

Soil Dynamics
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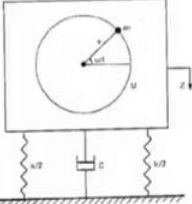
Module - 2
Vibration Theory
Lecture - 10
Transmissibility Ratio, Response to Arbitrary,
Step and Pulse Excitations

Let us start today's lecture of Soil Dynamics, we are continuing with our module 2 of Vibration Theory.

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SOIL DYNAMICS

Force of Excitation Due to Rotating Imbalance



$$(M - m) \frac{d^2 z}{dt^2} + m \frac{d^2}{dt^2} (z + e \sin \omega t) = -kz - c \frac{dz}{dt}$$

After rearranging the terms:

$$M\ddot{z} + c\dot{z} + kz = me\omega^2 \sin \omega t$$

$$M\ddot{z} + c\dot{z} + kz = Q \sin \omega t \quad [\text{where, } Q = me\omega^2]$$

The solution of this equation:

$$z = \frac{me\omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} \sin \omega t$$

$$Z_0 = \frac{me\omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} \quad \text{and} \quad \tan \phi = \frac{c\omega}{k - M\omega^2}$$

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What we had learnt in the next lecture, let us quickly go through a small recap, force of excitation due to rotating imbalance we have learnt in the previous lecture. That is if the total mass of a system is capital M, and there is a rotating components with eccentricity e from the center of that mass of a rotating mass of small m with exciting frequency of lambda we have used in our derivation, here it is omega it is given. So, the equation of motion boils down to this form, where the forced vibration is expressed in this pattern, which is harmonic in nature with amplitude of this much value. That is mass of the rotating component, times the eccentricity, times the square of the exciting frequency is the amplitude of that exciting load.

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SOIL DYNAMICS


Force Due to Rotating Imbalance (Contd.)

In nondimensional form, these equations may be arranged as,

$$Z_0 = \frac{\frac{me\omega^2}{M\omega_n^2}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\xi\omega/\omega_n)^2}}$$

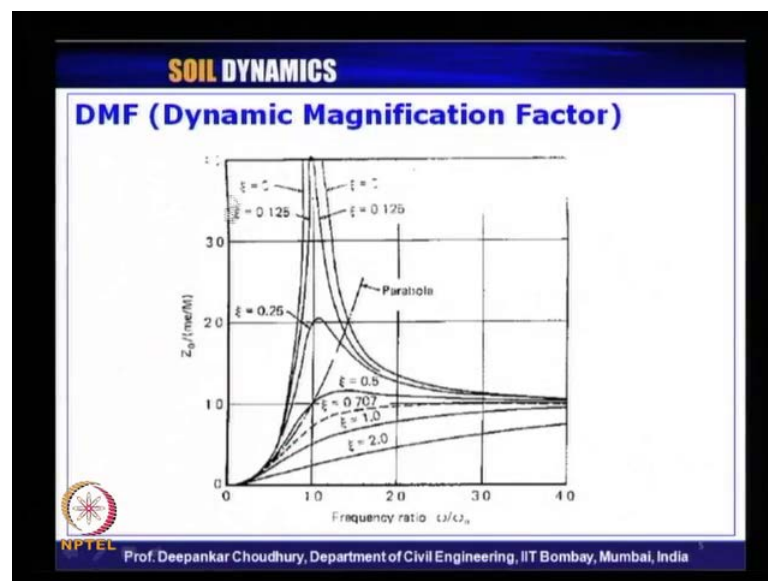
or, $\frac{Z_0}{\frac{me}{M}} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} = DMF$

and $\tan \phi = \frac{2\xi r}{1-r^2}$

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From which we have seen how we can express the dynamic magnification factor.

