Solid Dynamics
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Module - 5
Machine Foundations
Lecture - 30
Torsional Mode/Yawing Mode, Constant Force Type Excitation, EHS Theory

Let us start our today’s lecture on soil dynamics. We are continuing with module five, that is, machine foundations.

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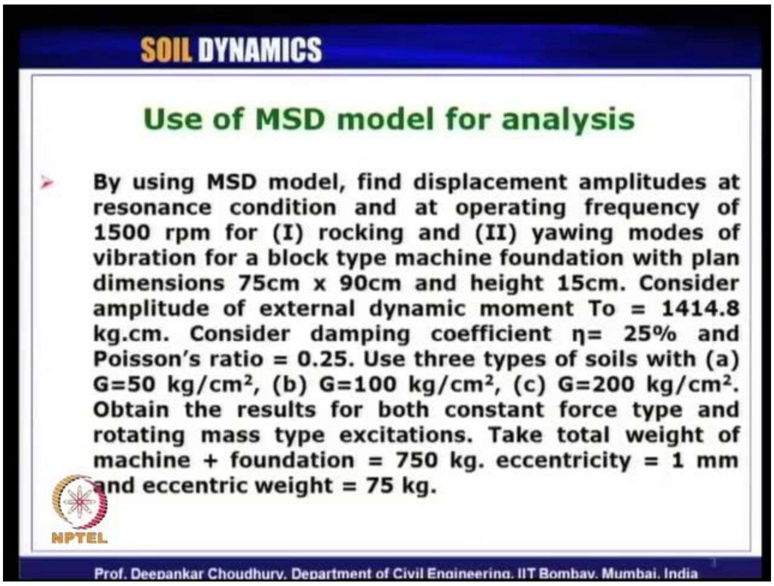

A quick recap of what we had studied in the previous lecture; we have solved the problem using mass spring dashpot model for computing the displacement amplitude at both resonance condition as well as at operating frequency condition for rocking and yawing mode of vibration. Let us now move to yawing mode of vibration. Yawing is nothing but torsional mode of vibration.

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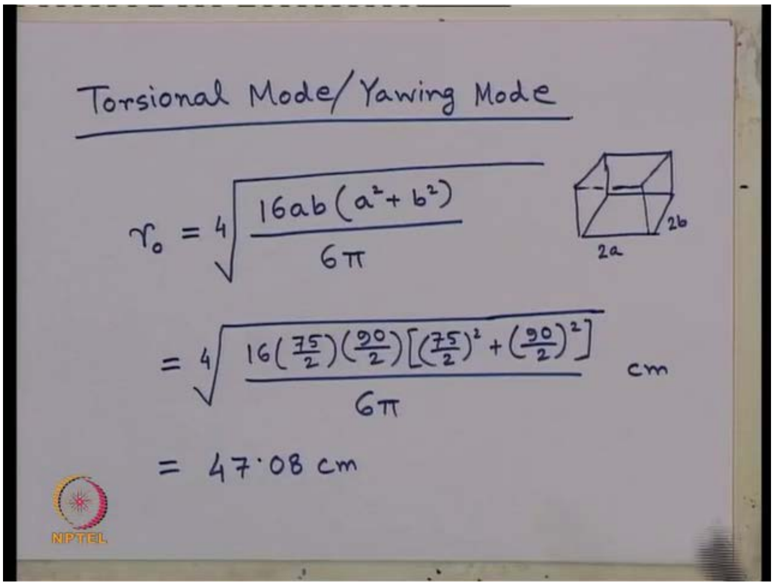

So, for torsional mode of vibration, let us see how the expressions, etcetera changes. Torsional mode or yawing mode of vibration – in this case, again the projected area has to be compared with respect to the moment of inertia – area moment of inertia for a circular footing and a rectangular footing. And the formula is given by this – r naught is obtained as fourth root of 16 ab a square plus b square by 6 pi. This is the formula, where planned dimensions are given as 2a and 2b that is what is a and b.

So, for the case of Torsional or yawing mode, this is the way we calculate the equivalent radius of a circular footing by equating the moment of inertia – area moment of inertia. So, if we calculate for our case, 16 – a is 75 by 2 – 90 by 2 – 75 by 2 whole square plus 90 by 2 whole square by 6 pi; so much of centimeter, because everything is in centimeter. Let us calculate this how much r naught we are getting; 47.08 centimeter. Look at here; this value of r naught neither same as for vertical and sliding mode of vibration nor same as rocking or pitching mode of vibration. So, in different modes of vibration, the equivalent radius of a circular footing will change. So, we need to be careful while starting or calculating a design problem. We have to find out everything with respect to an equivalent circular footing, because all our theory of elasticity formulae to compute the stiffness are given based on this r naught value.

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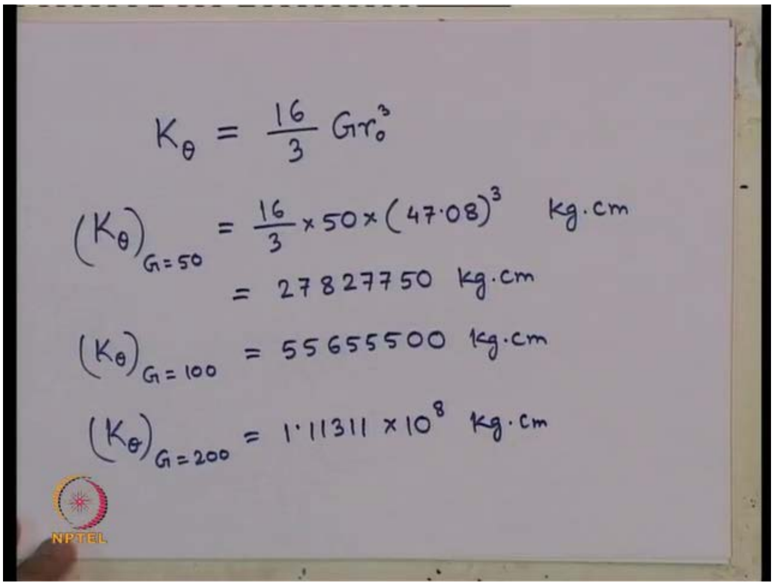

For the torsional mode case, what is the expression for k – as the symbol, let us use k theta. For torsional or yawing, we are now using theta. k theta is calculated using this expression 16 by 3 G r naught cube. So, from theory of elasticity, once again this expression is given and Timoshenko and Goodier. k theta can be calculated knowing the soil shear modulus and radius of equivalent circular footing. So, k theta – one observation you can note; in this case, it is not a function of Poisson's ratio of the soil unlike the vertical sliding and rocking mode of vibrations.

So, for three different types of soil given to us, let us compute the value of k theta; 16 by 3 – G is 50 and r naught is 47.08 whole cube. What should be the unit? kg centimeter again, because this is Torsional spring constant; 27827750 – so much of kg centimeter. Similarly, for other two values of G, other two types of soil, they are linearly related to G. So, this will be double of this; 55655500 kg centimeter. And k theta for G equals to 200 will be double of this; so 1.11311 into 10 to the power 8 kg centimeter. And how to calculate the mass moment of inertia? Next step is to calculate mass moment of inertia, so that we can calculate the natural frequency of the system.

