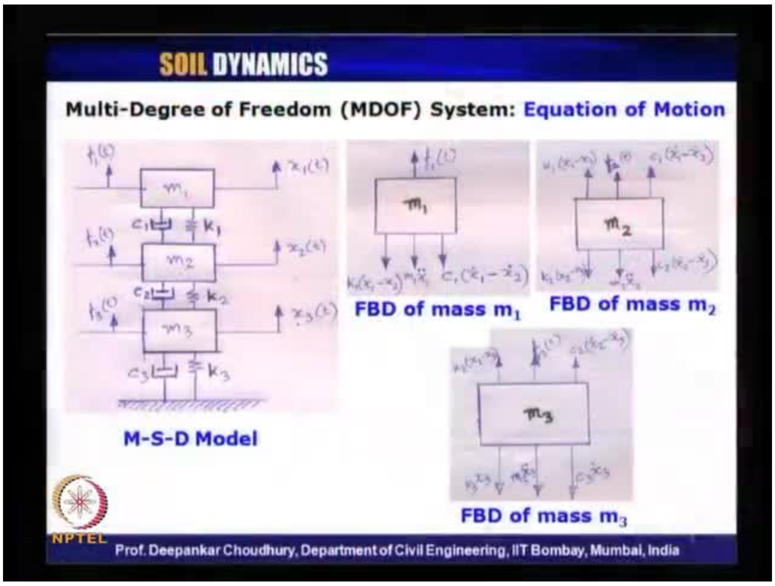
Soil Dynamics
Prof. Deepankar Choudhury
Department of Civil Engineering
Indian Institute of Technology, Bombay

Module - 2
Vibration Theory - (MDOF) System
Lecture - 14
Equation of Motion, Longitudinal Waves in an Infinitely long rod

Let us start with today's lecture of soil dynamics. We are continuing with our module two vibration theory. Just a quick recap what we have studied in the previous lecture, we have started with multi degree of freedom system.

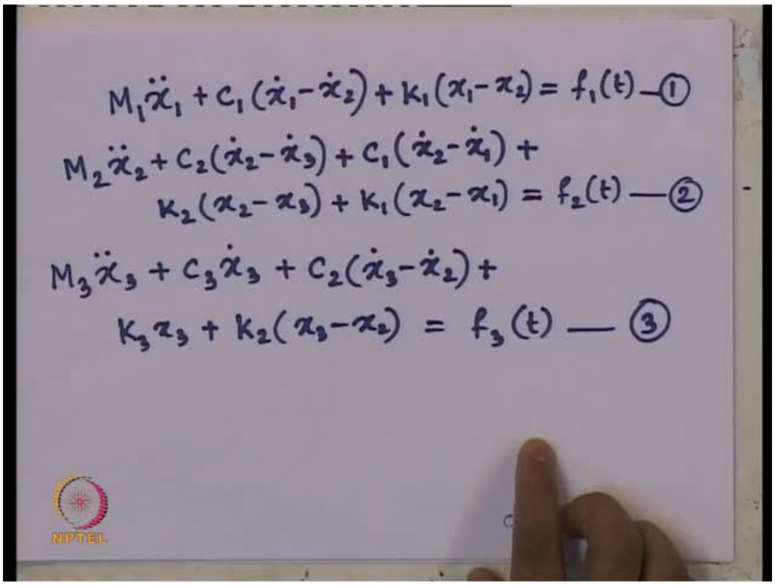
(Refer Slide Time: 00:42)



And how to formulate the equation of motion for multi degree of freedom system we have seen, the mass spring dashpot model for a multi degree of freedom system, we had considered a 3 degree of freedom system with mass m 1, m 2, m 3 and corresponding k and c values are k 1 c 1, k 2 c 2, k 3 c 3, and our degrees of freedom were x 1 t, x 2 t, x 3 t - three degrees of freedom. And applied dynamic loads to each of the units were f 1 t, f 2 t and f 3 t from which we have drawn the free body diagram for each of this mass m 1, m 2 and m 3, and forces acting for mass m 1 are shown here, forces acting for mass m 2 are shown here to maintain the internal equilibrium, we have taken all the forces and also the forces acting on mass m 3, where shown here. Then what we did? We applied the

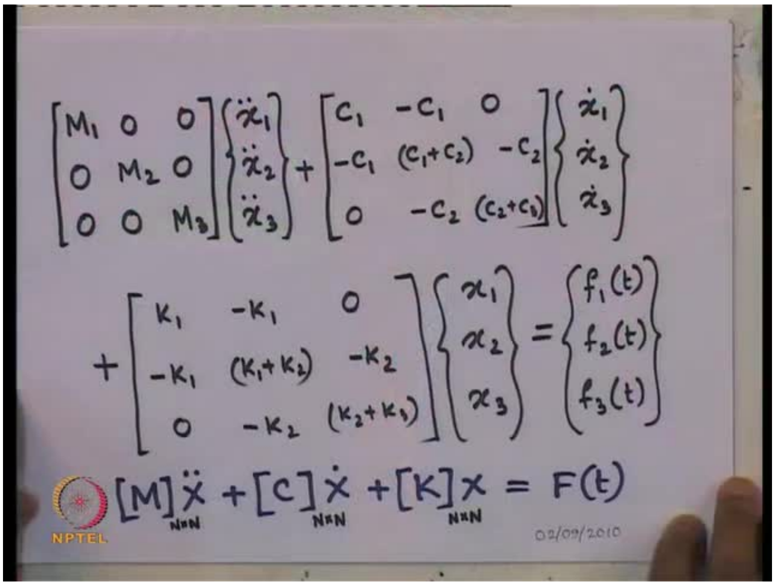
D'alembert's principle, that is equilibrium of all the forces in the vertical direction and we had written three equations for each of the mass and that equation was shown like this.

(Refer Slide Time: 02:03)



So, these were the three equations we obtained, from the free body diagram by applying D'alembert's principles. These equations in a simplified format, we had written in the form of a matrix.

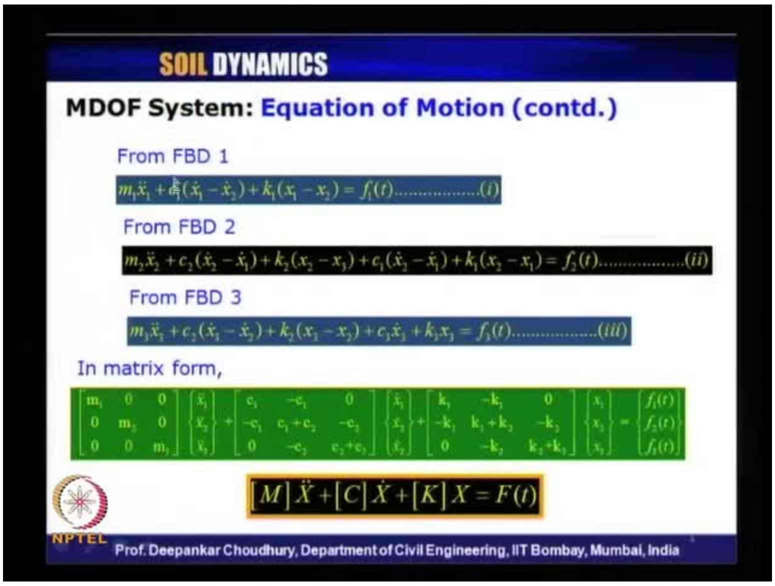
(Refer Slide Time: 02:23)



So, how the matrix can be written? Which are nothing but the representation of those three equations in a better format or simplified format. So, this is the mass matrix then acceleration vector, then this is the damper matrix velocity vector and this is the stiffness matrix displacement vector, equals to the externally applied dynamic load vector, which we had written in our known form of M x double dot plus c x dot plus k x equals to F of t. So, which is nothing but same as what we had studied for a single degree of freedom system, so equations of motion remains same, only difference is instead of these values are single numbered, now these are matrices or vector and the size of the matrix or the size of vector depends on how many degrees of freedom we are considering.

