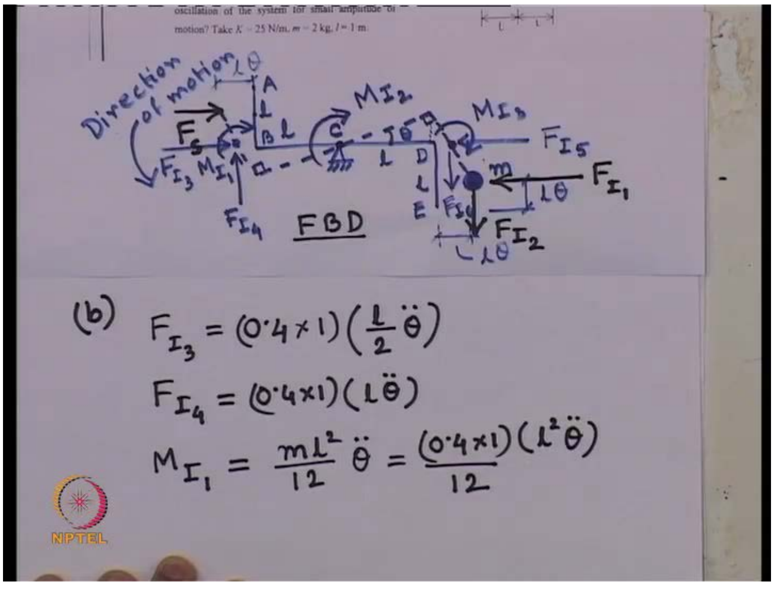

If we look at this free body diagram here, so, F I 3 is how much? mass times acceleration in the horizontal direction. Now, horizontal direction, how much it has been displaced about its center? l by 2 theta. Now, what is the mass of this segment A B? The length l is given as 1 meter and the mass per meter length is given as 0.4 kg per meter. So, the total mass is 0.4 kg itself times 1. I will write for clarity mass times acceleration. How much is the horizontal acceleration we have found out? l by 2 theta double dot is the acceleration in the horizontal direction. How much is F I 4, that is vertical inertia force? Mass times vertical acceleration and how much is the vertical displacement for this center point of the rod? That is l theta, vertical displacement is l theta. Vertically it has moved l theta.

So, center also has vertically moved l theta. So, acceleration is l theta double dot. Because look at the center. End point horizontal displacement is l theta. So, about center, this displacement is l by 2 theta, whereas vertically this has moved because of this movement, that is shifting of point B and that was l theta. So, entire rod A B has moved down by l theta. So, that is why center also has experienced the displacement of l theta. So, that is why the vertical acceleration is l theta double dot and not l by 2 theta double dot like horizontal one.

So, we to look at the picture very carefully in the free body diagram while writing this expression for inertia forces. How much is M I 1? We have learnt earlier that Torsional moment of inertia, mass moment of inertia that will be M l square by 12 above the center for any rigid bar times theta double dot, which means 0.1. 0.4 times 1 is the mass and l we will keep here, l square theta double dot and that is our M I 1.

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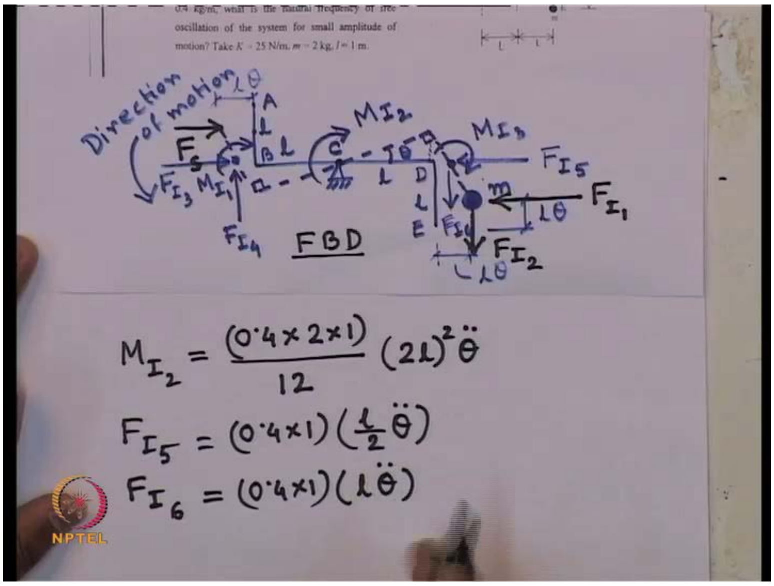

Now, let us see, what are the other inertia forces, new inertia forces which we have introduced? For the central bar, that is B C D, we have only M I 2. How much is M I 2? That will be 0.4 times length of B C D is 2 l. So, 2 into 1 M by 12 l square theta double dot. How much is l for this case? 2 l. So, this is M I 2 for the bar B C D.