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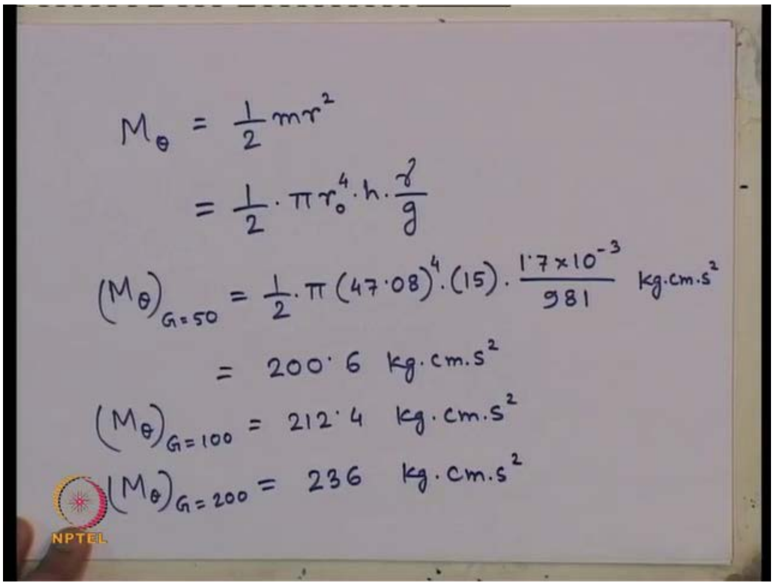

So, mass moment of inertia M theta for this torsional mode or yawing mode is given by half m r square. For a circular disc, the torsional mass moment of inertia is given by half m r square; we know this. So, half m will be pi r naught to the power 4 h gamma by g. m is how much? Pi r naught square h into rho. Am I right? pi r naught square h is the volume of that circular foundation times rho is the density will give us the mass. So, that is why again another r square is there; so r naught to the power 4. So, M theta for first type of soil – half pi – r naught is 47.08 to the power 4 – h is 15 centimeter – gamma we had already assumed 1.7 10 to the power minus 3 by G is 981; unit will be again kg centimeter second square. Now, let us calculate this; how much it is coming? 200.6 kg centimeter second square. In the same way, we can get M theta for other two types of soil for G equals to 100. These values remain all same except the change will be in the change in the value of gamma. So, 212.4 kg centimeter second square and M theta for G equals to 200 will be 236 kg centimeter second square.

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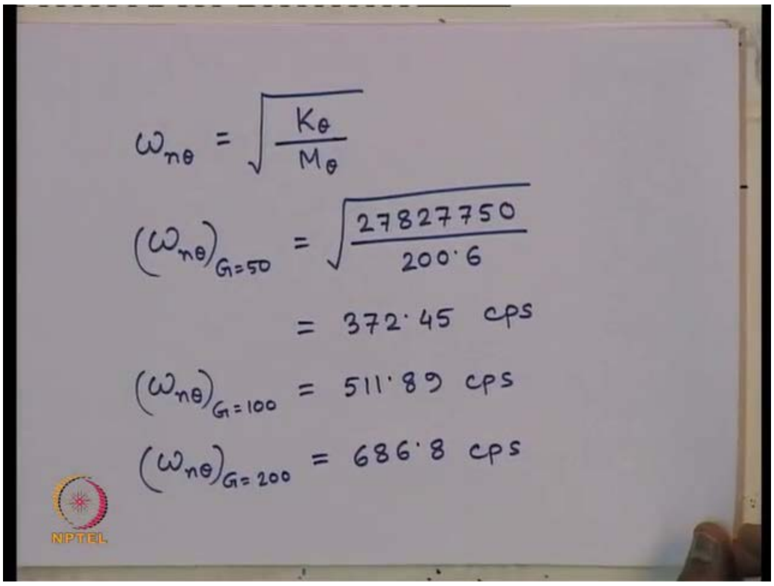

Now, next step is to calculate the natural frequency. So, omega n theta can be calculated using this k theta by M theta; that is, Torsional spring constant by mass moment of inertia with respect to torsion. So, omega n theta for G equals to 50; first type of soil will be… k theta for first type of soil, how much we got? 27827750 and M theta for first type of soil we have calculated 200.6. If we simplify this, how much we will get? It is coming about 372.45 cycles per second. For next type of soil, k theta will be double; and M theta we have computed for second type of soil – 212.4. So, by putting those values, we are getting 511.89 cps. And omega n theta for G equals to 200 will be 686.8 cps. So, once we obtain the natural frequency, now, we are ready to compute the value of displacement amplitude for both the modes of excitation.

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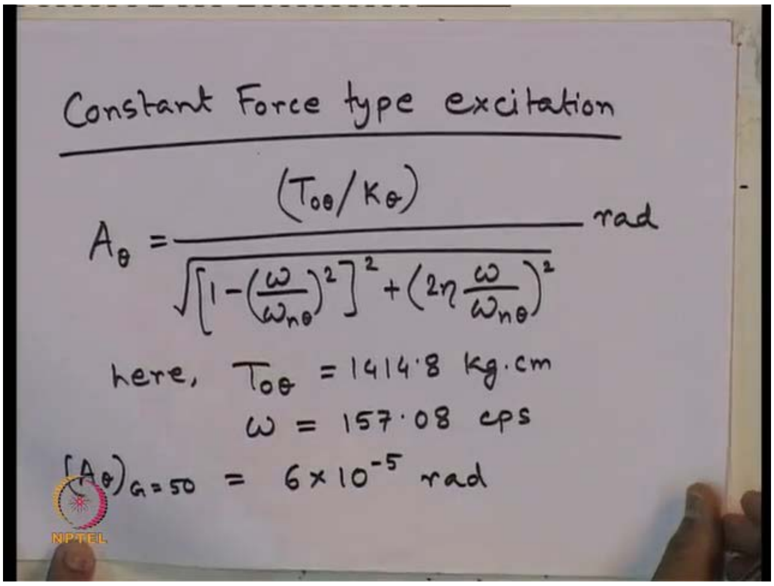

Let us start with constant force type excitation. Constant force type excitation will have the expression to compute the displacement amplitude A theta, is nothing but T naught theta by k theta divided by root over 1 minus omega by omega n theta whole square that square plus 2 eta omega by omega n theta whole square. Again the unit will be guided by this ratio, which will come as radian. Now, this t naught theta is given to us. Here t naught theta is 1414.8 kg force centimeter; and omega, that is, operating frequency we had calculated – 157.08 cps. So, by putting this value, we will get A theta for all the types of soil G equals to 50, we will put this value. How much we will get? We are getting 6 into 10 to the power minus 5 radian.

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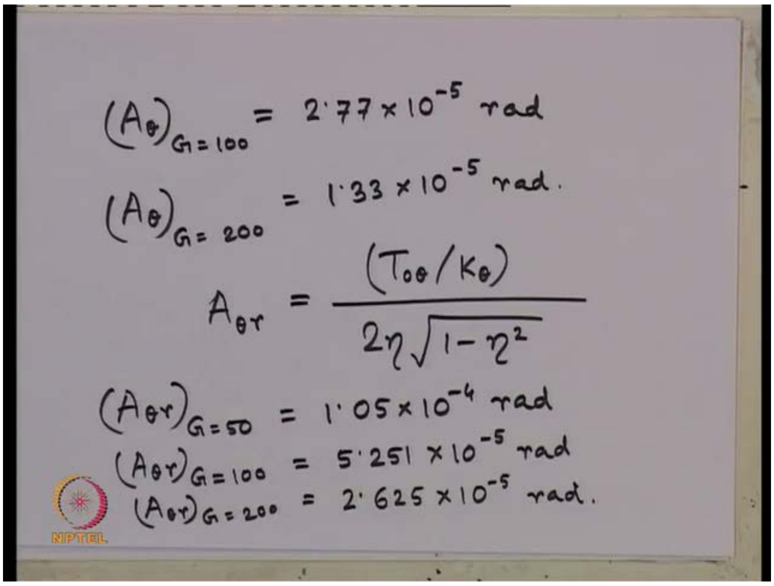

A theta for G equals to 100. If we put the values and calculate, we are getting 2.77 into 10 to the power minus 5 radian. And A theta for G equals to 200, we are getting 1.33 into 10 to the power minus 5 radian. Whereas, the expression to compute the displacement amplitude at resonance condition is T naught theta by k theta by 2 eta by root over 1 minus eta square similar to the expression for DMF – dynamic magnification factor. So, A theta r at G equals to 50 should be putting corresponding values; we get eta is 0.25; 1.05 into 10 to the power minus 4 radian. If you check this value, this has to be more than the at operating frequency whatever A theta we are getting. A theta r at G equals to 100. It is coming 5.251 into 10 to the power minus 5 radian. Look this is more than this value. And A theta r for G equals to 200; it is coming as 2.625 into 10 to the power minus 5 radian. This is again more than this at operating frequency. So, these are at operating frequency; these are at resonance frequency. So, in the similar way, we can find out for rotating mass type excitation also. So, that can be easily obtained in the similar fashion.