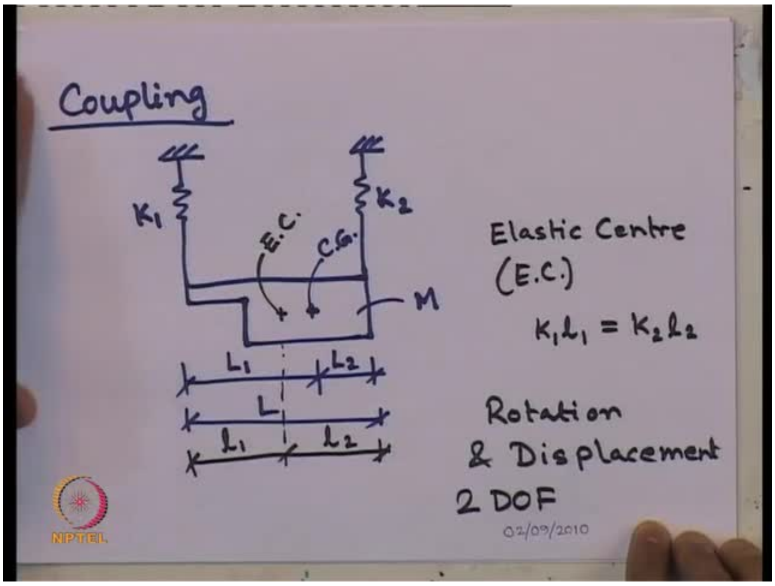
So for a N degree of freedom system the mass matrix will be of the size N by N and the acceleration vector will be of the size N by 1. Similarly, the damper matrix will be of the size N by N. And the velocity vector will be of the size N by 1 stiffness matrix will be of the size N by N. And displacement vector will be of size N by 1 and externally applied force vector will be of size N by 1.

(Refer Slide Time: 03:53)



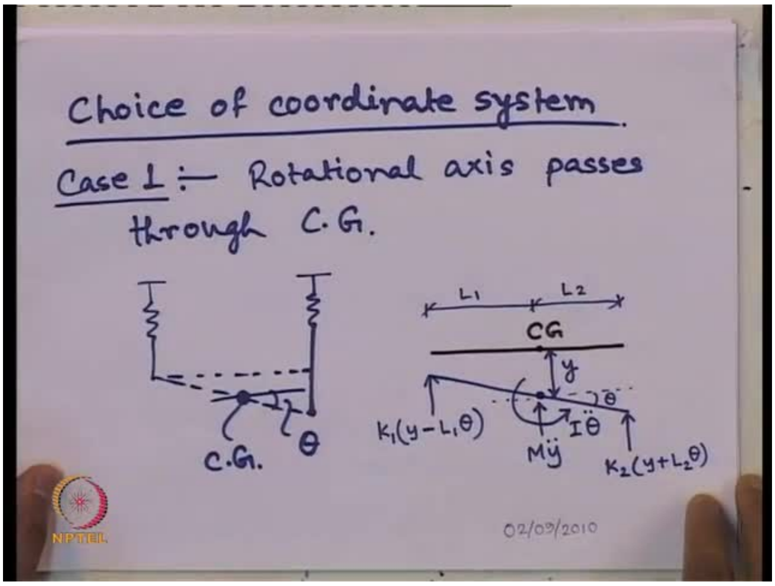
So, in this slide we are showing what we had obtained in the previous lecture, the same thing in the matrix form and this is the governing equation of motion. Now, in this matrix format what we had seen? Look at the mass matrix, damper matrix and stiffness matrix all are diagonal. That is with respect to this diagonal, they are symmetric and if you carefully look at the mass matrix, only the values are non-zero for a particular degrees of freedom for which we had written the equation of motion. What does it mean? The m 1 is non-zero when we are considering the fast degree of freedom that is x 1 t. Similarly, m 2 is non-zero when we are considering x 2 t, m 3 is non-zero when we are considering x 3 t, otherwise all other entries are 0. This is called uncoupled matrix, we are using the terminology uncoupled matrix. Now, we talked about this coupling in the previous lecture.

(Refer Slide Time: 05:10)



So, we had taken two degrees of freedom systems, which is un damped which is connecting one rigid beam of mass M of varied cross section like this through two springs K 1 and K 2 to a support and center of gravity of the beam is here that is C G and we have defined another point which is known as elastic center E C. How elastic center is defined? The distance from the two ends where the springs are connected if the distances are L 1 and L 2 from this elastic center to the corresponding springs connected to the beam. Then elastic center is defined as K 1 times this distance L 1 should be equals to K 2 times this distance L 2. And what are the degrees of freedom we are considered for this beam both vertical displacement and rotational displacement.

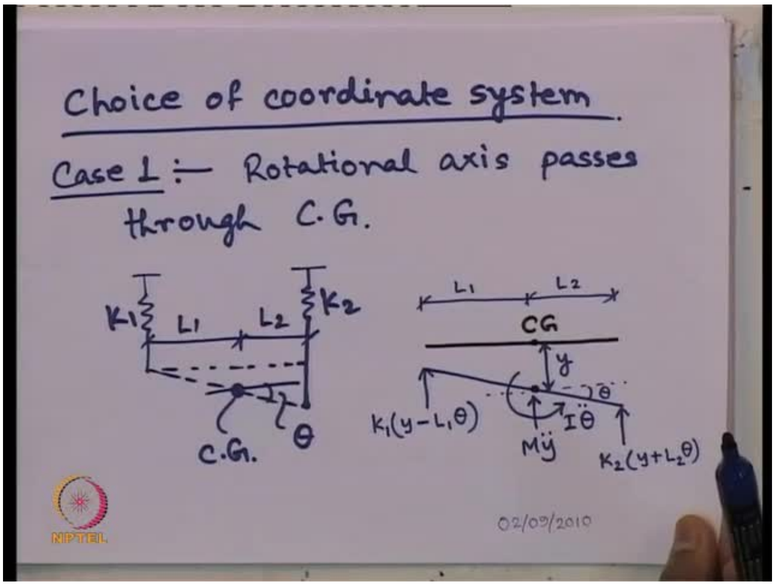
(Refer Slide Time: 06:18)



And then we said that, let us see what happens, if we choose different coordinate system for the rotation. Vertical definitely we have no choice, but with respect to the initial position it has vertically displaced, let us say at an instant of time vertically downwards. So, this is our reference level the black line with respect to which, with respect to this data we are considered case one, where the rotational axis passes through the center of gravity of the beam. So, it has vertically come down as well as it has rotated about it C G.

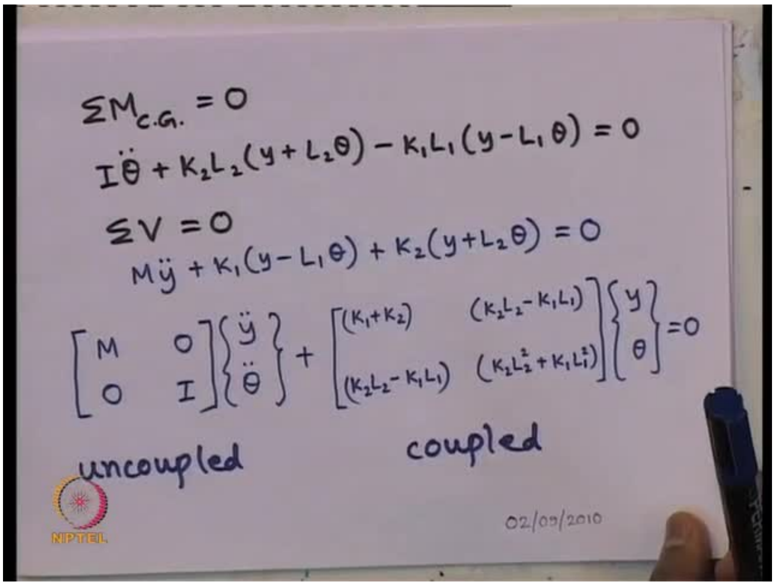
So, these are the two degrees of freedom and the vertical displacement is y of the point C G and the rotation above the C G is theta, as shown here. So, the corresponding forces acting on the beam are the inertia force M y double dot because of the vertical displacement corresponding acceleration is y double dot. So, inertia force is M y double dot and because of rotation, the rotational inertia force is I theta double dot. I is the moment of inertia of the beam and because of the two springs connected to the two ends, what are the spring forces acting?