Now, for the bar D E, the inertia forces, we can now easily write down quickly. F I 5 will be 0.4 times 1 n times acceleration. So displacement, horizontal displacement of this bar D E about its center is again l by 2 theta. So, acceleration is l by 2 theta double dot. F I 6, that is vertical inertia force is M times the vertical movement of the bar about its center, and that is again l theta. So, acceleration is l theta double dot.

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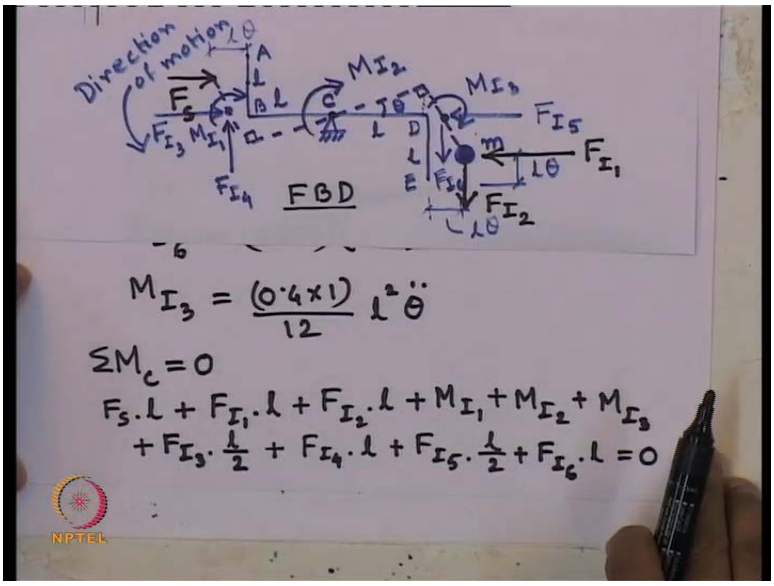

The Torsional inertia force about its center M I 3, that will be M by 12 l square theta double dot. So now, with all these forces, we have to again use the condition of equilibrium about that hinge support point C moment equilibrium equals to 0. So, what we can write down? This equation will now be F s times l clock wise plus F I 1 times l, that is also clock wise, plus F I 2 times l, that is also clock wise plus M I 1, it is also clock wise, plus M I 2 clock wise plus M I 3 clock wise plus F I 3 times, how much is this lever M? This lever M is l by 2 for this force, which is acting at the center of the bar, plus F I 4 times, its lever M is l, plus F I 5 times l by 2 plus F I 6 times l equals to 0. Now, all the expressions for these forces are known to us, so, we can put them in this expression.

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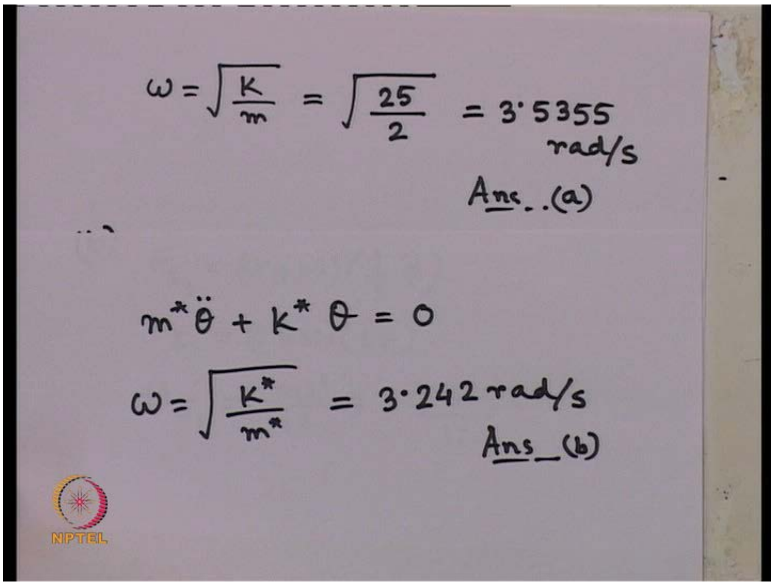

Then, we can put it in our canonical form of m star theta double dot plus K star theta equals to 0, and then compute the natural frequency root over K star by m star. If we put all the expressions and do the calculation, we will get the natural frequency is coming as 3.242 radian per second; so that is the answer for part B. Now, if we compare the natural frequency obtained in case A and case B, if I keep it here both the results, we can easily see in the first case, the natural frequency was 3.53 and in the second case, it is reduced to 3.24 which is quite expected because, the mass has increased. So, this is a kind of cross verification that whether our analysis or calculation is correct, because with increase in mass, natural frequency has to decrease. Now, with this we have come to the end of the case for undamped free vibration. So, we will continue our lecture in the next class. We will stop for today.