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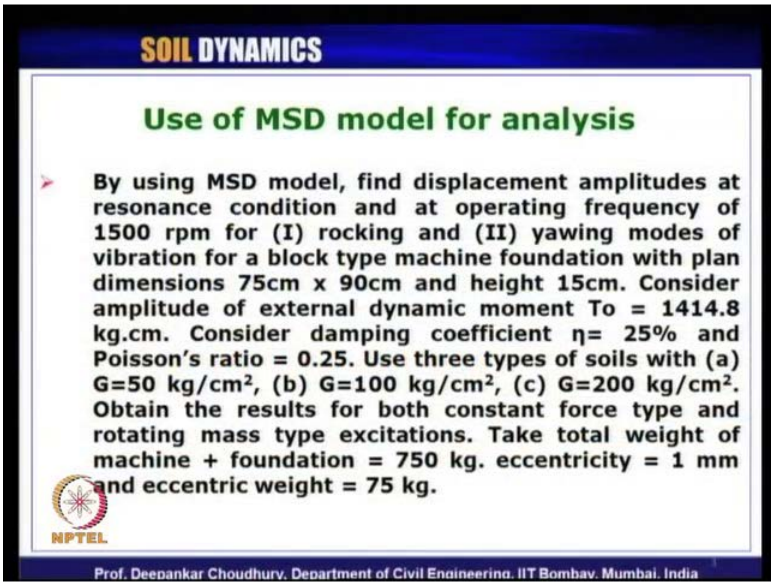

Now, let us start our next sub topic on this machine foundation, that is, elastic half space theory.

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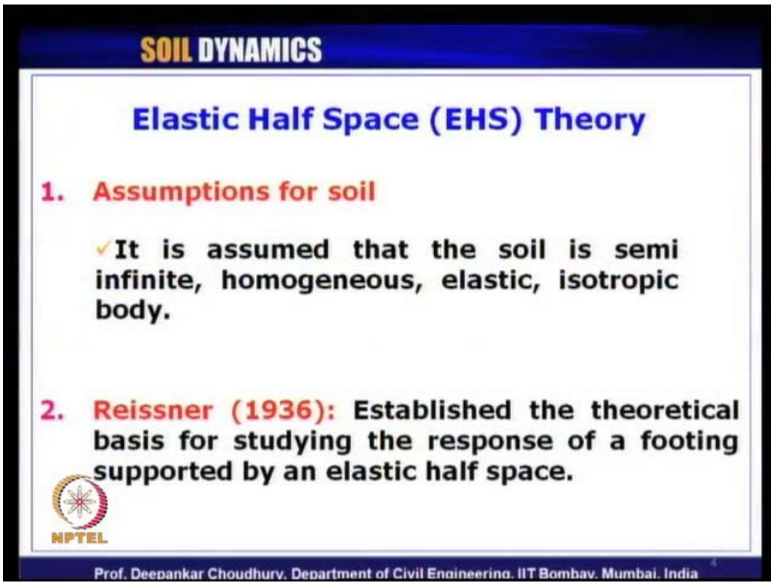

So, this is the third or the last method. We have discussed that, there are basically three methods to analyze or design a machine foundation. One is Tschebotari-off's reduced natural frequency method, which we have studied thoroughly and also we have solved the problems; we have practiced it. Then we have seen mass spring dashpot model; that also we have studied thoroughly and we have seen how its practice can be made. Now, we are coming to this third theory or which is known as elastic half space theory or EHS theory. So, what is the assumption for the soil in this EHS theory? The basic assumption for the soil is; it is assumed that, soil is semi infinite, homogeneous, elastic and isotropic body. So, first, the elastic half space theory was proposed in 1936 by Reissner.

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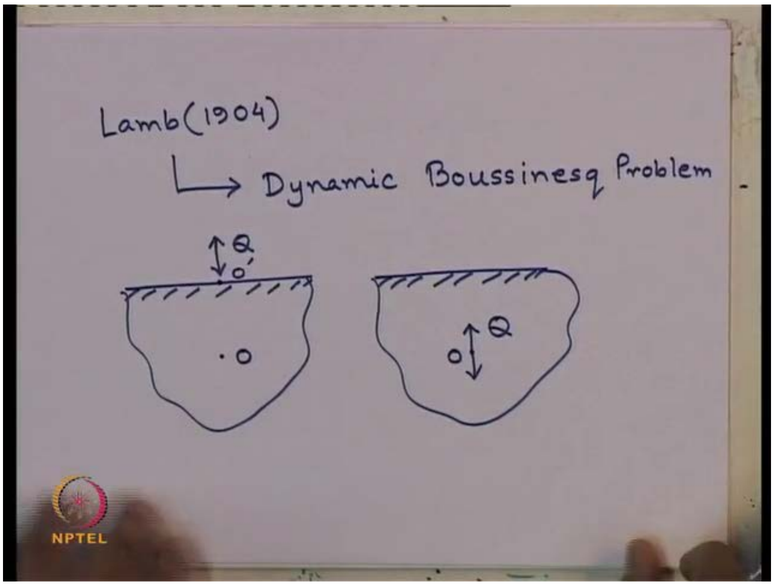

Before that, basically, if we see what is the utility of elastic half space theory for dynamic type of load, it was basically proposed by Lamb in 1904 mentioning that, it is nothing but a problem of dynamic Boussinesq problem. So, in this static case of loading, we know what is Boussinesq's condition for finding out the stresses below any footing; how the stresses varies within the soil mass both depth wise as well as radial distance wise.

So, for dynamic loading, Lamb proposed that, it can be considered as a dynamic Boussinesq's problem, where this is the ground surface. Suppose on this ground surface, we have some dynamic load in a vertical direction Q; say this point is O dash; it has been shown that, how the load or its effect can be transferred to any point within the soil mass and what are the different stresses are going to get developed because of the dynamic loading. So, its effect on the soil media from elastic half space theory was shown as if the dynamic load is transferred within the soil mass and then the behavior of the soil mass, that is, this stresses getting generated, then displacement going to form, etcetera has been studied. So, this is basically conceptualized from the static

Boussinesq’s problem. Instead of static load, now, it has been changed to dynamic load. So, this is for vertical mode of vibration.