(Refer Slide Time: 07:45)



This side spring K 2 is was connected, so this was K 2 and this was K 1 and the distances with respect to the C G we have seen this distances we had considered L 1 and this distance we had considered L 2 from the C G to the ends of the springs. So, this spring is subjected to the corresponding displacement is y plus L 2 times theta. So, K 2 into y plus L 2 theta is spring force and this spring force also will act vertically upward because the net displacement of this point is vertically downwards. So, it will try to push it back to its original position. So, this magnitude will be K 1 times y minus of this L 1 times theta.

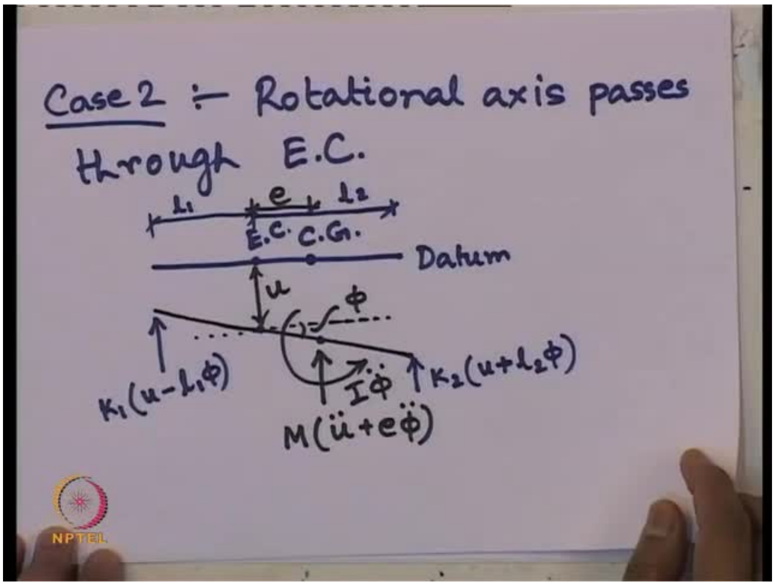
(Refer Slide Time: 08:39)



From which we had seen that by considering two equilibrium conditions that is nothing but applying the D'alembert's principle for moment equilibrium about the C G and the vertical force equilibrium for all the vertical forces, we can write these two equations which we can represent in matrix format in this way. So, what does it mean? Finally, for this case when the rotational axis passes through the center of , we got mass matrix. This is the mass matrix, which is uncoupled, so this is uncoupled. Why it is uncoupled? Earlier we had mentioned we called uncoupled if other than that corresponding degrees of freedom if all other entries are 0.

So, here this is non-zero corresponding to the degrees of freedom of y and here this is non-zero I corresponding to degrees of freedom theta, whereas, other entries in the matrix is 0 and they are diagonal. Whereas, look at the stiffness matrix this our modified stiffness matrix for the entire system, this is coupled because for the first degrees of freedom that is y here, the entry for theta is non-zero. So, theta is also having some effect on the degrees of freedom of y, so that is why we have mentioning it as a coupled matrix and we have defining this as uncoupled matrix. And as we had seen earlier, the size of the matrix should be the depending on the degrees of freedom. We have two degrees of freedom system, so this size of the matrix is 2 by 2 and this vector size is 2 by 1. So this is for the first case. Now, let us see what are the other possible cases for the same problem we can address to.

(Refer Slide Time: 10:53)



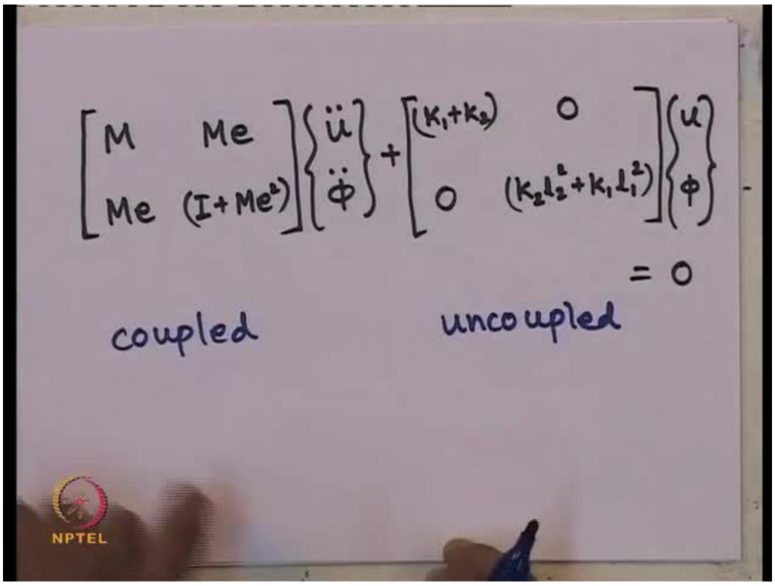
So, in the case two let us consider that rotational axis passes through elastic center. The first case we had considered the rotational axis passes through the center of gravity, in this case we are considering the rotational axis passes through the elastic center. So, how the free body diagram will look like, let us see. Now, we have datum this one we had our C G of the beam here and elastic center is somewhere here E C. And what happen in the new position it has come down here and it is rotated we had considering now about the elastic center. So, this is E C whereas, this is C G.

So, let us say the rotation about the elastic center is represented as phi and what are the forces acting on the beam now? The inertia force will be, let us say the vertical displacement that is from here to here is u and the distance between elastic center and center of gravity, let us say e. So, what should be the inertia force due to the vertical movement on the beam? And where it should act? As we know inertia forces will always act at the C G of the beam, so because of the vertical displacement EC has moved down by an amount of u whereas, the C G has moved down by an amount of u plus e times this phi, for small phi of course, so the inertia force vertical inertia force, will be M times the corresponding acceleration will be u double dot plus e phi double dot, am I right?

The displacement for the C G is u plus e phi. So, the corresponding acceleration is u double dot plus e phi double dot times mass will give me the inertia force because of the vertical displacement and what is the rotational inertia force? That should be again above this C G that is I phi double dot. And we had springs at this two corners, so what are the spring forces now acting? Remember the distance of this spring from the elastic center were L 1 from this side and L 2 from this side, from the elastic center. So, at this end the spring force should be K 1 times u minus small l 1 times phi and at this end the spring force should be K 2 times u plus small l 2 times phi. So, free body diagram is complete these are the forces now acting on the beam.

Now, again we will apply our D'alembert's principle for both the moment equilibrium and the vertical force equilibrium. So, vertical force equilibrium and moment equilibrium about say E C, if we apply and take the equations in the form of a matrix. How it should look like, let us see. So, the equations in form of a matrix, if we write it will be M M e M e I plus M e square times u double dot phi double dot plus K 1 plus K 2 0 0 K 2 l 2 square plus K one l 1 square times u phi equals to 0.