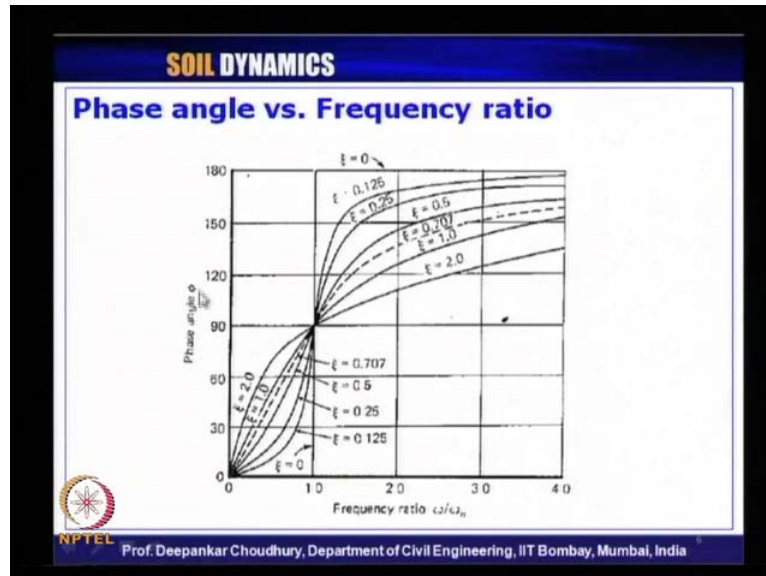
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And that dynamic magnification factor, how it varies with respect to the frequency ratio, and here the peak of the maximum dynamic factor or DMF occurs, little after the frequency or ratio of 1, that is frequency ratio little above 1. And when r tends to infinity, the value of that DMF approaches 1, and when r is very close to 0, than the DMF value is also very close to 0, at 0 it is exactly 0. And these are the patterns for different damping ratio, for damping ratio equals to 0 that is for undamped case, at frequency ratio 1 we get

the value as infinity. But for damped system that is non-zero values of damping ratio, we get a finite value or maximum values of this DMF at r little above 1.

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How the phase angle varies with respect to frequency ratio, that also we have seen how it varies, and it varies from frequency ratio at 0, the phase angle is 0, and when frequency ratio tends to infinity, the phase angle approaches 180 degree. And when frequency ratio is 1 at that point, the phase angle is 90 degree, and it varies accordingly the damping ratio of the system.

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The figure is a slide titled "SOIL DYNAMICS DMF (Force Due to Rotating Imbalance)". It contains the following text and equations:

Now, $DMF = \frac{1}{2\xi}$ for, $r=1$

(DMF)_{max} at $r = ?$

$\frac{d}{dr}(DMF) = 0$

from this, $r = \frac{1}{\sqrt{1-2\xi^2}}$

$(DMF)_{max} = \frac{1}{2\xi\sqrt{1-\xi^2}}$

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Also we have computed what is the maximum dynamic magnification factor, and at which value of that frequency ratio it occurs. We obtain that the value of frequency ratio where it becomes maximum is $1/\sqrt{1-2\zeta^2}$, where ζ is the damping ratio of the system. And the value of that maximum dynamic magnification factor is $1/2\zeta\sqrt{1-\zeta^2}$.

And when we have compared this system with our previous system, we have considered that is single degree of freedom system subjected to harmonic forced vibration. In that case whatever DMF value we obtain, and in this case of rotating imbalance, we got the maximum value of the DMF, magnitude of the maximum value of DMF is same. Whereas, the location of the frequency ratio at which point that maximum is reached is varies; one is reciprocal to the other one.

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SOIL DYNAMICS
Half Power Band Width Method
 (Experimental method to determine ζ)

For small value of ζ ,
 $(DMF)_{max} = 1/(2\zeta)$
 for, $r = 1$

$(DMF)_{min} = \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}\zeta} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

or, $\frac{1}{8\zeta^2} = \frac{1}{(1-r^2)^2 + (2\zeta r)^2}$

or, $\zeta = \frac{\omega_1 + \omega_2}{2\omega_n} \left(\frac{\omega_1 - \omega_2}{2\omega_n} \right)$

$\zeta = \frac{\omega_1 - \omega_2}{2\omega_n}$ since, $\left(\frac{\omega_1 + \omega_2}{2\omega_n} \right) \approx 1$

$\zeta = \frac{\omega_1 - \omega_2}{2\omega_n}$

The slide also features a graph of DMF vs ω showing a resonance peak at ω_1 and ω_2 where DMF is $DMF_{max}/\sqrt{2}$.

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Also we have seen the concept of what is known as half power band width method, it is an experiential method to determine the damping ratio of a system. And using this concept we can find out the damping ratio, based on the applied external frequency and the natural frequency of the system expressed in this way.