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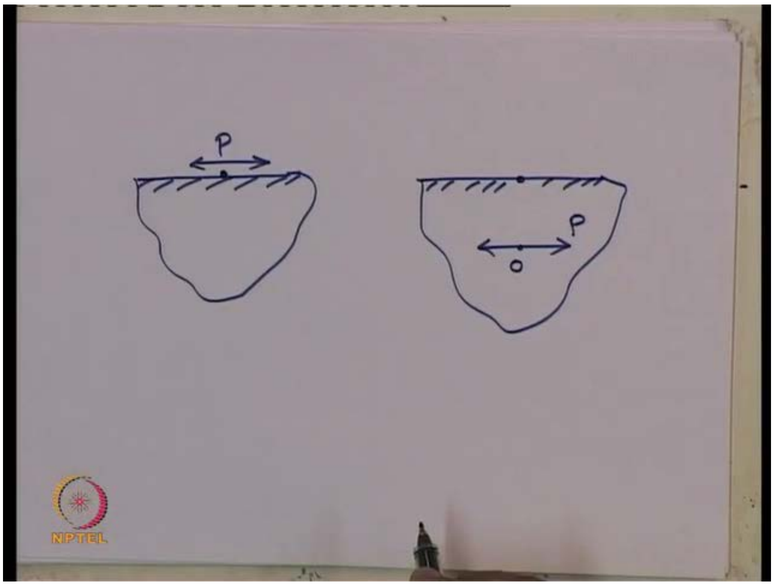

Similarly, for horizontal mode of vibration, the problem can be stated as if the horizontal load is acting at this point at the ground surface, which can be transferred and its effect can be obtained by transferring its position within the soil mass at any point; and that dynamic effect can be considered. So, like that, for each individual modes of vibration, its effect can be studied thoroughly. So, that is nothing but the Boussinesq’s problem – the dynamic Boussinesq problem as proposed by Lamb. So, how Reissner had modified this, that is, using the concept of dynamic Boussinesq’s problem? Reissner established the theoretical basis for studying the response of a footing supported by an elastic half space. So, what was the model considered by Reissner?

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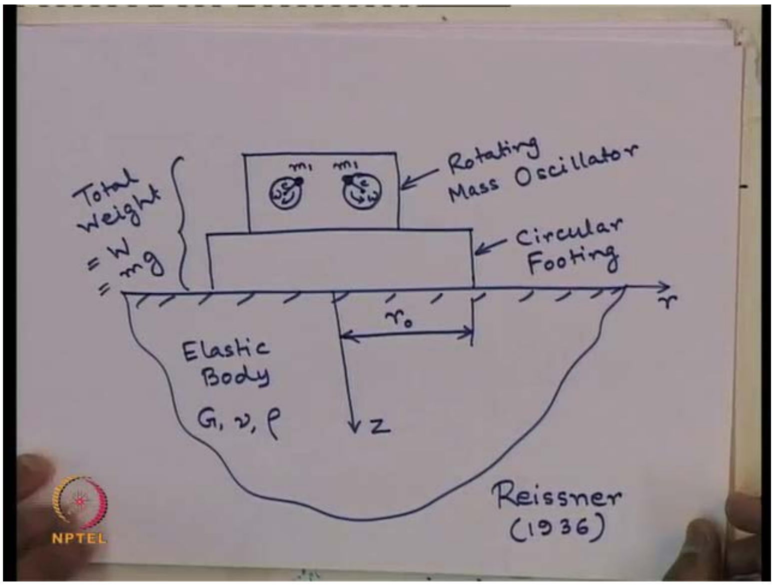

Let me draw that. So, this is depth wise z axis; this is radial distance wise r axis. For a circular footing… So, this is the circular footing, was considered in the Reissner model. And say this is the machine with rotating mass type excitation; it can be anything as you known either constant force type or rotating mass type. Let us say we are considering for rotating mass type with e as eccentricity, m 1 as the mass of eccentric mass; and the frequency with which it is rotating is omega. So, this one is nothing but rotating mass oscillator. And the total weight of this machine plus footing total is w, which is nothing but m times g; m is total mass of the footing plus machine. And this distance is nothing but r naught as the equivalent radius of the circular footing. And what is this soil is considered as? Elastic body with known properties of G, nu and rho; that is, shear modulus, Poisson's ratio and density of the soil. So, these things are known. So, the vertical displacement when it is subjected to vertical mode of vibration for this type of model as originally was proposed by Reissner in 1936.

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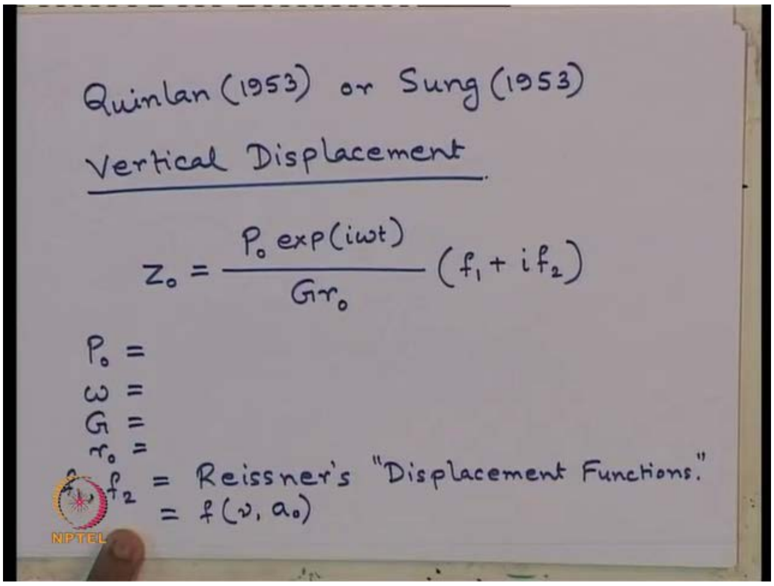

That was further analyzed and given by two researchers differently from different places, but their format was same. So, Quinlan in 1953 or Sung – 1953; so separately, they have done the research and came out in the same year with the expression for vertical displacement for such type of problem, that is, a machine foundation subjected to a vertical mode of vibration. So, what is that vertical displacement expression was given by them? z naught is P naught exponential i omega t by G r naught times f 1 plus i times f 2. i is nothing but that imaginary number. In this, what are the different terms? This P naught is called amplitude of total force applied to the circular contact area.

So, P naught is amplitude of dynamic load – amplitude of dynamic vertical load. What is omega? Omega is exciting circular frequency for the force applied. G is nothing but shear modulus of the elastic half space. G is shear modulus of the elastic half space means shear modulus of the soil. r naught is the radius of the circular contact area as I have shown just now in the picture also. Whereas, this f 1 and f 2 – these two are called Reissner's displacement functions. So, Reissner's displacement functions are used in the expression to get the vertical displacement of such machine foundation when it is subjected to vertical mode of vibration.

Now, how to get this Reissner's displacement function? Reissner has already given this displacement function in the form of a design chart. So, these are available with us. We will discuss this shortly. And it is found that f 1 and f 2 – these are nothing but… These displacement functions are nothing but some function of two parameters: one is Poisson's ratio; and another term Reissner has defined a naught. What is a naught? Let me define that.