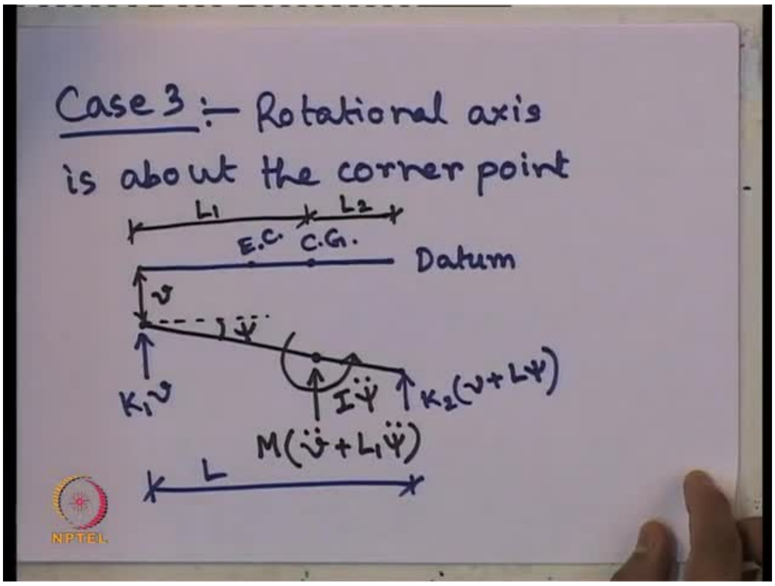
(Refer Slide Time: 15:14)



So, after applying moment equilibrium and vertical force equilibrium that is D'alembert's principle, we get the equations in this format. Where mass matrix and stiffness matrix, if we look at them carefully, what we have obtained in this case? The mass matrix is now coupled, because for the degrees of the freedom of u that is the vertical displacement, the entry for the rotational displacement component is non-zero. So, the mass matrix is coupled whereas, look at the stiffness matrix for the entry due to the vertical displacement it is non-zero value.

However, the other entry because of the rotational displacement is 0 here, so stiffness matrix is uncoupled in this case. So, by changing our coordinate system for the same beam problem for the rotational axis instead of passing through C G to the elastic center, the equation of motion basic equation of motion change to the form of mass matrix uncoupled to coupled and stiffness matrix from coupled to uncoupled. So, this is the second case, let us say and let us select another coordinate system.

(Refer Slide Time: 17:42)



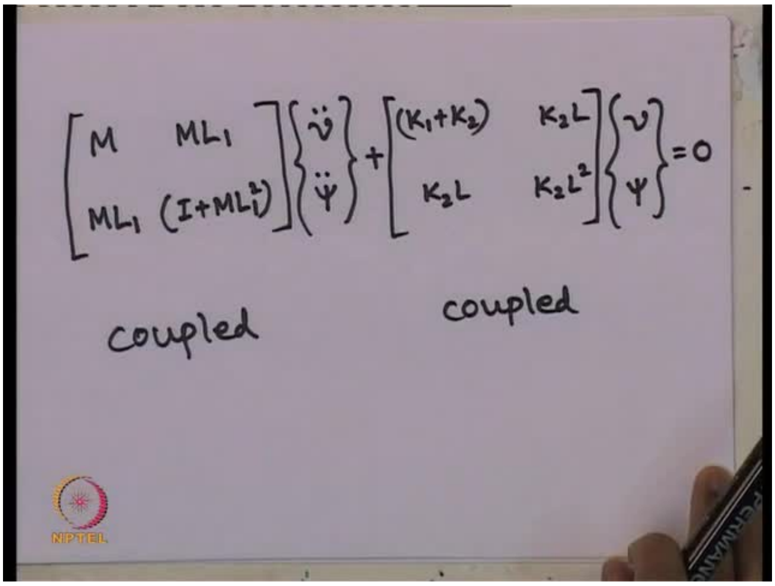
So, case three we are considering the third coordinate system, that is rotational axis is about the corner point, that is the end point of the beam. Suppose, this was our initial position of the beam datum C G of the beam were somewhere here and E C of the beam were somewhere here what we are considering. Now, the new position like this where we are telling it has rotated about this end point about this corner point and let us say that rotational value is psi and vertical displacement from original position is given by small v now. What are the forces acting on this beam?

That if we want to draw the free body diagram of course, we know all the forces will acting about the C G, the inertia force. Inertia force due to the vertical movement of this point will be how much? C G we know from this corner is at a distance of L 1 and from this corner it is at distance L 2 capital L 1 and capital L 2 and this value of inertia force should be this value of inertia force is mass times corresponding acceleration. So, what is the displacement of the C G that is v plus L 1 phi L 1 psi. So, v plus L 1 psi is the correspondence displacement of the C G vertically.

So, corresponding acceleration should be v double dot plus L 1 psi double dot and the rotational inertia force should be I psi double dot and what are the spring forces? Spring force at this corner, where K 1 is connected, that is simply K 1 times small v because just v is the vertical displacement of the spring. However, at this end the spring force should be K 2 times v plus L 1 plus L 2 psi and how much is L 1 plus L 2? We had considered earlier the total length of the beam is say capital L, so L psi. So, this completes the free body diagram of the beam.

Now, again we will consider the vertical equilibrium of all the forces and the moment equilibrium about any point to corner point. So, we are basically applying the D'alembert's principle once again for moment equilibrium as well as the vertical force equilibrium, by doing that what we will get the equations.

(Refer Slide Time: 21:18)



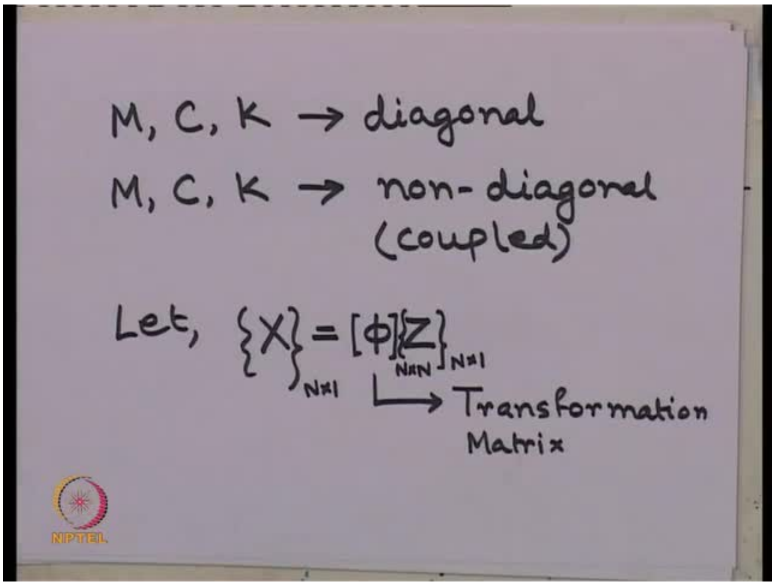
Let me write it in the matrix form, if you write the equations in matrix form it will look like this M M L 1 M L 1 I plus M L 1 square times v double dot psi double dot plus K 1 plus K 2 K 2 L K 2 L K 2 L square times v psi equals to 0. So, this is the form of the matrix or form of the equations governing equation of motion in matrix form. What we have seen from this? Looking at the mass matrix it is coupled, looking at this stiffness matrix it is also coupled because in the mass matrix due to the degrees of freedom of v the effect of psi is present very much present it is non-zero. Also for this stiffness matrix because of the degrees of freedom v the effect of psi is present it is non-zero.

So, what we we can conclude from these three cases for a same problem? The choice of a coordinates system is extremely important for solving any problem or I should say that to define the equation of motion for any system, the selection of coordinate system is very very important because based on the selection of the coordinate system. You can get your mass matrix damper matrix stiffness matrix either coupled or uncoupled

So, the behavior of these matrices that is whether coupled or uncoupled is not an intrinsic property. It depends on your assumption of the coordinate system. Now, how you may ask that then how we will know which coordinate system will be better, it does not matter. The answer is, it does not matter you can take any physically most appealing coordinate system. What does it mean physically most appealing? That is looking at the problem, looking at the system you understand that in which directional mode shape it can rotate or in which form it will be more correct way to consider its direction of rotations and things like that.

So, based on that you define your coordinate system at the beginning and then let us see what happens suppose by choosing any type of coordinate system you finally, got an equation of motion, which is giving you mass damper and stiffness matrices all are coupled, okay? So, what will happen remember in the first case, that is when we have solved the basic multi degree of freedom problem. What we got all the mass damper and stiffness matrices are uncoupled, so that is called the natural coordinate system.