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SOIL DYNAMICS

Transmissibility (TR)

The transmissibility (TR) is defined as the ratio between the amplitude of the transmitted force f_T and the amplitude of force applied to the structure.

$$TR = \frac{f_{T \max}}{P_0} = \frac{1 + [2\zeta(\omega/\omega_n)]^2}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \quad TR \text{ is dimensionless.}$$

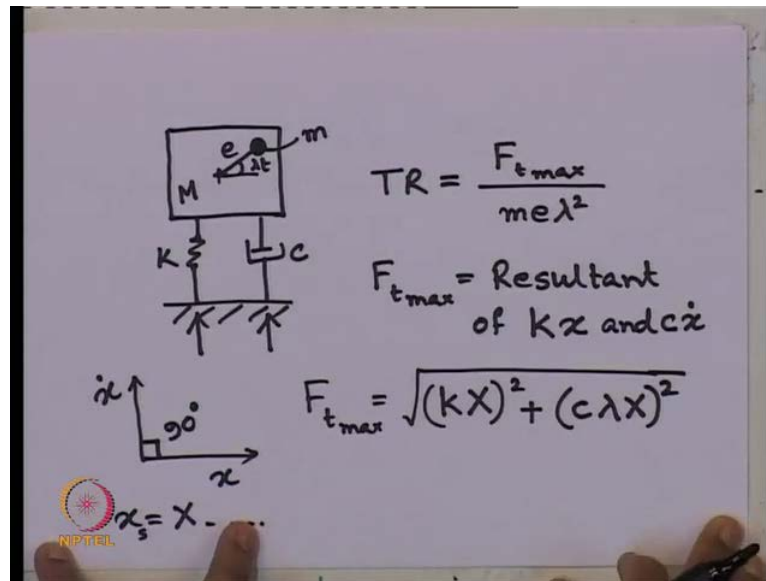
- The objective is to choose the support (k , c) such that TR is as small as possible.
- The mass m of the structure and the frequency of the load ω are imposed. The problem is to choose a support with k and ζ such that the transmissibility TR is minimal.

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Now, coming to our new topic today, we are starting with a terminology which is called Transmissibility Ratio or in short it is expressed as TR. So, what is transmissibility ratio, transmissibility ratio is defined, as the ratio between the amplitude of the transmitted force f_T , and the amplitude of force applied to the system or to the structure. So, in mathematical form, the transmissibility ratio is expressed as the maximum value, this is the amplitude of f_T max, which is transmitted to the system divided by the amplitude of the force, which is applied to the system.

Now, let us see how we can obtain the expression for transmissibility ratio, it is deriving.

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So, let me draw a single degree of freedom system here, with a rotating imbalance, so I am considering a system which is having a rotating mass of small m , and this is λe , e is the eccentricity and capital M is the mass of the total system, this is stiffness k , this is damping c . So, the transmissibility ratio TR , it is expressed as the maximum value of force transmitted divided by the amplitude of the force applied to the system.

And how much is the amplitude of the force applied to the system; that is $m e \lambda^2$ square, we have derived in the previous class. That, what is the amplitude of the force applied to the system, that is mass times eccentricity times, that applied frequency square that gives us the amplitude of the force applied to the system. And how much force is transmitted to the system, force transmitted to the system means, it is coming from this spring and this dashpot.

So, this spring and dashpot, how much force maximum force it can transmit that will be nothing but resultant of how much force the spring is transmitting to the base of the system, that is K times x , if x is the displacement of the system. And how much force is transmitted by the damper that is C times \dot{x} , so resultant of this two force will give us the force transmitted to the base of the system. Now, remember these two forces are not acting in the same direction; they are orthogonal to each other, because we know displacement function and velocity function, and they are acting orthogonal to each other right.

So, what will be the value of $F_{t \max}$ that will be, this x if we express in terms of maximum value we are taking, so K times say capital X times some function, some harmonic function. So, we are expressing x as amplitude times some function, whatever resultant we obtain in the dynamic case, after solving we got some steady state response x s t, in terms of some amplitude times, some harmonic function, do you remember that. So, if it is that then the maximum value will occur only for the amplitude component, so that is what I am taking here.

So, it will be the resultant of these two, so $K X$ whole square plus C , then λX whole square, because if we differentiate this expression of displacement function, steady state displacement what we get, the velocity will come, the λ factor involved in the harmonic function will come out. So, λ times X will be the amplitude for the velocity. So, the maximum value of the velocity will be C times, λ times X , now these are the two forces acting orthogonal to each other.

So, the resultant of them should be just root over square of these two forces. So, that is the resultant force maximum, which is getting transmitted to the base of the system am I right, see if we simply this is what we can get on simplification, let me write it here again.