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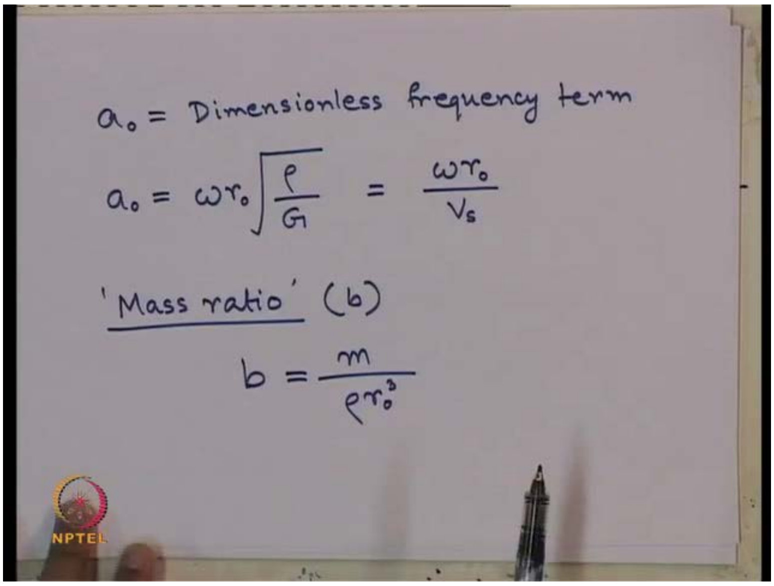

a naught is defined by Reissner as dimensionless frequency term. So, a naught is calculated like this – omega r naught root over rho by G. So, if you rewrite this in terms of shear wave velocity, it comes out to be omega r naught by v s, because v s is nothing but root over G by rho; remember? So, you can see omega is applied external exciting frequency; r naught is the equivalent radius of the footing; v s is the shear wave velocity in the soil media. So, this is having the unit of say meter per second. This is in meter; this is in cycles per second. So, it will also be meter per second. So, finally, a naught is nothing but dimensionless. So, that way, Reissner has defined this parameter a naught as the dimensionless frequency term using this mathematical expression.

And, another term was established by Reissner, which is called mass ratio. Mass ratio is denoted by small b. It is given by the expression m by rho r naught cube. This is the expression for mass ratio. Again, this is a non-dimensional parameter. As this is a ratio mass in kg unit suppose; rho density kg per centimeter cube, r naught in centimeter. So, it will also be in kg. So, that is why it is a non-dimensional term. So, Reissner's entire displacement functions – this f 1, f 2 are expressed in terms of these non dimensional parameters, that is, dimensionless frequency term and mass ratio.

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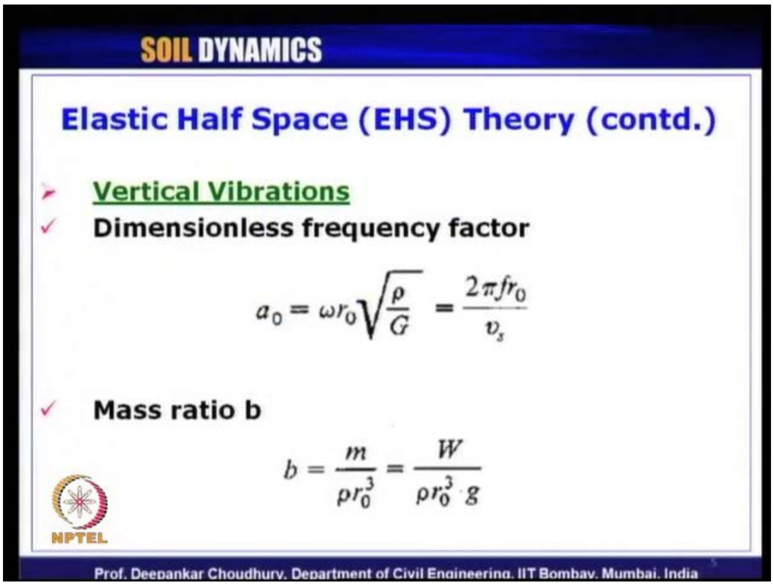

So, that is how for vertical mode of vibration, you can see in this slide, the dimensionless frequency factor is calculated like this. So, if you change from omega to f as you know in the case of hertz if you use the unit of hertz, it will be 2 pi f naught – nothing but omega r naught by v s – shear wave velocity. And mass ratio b is m by rho r naught cube. If you use the weight, then weight by gamma r naught cube you can use.

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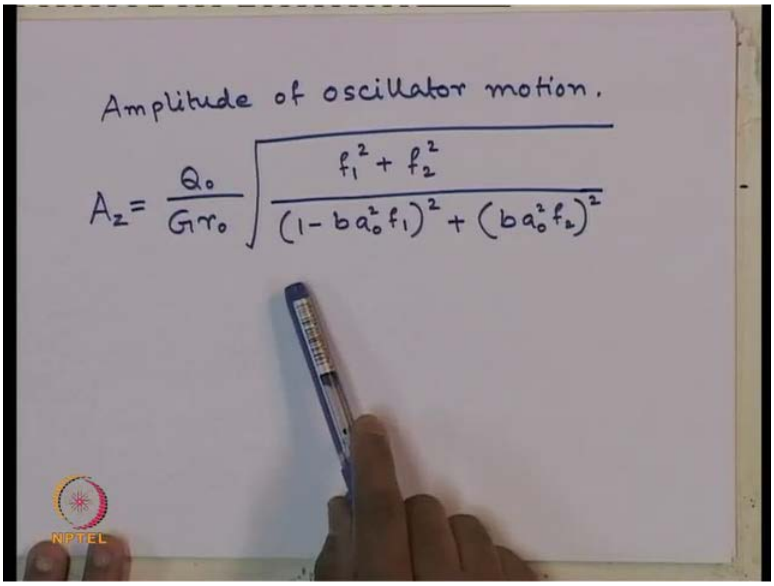

Now, Reissner has established the amplitude of oscillatory motion. It is given by this expression – a z is equals to Q naught by G r naught root over f 1 square plus f 2 square by 1 minus b a naught square f 1 whole square plus b a naught square f 2 whole square. So, this is the expression for the amplitude of the oscillatory motion. Can you correlate this expression with any other known expression of amplitude of motion whatever is known to you earlier? These are nothing but derived basically from the similar concept of mass spring dashpot model. Can you see? This Q naught is the amplitude of applied dynamic load – Q naught by G r naught; f 1, f 2 are the displacement functions; these terms will take care of that frequency ratio what we had used in the mass spring dashpot model. So, once we known this f 1 and f 2 from the design chart proposed by Reissner, Q naught is known to us, because it is supplied by the machine manufacturer; G r naught from the given soil property and design or whichever foundation we are going to check from that size of the foundation, you know r naught. b and a naught you can calculate from the expressions of dimensionless frequency factor and mass ratio expression whatever we have discussed just now. So, a z can easily be obtained.

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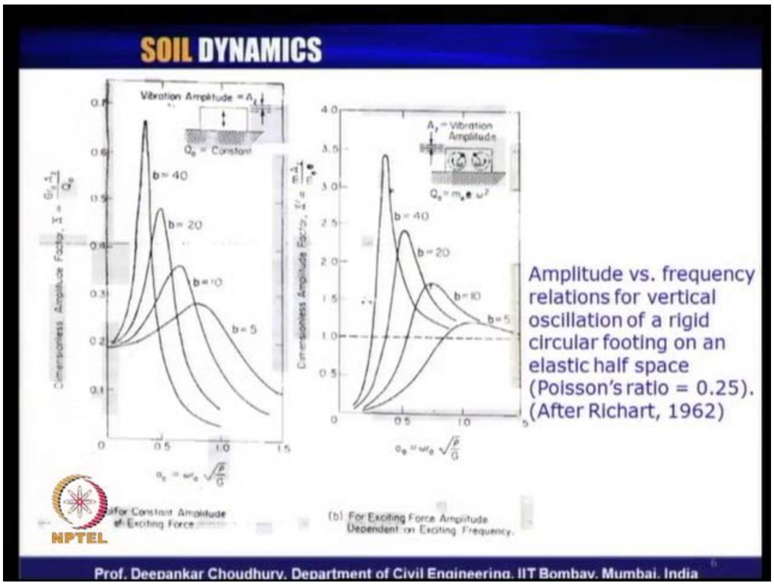

So, now, let us look at this chart, which is given in this Richart chart book of 1962 for a particular value of Poisson's ratio of 0.25 for two different cases. If you look here carefully, the left-hand side picture – it gives the variation of dimensionless amplitude factor. So, vibration amplitude A z – expression whatever we have given; if you put it in this way that, G r naught A z times Q z; it will become dimensionless, because in the previous expression, if we take this G r naught here and Q naught here; it will become dimensionless, which are functions of f 1, f 2, b and a naught. So, for different values of b and for different values of a naught, they will change. So, this is the axis for a naught – different values of a naught; and these are curves for different values of b – mass ratio. So, from this graph, one can easily read after calculating a naught. a naught can easily be calculated, because omega is given to us; r naught of the foundation is known; rho of the soil is known; G of the soil is known. So, a naught can be calculated. So, b also can be calculated – mass ratio. So, you go to this chart for constant force type vibration, vertical mode of vibration; you can get these coefficients. And once you known G r naught by Q naught – this term, you can calculate the vertical amplitude A z also. So, that is the use of this design chart, which takes care of that f 1 and f 2.