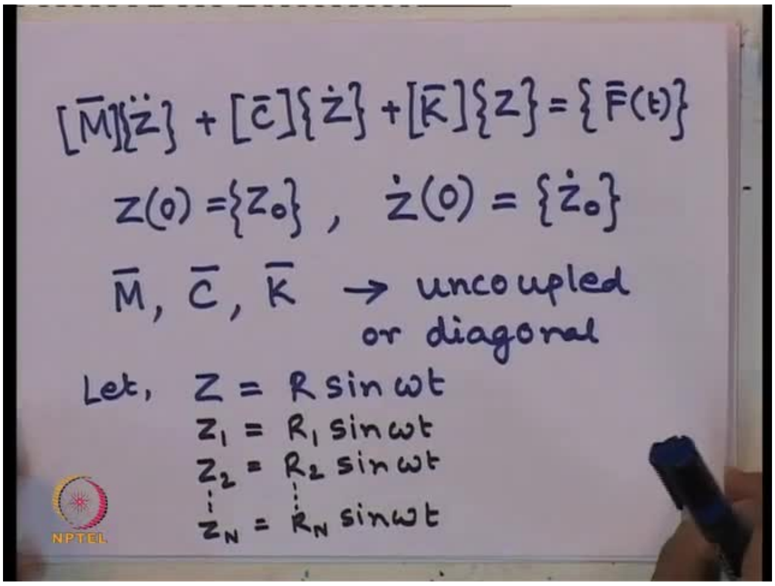
(Refer Slide Time: 25:31)



So, when M C K all these matrices are diagonal, we are calling it as the natural coordinate system, but if M C K are non diagonal or let us say in other words if they are coupled then it is not a natural coordinate system. So, in that case we have to do something to transform from our assumed coordinate system to natural coordinate system. So, let us say the solution x this is the x vector. Remember I am writing this as vector and these are matrices. Now, these are not the single number, so x vector let us say this is given as phi times z vector phi is a matrix, which is called transformation matrix.

So, if we have considered n degrees of freedom system. This x vector will be of size N by 1 the transformation matrix size will be N by N and the new coordinate system z will be of vector will be of size N by 1. What does it mean x is our chosen degrees of freedom in our assume coordinate system, whereas, z is nothing but the natural coordinate system, where we are getting M C K all are diagonal. So, we are transforming our assumed coordinate system to the natural coordinate system because then our solution will be easier. So, that is why we are using an operator which is called nothing but transformation matrix, so with using the transformation of the system, let us say our new equation of motion takes the shape of M bar z double dot this is matrix.

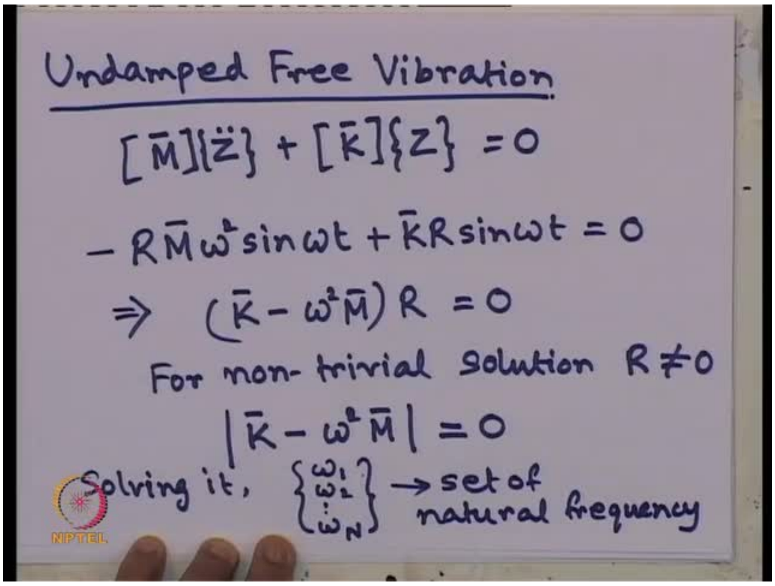
(Refer Slide Time: 28:23)



This is vector plus C bar z dot plus K bar z equals to vector f of f bar, f of t. So, what does it mean? Using the transformation matrix, the new coordinate system which is we are referring as natural coordinate system corresponding changes in our mass stiffness and damper matrices are represented M bar C bar and K bar. So, they have transformed from M, C and K to M bar, C bar and K bar corresponding forced vector also has been transformed to f bar from the original f. And let us say the initial conditions are z at times 0 is given as z naught. Remember all these are vectors, these are not single numbers.

We are considering generalize multi degree of freedom case and the initial velocity vector is say z naught dot these are given conditions to us and what we said this M bar, C bar and K bar are uncoupled or diagonal. That is by using the transformation matrix, we make each of them as diagonal. Now, the form of solution, let us consider it is given by z equals to r sine omega t is the form of the solution for the system. What does it mean? That means these are vectors, so we have z 1 is equals to r 1 sine omega t z 2 is r 2 sine omega t. Similarly, up to z N will be r N sine omega t these are assumed form of solution. Now, let us see for un damped free vibration. How we can solve the problem? Let us consider the simplest case, then it can be applied to other cases also.

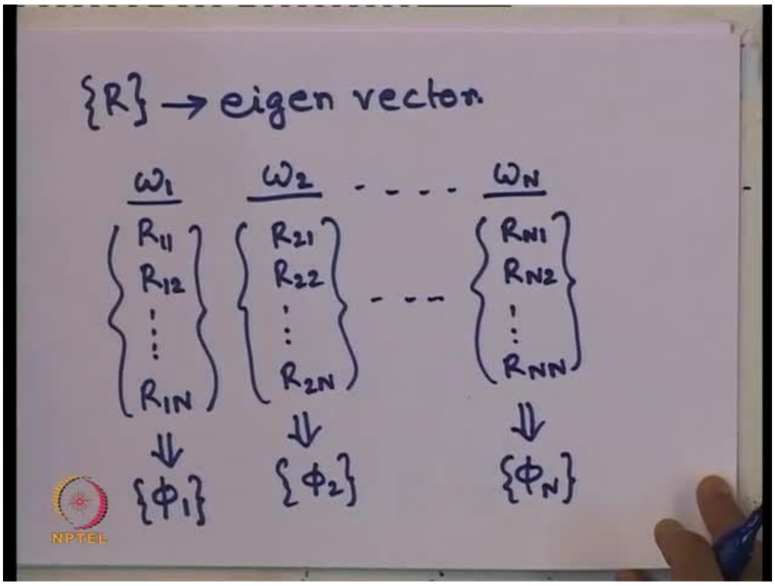
(Refer Slide Time: 31:30)



So, for un damped free vibration we are trying to find out the solution. What we have M bar z double dot plus K bar z equals to 0 because un damped, so C is not present C matrix is not present and free vibration. So, f bar vector is not present, what we can write by putting the assume solution of z. If we write down the equation, it will give us minus r M bar omega square sine of omega t plus K bar r sine of omega t equals to 0 because z we have assumed in the form of r sine omega t. So, obviously z double dot will be minus r omega square sine omega t, right? Which gives us K bar minus omega square M bar times r equals to 0.