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The image shows a whiteboard with handwritten mathematical derivations. At the top, the maximum force $F_{t \max}$ is given as the square root of the sum of two terms: $(KX)^2$ and $(c\lambda X)^2$. This is simplified to $KX \sqrt{1 + (\frac{c\lambda}{K})^2}$. Below this, the transmissibility ratio TR is defined as $\frac{F_{t \max}}{m e \lambda^2}$, which is boxed and simplified to $\frac{\sqrt{1 + (2\eta r)^2}}{\sqrt{(1-r^2)^2 + (2\eta r)^2}}$. At the bottom, three specific cases are listed: $r=0 \Rightarrow TR=1$, $r \rightarrow \infty \Rightarrow TR \Rightarrow 0$, and $r=1, \eta=0, TR=\infty$. A small logo is visible in the bottom left corner of the whiteboard.

$$F_{t \max} = \sqrt{(KX)^2 + (c\lambda X)^2}$$

$$= KX \sqrt{1 + \left(\frac{c\lambda}{K}\right)^2}$$

$$TR = \frac{F_{t \max}}{m e \lambda^2} = \frac{\sqrt{1 + (2\eta r)^2}}{\sqrt{(1-r^2)^2 + (2\eta r)^2}}$$

$$r=0 \Rightarrow TR=1$$

$$r \rightarrow \infty \Rightarrow TR \Rightarrow 0$$

$$r=1, \eta=0, TR=\infty$$

So, $F_{t \max}$ is given by root over $K X$ whole square plus C λX whole square, which we can simplify like this $K X$ we can take out, 1 plus c λ by K whole square,

I have taken out KX , so I am dividing it by KX , so that is why X vanishes. So, the transmissibility ratio is nothing but $F_t \max$ by $m e \lambda^2$, if we put this expression as simplify finally, we can get this expression it will give $1 + 2\eta r$ whole square by root over $1 - r^2$ whole square plus $2\eta r$ whole square. So, that is the expression for transmissibility ratio for any system.

And you can see, transmissibility ratio as it is expected will be a dimensionless parameter, because it is a ratio of the two forces, force transmitted to the base of the system to the force applied to the system. And it is function of the frequency ratio, and the damping ratio of the system. So, now if we look at the slide, you can see here, the transmissibility ratio what we have given, this is the final expression in terms of this is the frequency ratio, and this is the damping ratio, and it is dimensionless.

And why this transmissibility ratio is important for our design aspect, because the objective is to choose the support that is support involved with the stiffness, and the damper. When we provide a foundation what we make sure, what is the stiffness provided by the foundations soil system, and how much damping is provided by the foundation and the soil system.

So, we have a control on this parameters K and C in the design such that, the transmissibility ratio is as small as possible why, because transmissibility ratio small means, this is your given force to the system on which hardly you have any control as a civil engineer, that is provided by the machine manufacturer or the supplier of the machine.

What you can do you can design a system in such a way that the force transmitted to the base of the system, in other words when we are handling with the foundation system, the force transmitted to the base of the foundation should be minimum right. So, that minimum dynamic load is transmitted finally, to the system or to the foundation. So, we want this numerator $F_t \max$ to be minimum or as small as possible, to have that we can have a control on either support condition K or C .

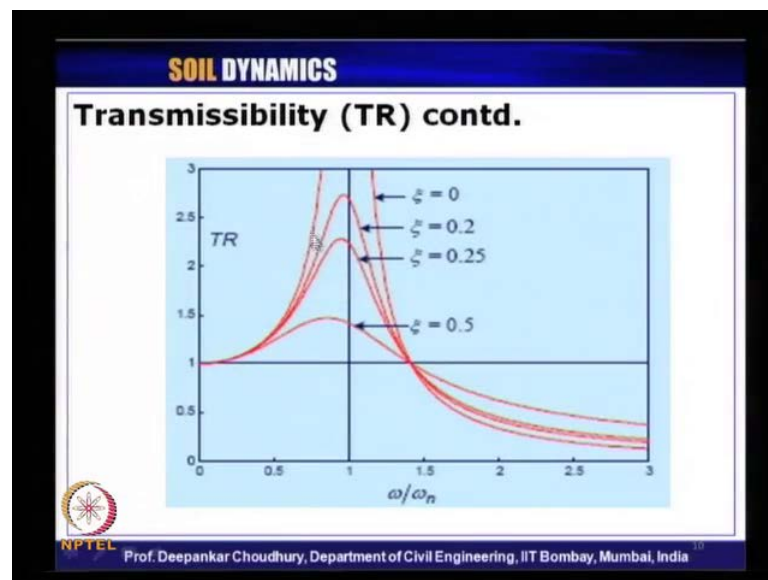
Now, the mass aim of the structure and the frequency of the load that is λ for our case when we have derived, these are imposed it is given. The problem is to choose a support such that, the stiffness and the damping ratio gives us the minimum value of this transmissibility ratio; that is the basic concept of the design. We will again discuss this

and we will come to the concept of vibration, isolation, the same methodology used later on. Now, we know the expression of this transmissibility ratio.