Similarly, for rotating mass type of vibration, the dimensionless amplitude factor is defined as m; m is the total mass of machine plus foundation. Whereas, m e is the mass of eccentric loading; e is the eccentricity. And A z we can compute. And a naught we know; b also we can calculate. So, go to any appropriate curve and read the value of this dimensionless amplitude factor. And correspondingly, you will get the value of A z.

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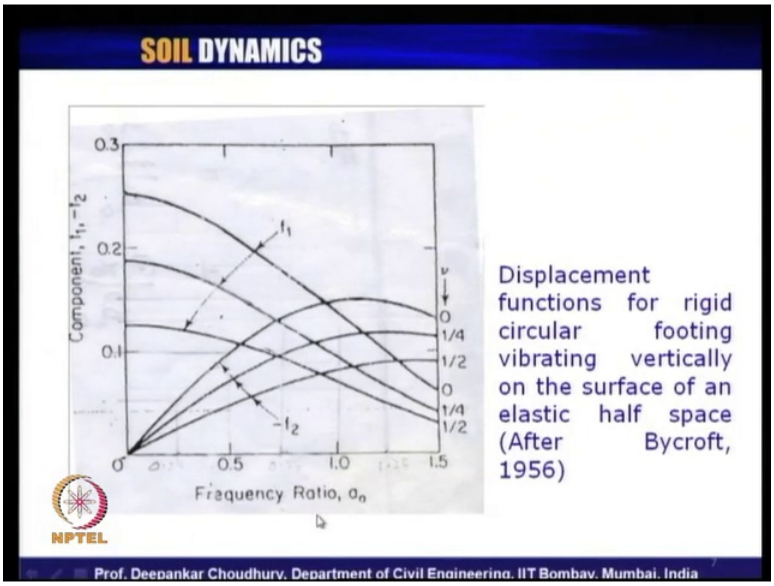

And, how this f 1 and f 2 – they vary with respect to the frequency ratio and Poisson's ratio? I have already mentioned, this f 1 and f 2, that is, Reissner's displacement functions are function of Poisson's ratio of the soil and dimensionless frequency factor. So, this is dimensionless frequency factor; these are curves for f 1; these are curves for f 2 for different values of Poisson's ratio of 0, 0.25 and 0.5, because for soil, any elastic body, all ranges are already covered for our soil mechanics; all ranges have been covered. Also, for f 1, this is for Poisson's ratio 0, 0.25 and 0.5. So, either two ways you can do the design using this Reissner concept. Ether you can use this chart directly or from this design chart, you get f 1 and f 2 value for a calculated value of a naught and for a known value of Poisson's ratio. Then put those values in this equation. That also will give you the value of A z. So, either way; either you can calculate using this equation, then you have to use this chart of f 1, f 2 or directly you can use these two design charts, where the effect of f 1 and f 2 are already taken care of. Clear, how we need to design this one?

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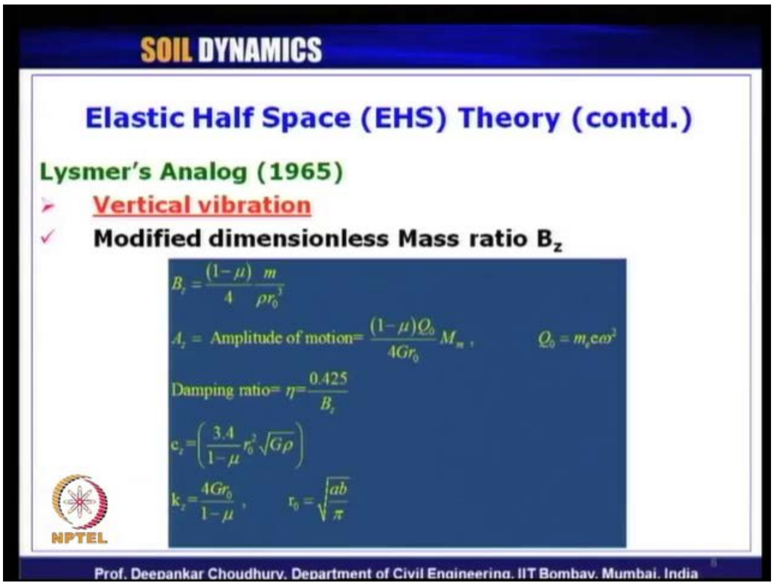

Now, this Reissner model was further modified by Lysmer – Lysmer's analog. Lysmer has proposed a simplified analog in 1965, that is, known as Lysmer's analog; where, for vertical mode of vibration, the modified dimensionless mass ratio is denoted by this symbol B z; z is for vertical mode of vibration as you know; so capital B. Reissner proposed the mass ratio as small b; whereas, Lysmer has modified it to capital B – B z. And how he has changed this? Do you remember? The frequency term or the Reissner's displacement functions, which we have discussed just now – they are function of two parameters: dimensionless frequency factor and Poisson's ratio. So, what is the change Lysmer did? Lysmer have considered the effect of Poisson's ratio in the calculation of non-dimensional mass ratio factor, so that Poisson's ratio factor with different Poisson's ratio, whatever changes, etcetera are already inside the expression of B. So, no need to plot different values of small f 1 and f 2 – that Reissner’s displacement functions; we can get a single displacement function f 1 and f 2 as given by Lysmer. So, that is how.

Remember this small b proposed by Reissner was this expression – m by rho r naught cube, which is small b. What Lysmer proposed let us multiply it with 1 minus mu by 4; mu is Poisson’s ratio of the soil. Then effect of Poisson’s ratio is also taken care of and still it remains a non-dimensional. And using that concept, the amplitude of motion can be calculated using this expression – that is, 1 minus mu by 4 G r naught times Q naught times M m; M m is nothing but a magnification factor for which the design chart was proposed by Lysmer for different values of dimensionless frequency term. And this Q naught can be for either constant force of vibration or rotating mass type of vibration; can be calculated in this fashion.