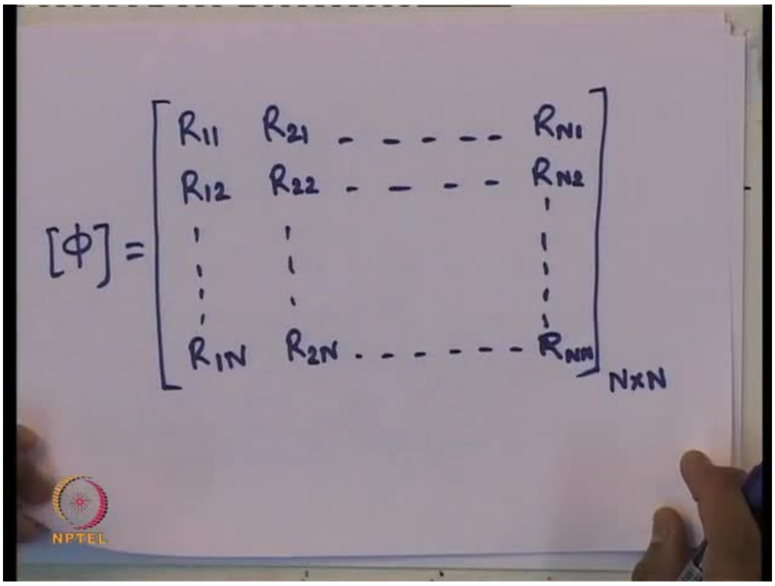
Now, for non trivial solution for non trivial solution, what we know r cannot be equals to 0, right? What does it mean? This part should be equals to 0, what we represent this 1? If this is equals to 0, these all are matrices now. So, it means K bar minus omega square M bar equals to 0, that is determinant of this 1 should be equals to 0. So, if we find out the determinant because K is matrix, M is a matrix, what we will get? We will get solving this we will get the solution, in terms of vectors omega 1 omega 2 up to omega N. For N numbers of degrees of freedom, what it is called? It is called set of natural frequencies. So, what we got for an N degree of freedom system, it will have N numbers of natural frequency and another terminology I am adding here among those N natural frequencies.
The lowest value of the natural frequency is called fundamental natural frequency. So, among N natural frequencies the value of the natural frequency, which is the lowest one is called the fundamental natural frequency. And if we want to draw the response displacement response of this vibrating system for N natural frequencies. We will get N different responses, which are called as mode shapes, so N mode shapes we can get. Now, from these values of Eigen values, what we got here from this determinant it becomes a Eigen value problem, which will give us this value of r which is nothing but a Eigen vector, right?

(Refer Slide Time: 35:58)



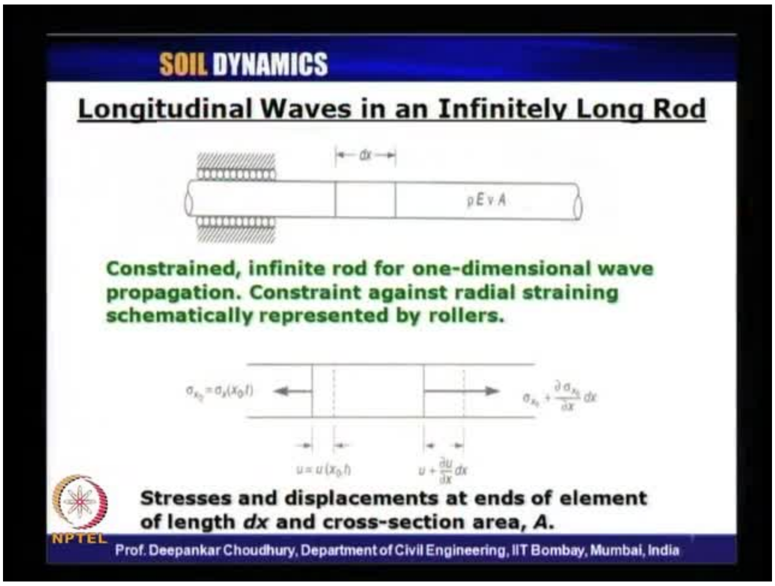
So, by putting those we will get r vector which is nothing but Eigen vector. So, what does it mean? If we select a solution of say omega 1, that is one particular Eigen value corresponding to that we will get the values of r 1 1, r 1 2 up to r 1 N, which will nothing but give us what is it? The transformation matrix first element phi 1. Similarly, corresponding to Eigen value omega 2, we will get r 2 1, r 2 2 up to r 2 N, which will give us phi 2. Like that for omega N we will get r N 1, r N 2 up to r N N, which will give us phi N. What does it mean? Now, our transformation matrix is ready, so the transformation matrix is nothing but the phi can be written as r 1 1, r 1 2, r 1 N, r 2 1, r 2 2, r 2 N, r N 1, r N 2, r N N.

(Refer Slide Time: 37:37)



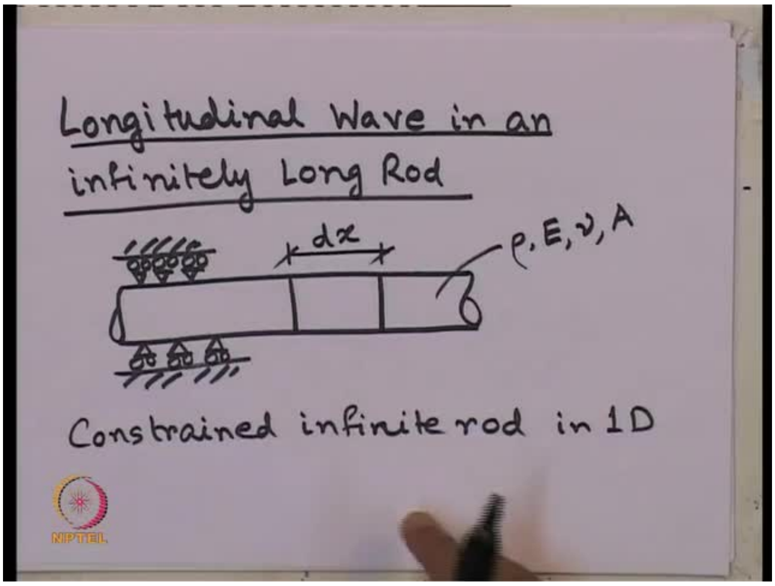
So, this will be transformation matrix of size N by N, as we have mentioned. So, like this using transformation matrix easily for any coordinate system, whatever we have assumed, whatever equation of motion, we got if all the matrices are coupled or some of them are coupled. And some of them are uncoupled, what we can transform them to our natural coordinate system where we will get all matrices are uncoupled or diagonal. And then we can solve it for the N numbers of degrees of freedom which will be obtained from its natural frequencies and then the corresponding solutions can also easily be obtained. So, with this we have come to the end of our chapter, module two of vibration theory. So, it ends our module two of this course on vibration theory. Now, let us start with our next module that is wave propagation.

(Refer Slide Time: 39:36)



For wave propagation, let us first start with the simplest case of longitudinal wave longitudinal wave in an infinitely long rod.

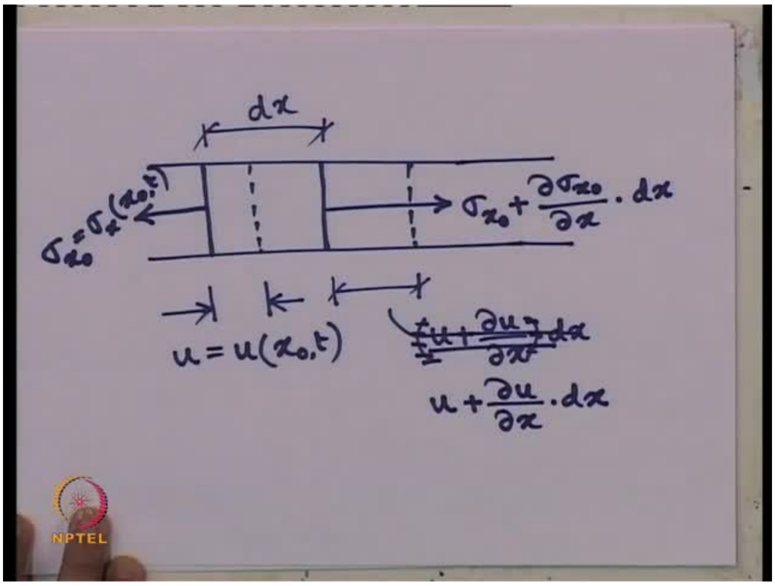
(Refer Slide Time: 39:39)



Let us see, what we can say about these waves? This is an infinitely long rod, we are considering and the boundary conditions we are taking like this, that is, what does it mean? We have put the roller connection to the surfaces of the rod, so that it cannot expand in this direction. So, we are considering basically constrained infinite rod rod in one dimension, in one d, for one dimensional wave propagation and the radial straining that is strain along this radial direction, is restricted by putting these rollers. So, that means we are only talking about the wave travelling in this direction. So, that is why one dimensional longitudinal wave, we are taking care of for this constrained infinite long rod.