Let us plot how it varies with respect to the frequency ratio, how it should look like, let us see here, this is our frequency ratio and this is the values of transmissibility ratio. Let us look at this sheet here, when r value is 0, frequency ratio is 0, whatever be the value of our damping ratio, how much is the value of the transmissibility ratio, this is 0 so this comes 1, this also 0 vanishes, this 0 vanishes.

So, one transmissibility ratio at frequency ratio, so we can see r equals to 0 gives us TR equals to 1, whatever be the value of damping ratio, and when r tends to infinity what happens to the value of transmissibility ratio, it tends to it tends to infinity it tends to 0. And when r equals to 1, in that case if we consider the undamped system equals to 0, and r equals to 1 what should be the value of TR , infinity. If we put r equals to 1 here this becomes $0 \div 0$. So, $1 \div 0$ we get transmissibility ratio as infinity, but if η is non-zero obviously, we cannot get a infinite value, but it will give us some maximum value.

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So, let us look at the plot on this slide once again, you can see it starts from transmissibility ratio equals to 1, when frequency ratio is 0 irrespective of the value of the damping ratio. And it attains its maximum for undamped case it goes to infinity at r equals to 1, and for damped system it attains a peak little before r equals to 1 that is r little less than 1, the maximum is obtained. Here also you can easily find out, what is the

value of the maximum TR at which value of the r , just you need to differentiate the expression of TR with respect to r .

And then it approaches to 0 at very high value of r , and you can see there is another point where the transmissibility ratio becomes 1, you can check at which value of the frequency ratio, the transmissibility ratio will become 1 irrespective of the value of damping ratio. So, please do this assignment that check what is the value of r , when transmissibility ratio becomes 1 if you solve, it is very easy exercise, I can give you the hint, let us look at this sheet.

We will put transmissibility ratio TR equals to 1 here, equate it with equals to 1, then you simplify it it will give you two values of r , because it is a second order equation. So, you will get two roots of r , one value will be r equals to 0 which already we have seen, and the other value will be non-zero value of r that value you need to find out.

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SOIL DYNAMICS

Response to Arbitrary, Step and Pulse Excitations

Now,

$$\ddot{u}(t) = u(t - t_d)$$

$$= 1, t > t_d$$

$$= 0, t < t_d$$

$$= 1/2, t = t_d$$

Equation of motion

$$m\ddot{x} + c\dot{x} + kx = Fu(t)$$

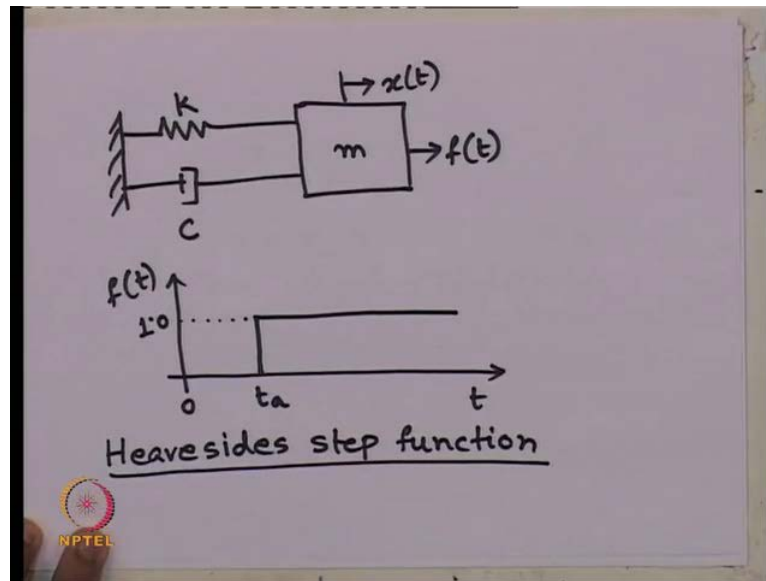
Initial conditions $x(0) = x_0, \dot{x}(0) = \dot{x}_0$

The slide also features a diagram of a mass-spring-damper system with a mass m , spring k , and damper c , and a graph of a step function $f(t)$ that jumps from 0 to 1 at time t_d .