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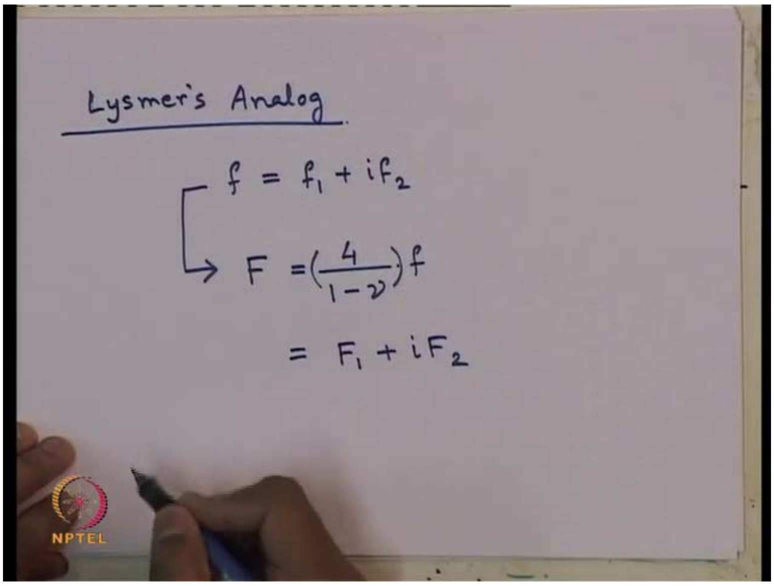

So, the basic change in the Lysmer’s analog was earlier the displacement function proposed by Reissner; it was given by f 1 and f 2. What Lysmer proposed? That he had modified it to another displacement function capital F, which can be calculated like this; that is, Reissner’s displacement function – multiply it with this term 4 by 1 minus mu to get rid of the effect of Poisson’s ratio. And then he proposed the displacement functions as capital F 1 and capital F 2. So, capital F 1 is nothing but 4 by 1 minus mu times this small f 1. And capital F 2 is nothing but 4 by 1 minus mu times small f 2. So, it is nothing but just a change from Reissner's model to Lysmer's analog by taking care of this Poisson's ratio effect in the expression of the displacement function itself.

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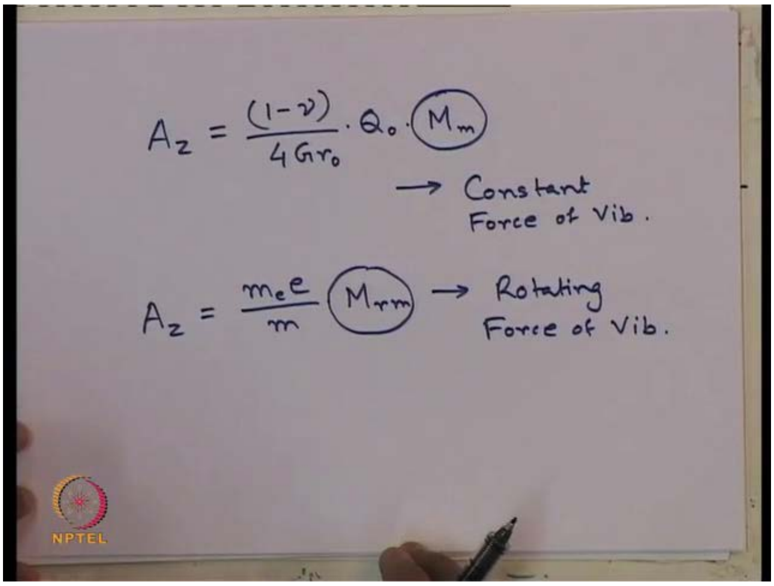

So, A z as I have shown just now, A z can be calculated using this expression and… This is for constant force of vibration. Whereas, A z can be calculated as m e e by m times M r m for rotating force of vibration. So, the design charts for these two coefficients or magnification factors are provided by Lysmer based on the analysis. What is Lysmer's model? Basically, it is similar to the equation of motion of Lysmer's analog, is similar to the mass spring dashpot model.

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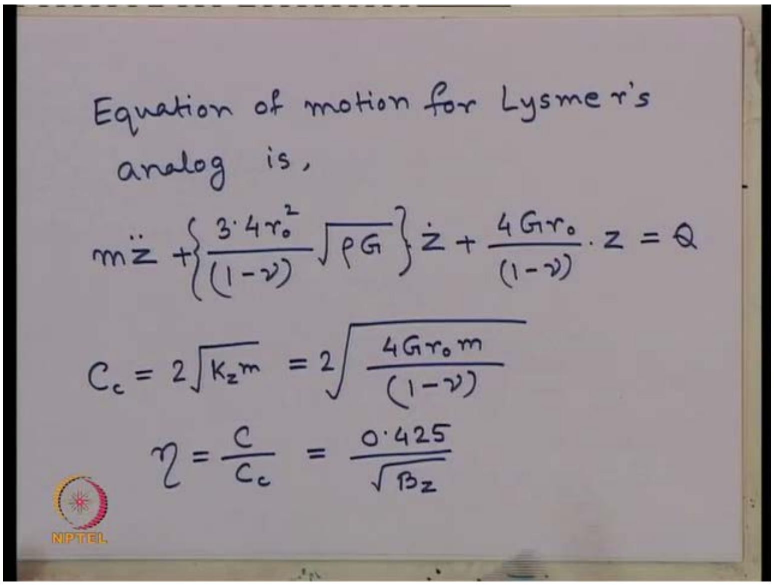

So, equation of motion for Lysmer's analog is given by this. For vertical mode of vibration, if we say m z double dot plus 3.4 r naught square by 1 minus mu root over rho G times z dot plus 4 G r naught by 1 minus mu times z equals to Q. That is the equation of motion of Lysmer's analog. So, let us see the different terms as proposed in by Lysmer. The spring constant k z can be calculated as we have already mentioned for vertical mode of vibration, 4 G r naught by 1 minus mu; where, r naught for any rectangular foundation can be calculated as root over ab by pi. And c z, that is, the damping coefficient can be calculated using this expression, whatever I have written just now. And the damping ratio can be calculated using this expression 0.425 by B z, because if you see the expression of C c, that is, critical damping coefficient, it can be expressed as 2 root over k z m, which can be written as 2 root over 4 G r naught m by 1 minus mu. So, the damping ratio is nothing but C by C c. C is this one. So, if you put these things and then simplify in terms of B z, it will give us finally, 0.425 by root over B z. This is the expression. This is the expression for damping ratio – 0.425 by root over B z.

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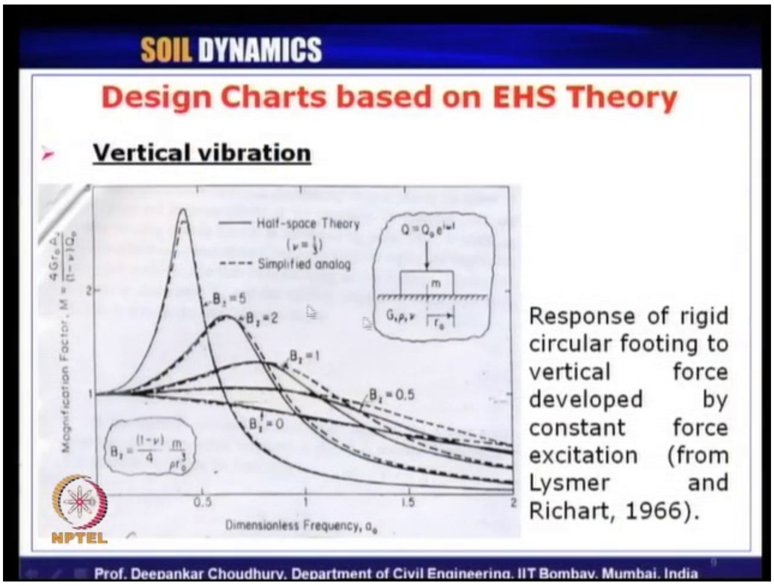

And, if you look at the Lysmer chart here; this shows a comparison between the elastic half space theory proposed by Reissner and the simplified analog proposed by Lysmer; that is, in dotted line and the other one is in formed line. So, for different values of dimensionless frequency factor, this is how B z is calculated for Lysmer analog. The magnification factor M can be calculated in this way. And for different values of B z, they are found to be matching very well, because basically it is nothing but taking care of the Poisson's ratio in the equation. Finally, they have regenerated this design chart. That is why it has to match. So, now let us look at the different expressions for different modes of vibration.