Let us say the properties of the rod are rho is the density of the material, E is Young's modular of the material, mu is the portions ratio of the material and A is cross sectional area of the rod. Now, when wave is travelling, the longitudinal wave is travelling in this one dimension in this rod, let us take one small infinitely small segment with a length say d x in this direction. Now, we are concentrating on this infinitely small segment of length d x, through which the wave is passing through. So, when the wave passes through this portion of the rod, what happens? The stresses are getting generated and as the wave is travelling through this section, the stress at this end and stress at this end will be different because the particle is also moving and corresponding displacement of the particle in this direction, let us say u. So, what is the displacement and stress function for this infinites element within that a long rod.

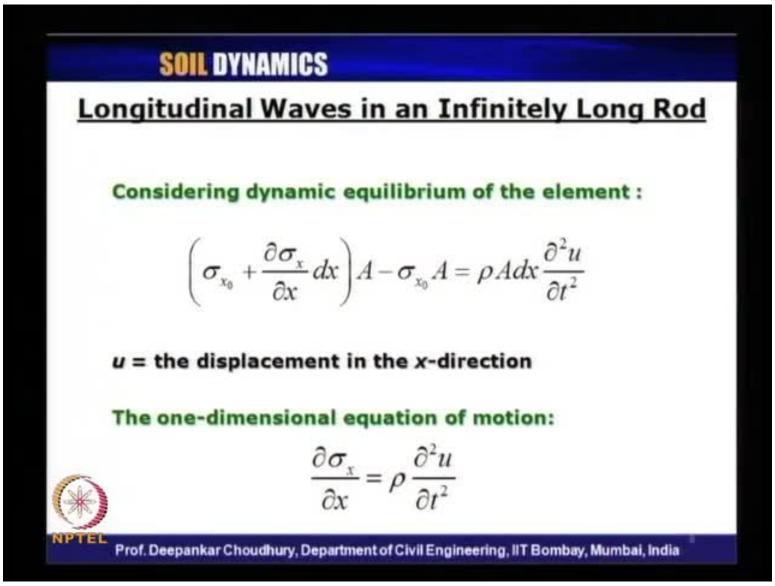
(Refer Slide Time: 43:26)



Let us see, let us exaggerate the figure. So, initial position of the particle when the wave is travelling, let us say was here of length d x and final position, let us say after a time because this wave travelling with respect to time, we are considering and its position new position is given by this dotted line. Let us say when the displacement, this is the displacement u in this direction is given as a function u of x not and t. So, it is a function of x not t which depends on the initial initial displacement. It is a function of initial displacement and time, with respect to time the displacement is keep on changing, clear? And at the end of the element d x, what we are considering?

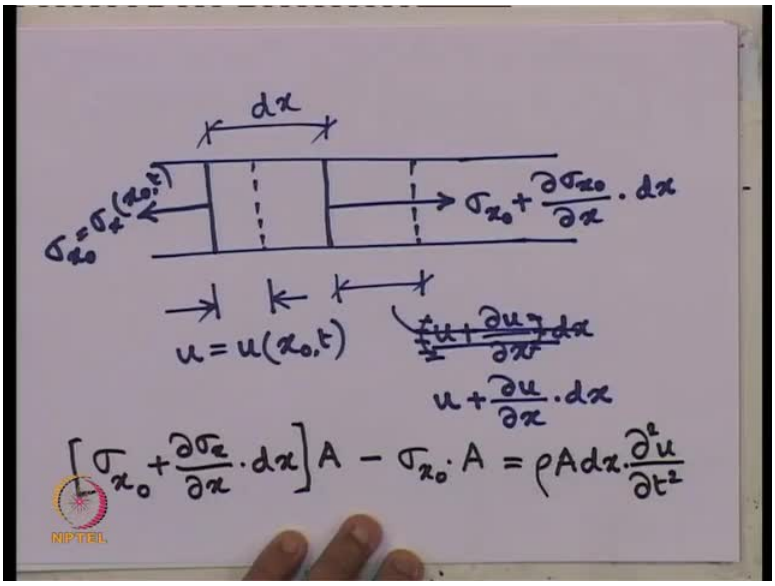
The displacement of this end, the travel of the longitudinal wave is u plus del u del x times d x. Here the displacement was u and the other phase or the other side other end of the section, we have chosen the movement or the displacement of the particle is u plus del u del x into d x. And hat are the stresses acting on the element this side? Let us say sigma x naught which is nothing but sigma x as a function of x naught and t. Whereas, on the other end the stress should be sigma x naught plus del sigma x naught by del x into d x. I do not think, I should put it here. It should be the differential, right? So, let me re-write it properly u with increment del into for the distance d x, that is what we should write. Now, this happens for a distance of d x with the cross sectional area of the element as A.

(Refer Slide Time: 46:34)



So, what we can write from this the considering the dynamic equilibrium in this direction of travel of the wave, we can write it here for the stress equilibrium or the force equilibrium in the x direction sigma x not plus del sigma x del x into d x.

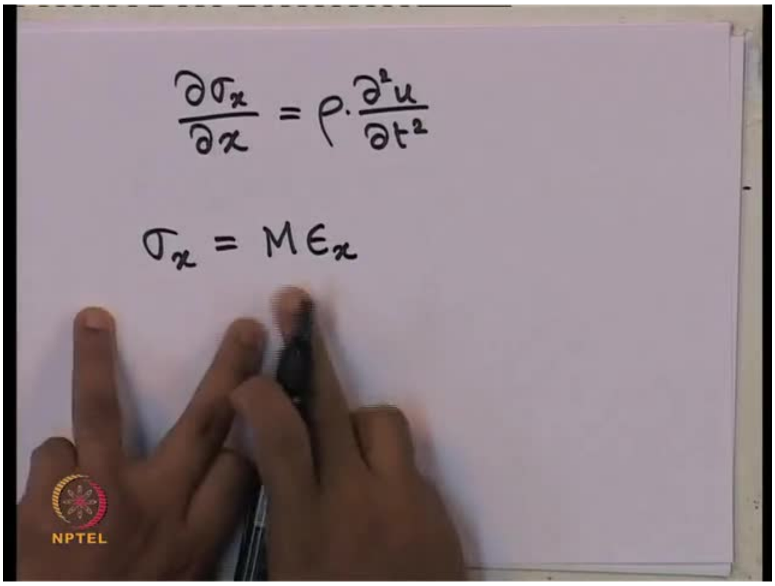
(Refer Slide Time: 46:51)



This is the stress times cross sectional area will give us force in x direction, right? Minus this is the force acting on the right hand side of the element and on the left hand side it is sigma x naught times A. This force, that is the difference of this force between the two ends it is occurring because of what? Because of the movement of the particle by the displacement of u, so definitely there is an acceleration associated with it because we are considering the dynamic equilibrium. And what is that acceleration? Del 2 u by del t square, times mass will give us the inertia force.

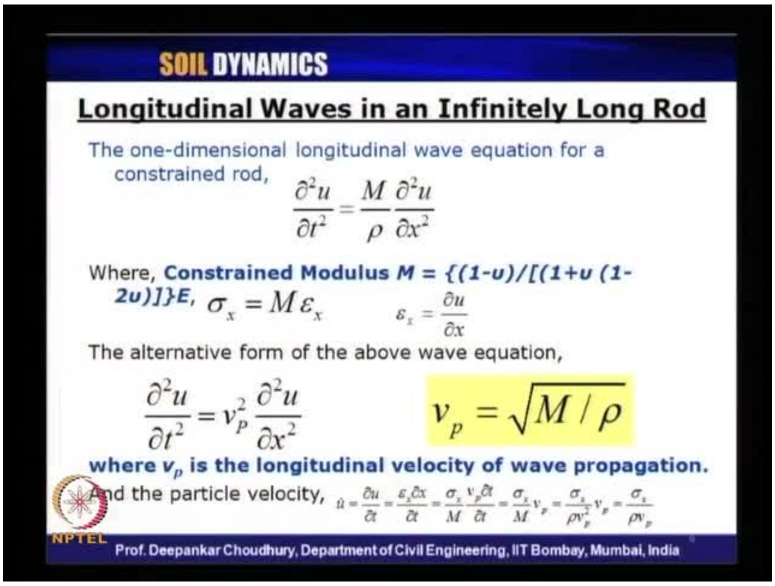
So, these balance of the stresses are internal force, will be equating with the inertia force of the system. So, this should be equals to rho A d x times del 2 u del t square. Actually strictly speaking, when we are talking about one dimension you can use d 2 u by d t square also, but when we are taking the multi dimensions 2 d or 3 d, then it will be partial differential, so remember this one. So, that one is not depended on the other depending on that we have to take whether it is a del or d. Anyway, so this is considering the dynamic equilibrium of the system, we can write the internal force equilibrium should be equated with respect to the inertia force of the system. This will give us by solving if I keep the previous equation here. So, that it is visible. Now, let me solve this one this get canceled, now area of cross section is non-zero d x is non-zero, so A d x A d x get canceled.

(Refer Slide Time: 49:35)



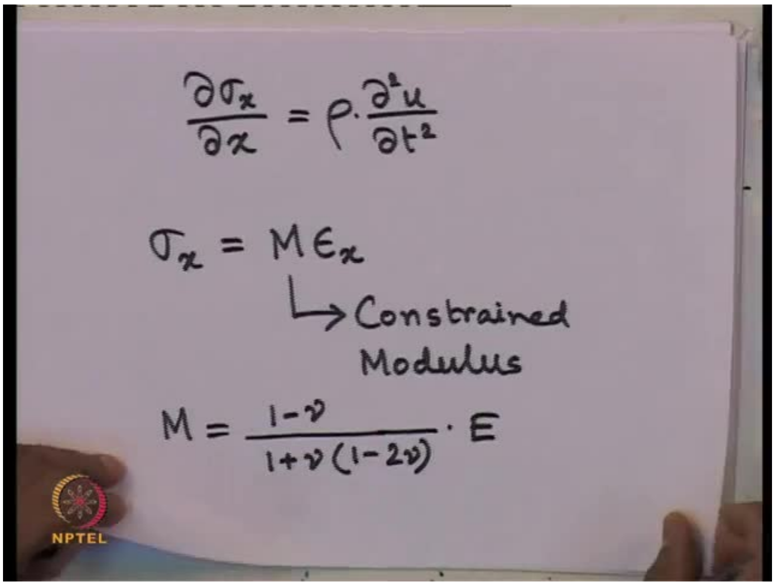
So, we can write it as del sigma x del x equals to rho times del 2 u by del t square. So, this is called basic equation of motion for in one dimension for a wave propagation. So, this is basic equation of motion for longitudinal waves travelling in one dimension. Now, what is sigma x? Sigma x we can re write it as, say some modulus times the strain stress equals some modulus times strain.

(Refer Slide Time: 50:28)



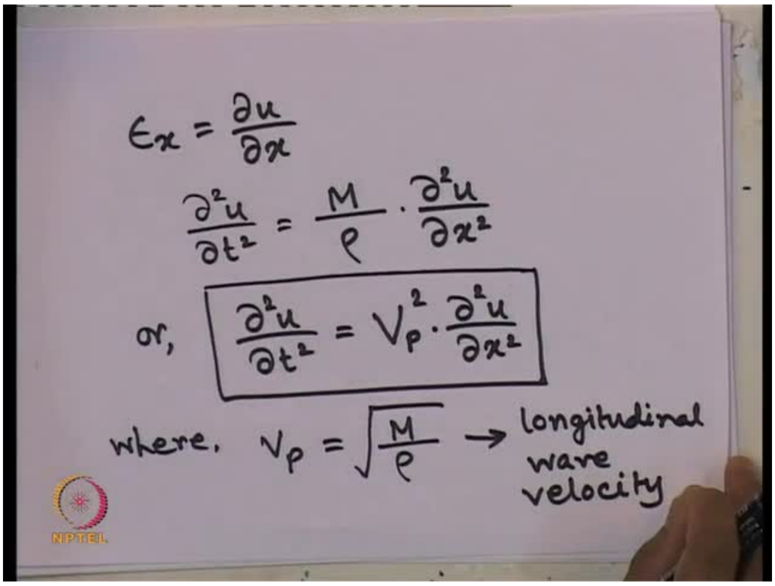
And this modulus m is called for this kind of infinite rod is constrained modulus.

(Refer Slide Time: 50:33)



Let me write it here constrained modulus. Why we have considered one dimensional long rod which is constrain in the direction of its radial movement? So, it is moving only in one direction that is in the x directions, so the corresponding modulus we called as constrained modulus. M is given as in terms of E 1 minus mu by 1 plus mu into 1 minus 2 mu times E. So, that is the expression for constrain modulus and the strain.

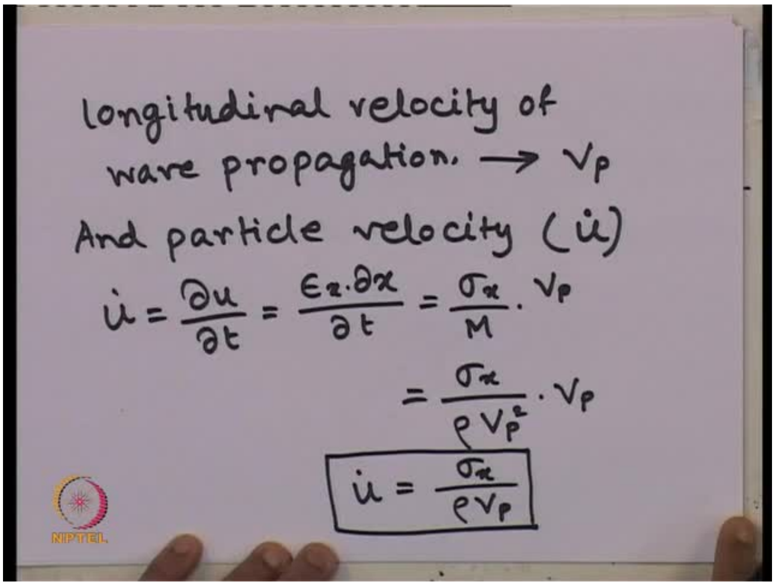
(Refer Slide Time: 51:43)



We know strain is given as, epsilon x is given as del u del x because u is the movement and x is the direction we are considering. So, if we put this expressions in our first equations, that is del sigma x del x equals to rho times del 2 u by del t square, what we can get? In terms of these two we can simplify it and rewrite it as del 2 u del t square, that is this side I am writing first, is equals to M by rho, rho goes here del 2 u by del x square, okay?

So this is the alternative form of the one dimensional equation, what we have written for longitudinal wave. In other words many times it is written as del 2 u by del t square equals to V P square times del 2 u by del x square, this is the final form of the longitudinal waves in one dimension. Where this V P is nothing but root over this M by rho and this is called longitudinal wave velocity, wave velocity. Later on we will see this is also called primary wave velocity, which is nothing but equals to root over constrained modulus by the density of the material.

(Refer Slide Time: 53:54)



And because of this movement whatever is the longitudinal velocity of wave propagation, longitudinal velocity of wave propagation and that is V P and particle velocity say, u dot u dot is the particle velocity which is nothing but del u del t, which can be given as epsilon x del x del t, which is equals to sigma x by M V P. And this will give us sigma x by rho V P square times V P equals to sigma x by rho V P, this is our u dot. So, what does it mean? If the longitudinal wave in one dimension for a infinitely long rod, the velocity of wave is V P, then the particle velocity is given by this stress divided by rho into V P. So, let us stop here today, we will continue our lecture with other directional equations in the next class.