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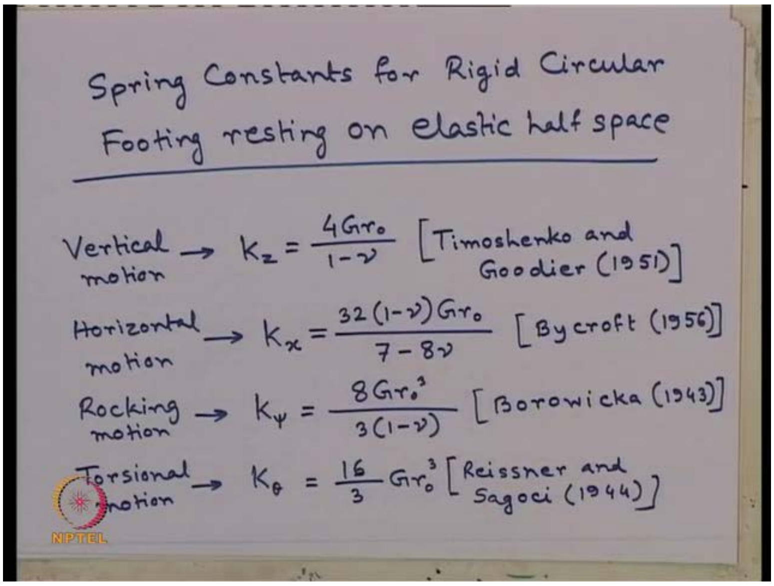

Let me give you the expressions for spring constant. Spring constants for rigid circular footing resting on elastic half space can be calculated for different modes of vibration; like for vertical motion, it is called k z, which is given by 4 G r naught by 1 minus mu; mu is Poisson's ratio; G is shear modulus; r naught is equivalent radius of the… Where from it has come? You can see the reference – Timoshenko and Goodier – 1951. For horizontal motion, the symbol is k x; k x can be calculated using the expression 32 1 minus mu G r naught by 7 minus 8 mu. For this, you can see the reference – by Croft – 1956. For rocking motion, the symbol k psi – 8 G r naught cube by 3 times 1 minus mu. This was given by Borowicka in 1943. And for Torsional motion, it is k theta, is calculated as 16 by 3 G r naught cube. This was given by Reissner and Sagoci in 1944. So, these things can be obtained for different modes of vibration. For horizontal motion, as you know, it can be k x or k y both. Accordingly, the r naught expression will change. Rocking and pitching also same calculation for r naught will change. So, let me give you the expressions for calculating the equivalent radius, which will help you to find out the expression for r naught correctly.

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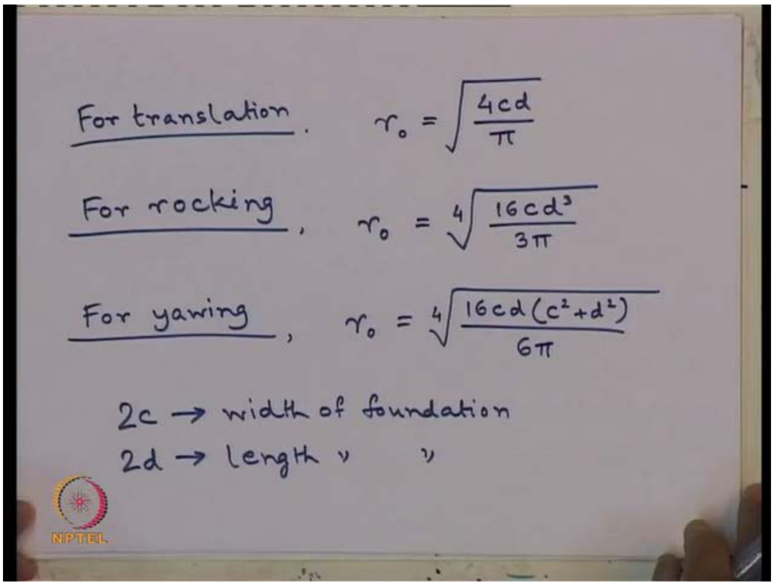

For translation mode of vibration, r naught is calculated as root over 4 c d by pi. For rocking mode of vibration, r naught is calculated using this expression – fourth root of 16 c d cube by 3 pi. And for yawing mode of vibration, r naught is calculated using this expression – fourth root of 16 c d c square plus d square by 6 pi. What are the c and d? 2 c is the width of foundation along the axis of rotation for the case of rocking; and 2 d is length of foundation in the plane of rotation for rocking. So, when it is pitching, what will happen? c and d will interchange. In that case, it will be c cube d. So, these are the expressions, which can be used to obtain from rectangular footing dimensions to equivalent radius of footings required to be considered for design. Other than this also, there are expressions to compute the spring constant for rectangular footing as well. So, let me give you those expressions also, which can be used.

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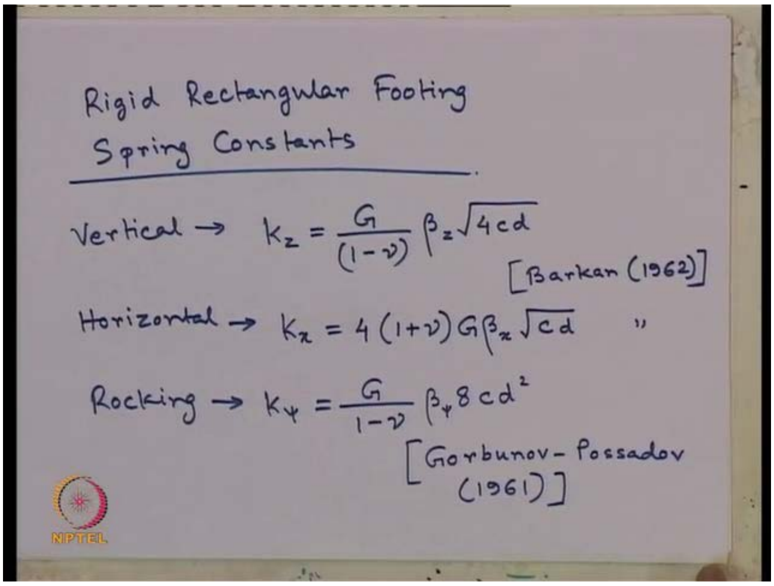

So, for rigid rectangular footing, the expressions for spring constants; for vertical mode of vibration, it is k z equals to G by 1 minus mu times beta z root over 4 c d. This expression was given by Barkan in 1962. In this, this beta z – these are coefficients of… depends on the size of the footing. For horizontal mode of vibration, it is given by k x as 4 1 plus mu G beta x root over c d. This is also given by Barkan. For rocking mode of vibration, k psi can be calculated as G by 1 minus mu beta psi 8 c d square. This was given by Gorbunov-Possadov in 1961. So, these expressions are also available. But, what generally for design we practice instead of going for these coefficients, because if you want to use the coefficients, we have to see these references and get this chart for the coefficients; so better to avoid these expressions and use the simple expression for the circular footing. And in that circular footing, you convert the rectangular dimensions to equivalent circular radius and then follow up the calculation. So, I am showing this methodology is also available. But, generally, in practice, we do not adopt this one.

Now, coming to this slide again, what we have discussed just now; for vertical mode of vibration with constant force type excitation, this is how the Reissner’s model has been compared with Lysmer’s analog – simplified analog. Similarly, for rotating mass type excitation also, they have been compared pretty well for vertical mode of vibration. So, with this, we will stop our lecture today. We will continue further in the next class.