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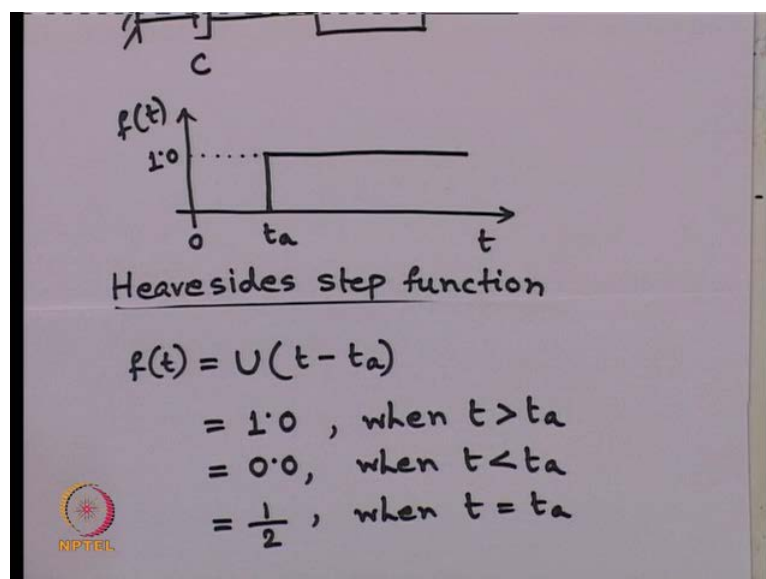
So, now let us move to our next subtopic, that is response to arbitrary step and pulse excitation. So, we have seen the cases of forced vibration with harmonic excitation with rotating imbalance. Now, we are coming to arbitrary excitation, state excitation, pulse excitation; that is all the possible types of excitations; and in those cases, how to obtain the solutions and things like that, we will see now.

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So, let us start drawing the basic system here, basic single degree of freedom system with mass m , stiffness K , damper C , we have single degree of freedom x of t which is subjected to force f of t . Now, f of t is any kind of force, let us say we are defining f of t is something like this, what does it mean that the forced vibration is starting at time t_a , and then it maintains a constant value with respect to time. And let us say the magnitude of this is unit magnitude, now I will define it in mathematical function which is known as this type of loads is called Heaviside step function.

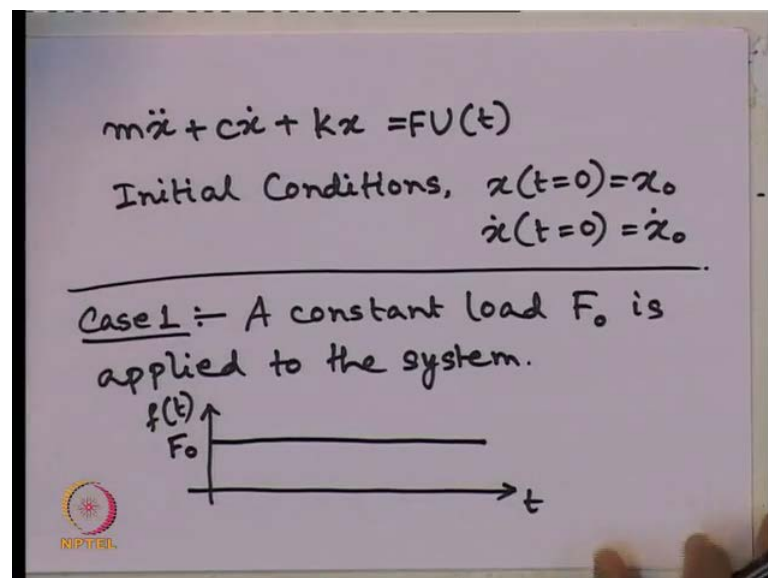
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So, what is heaviside step function, let us move it little up, this type of function is known as heaviside step function and how it is expressed, it is expressed mathematically like this, this is equals to some function $U(t - a)$. The heaviside step function is expressed mathematically in this form. The magnitude of which is equals to unit magnitude, when t is greater than $t = a$ right, from this picture we can write this, when t greater than (a) magnitude is unit.

And its magnitude is 0, when t is less than $t = a$, and what happens when t equals to $t = a$ how we compute, we just take the average of these two right, so we write it is half when t equals to $t = a$ to maintain the continuity at this point, we take that $F(t)$ is half when t equals to $t = a$.

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So, now how we write down the equation of motion for this system, the equation of motion is $m \ddot{x} + C \dot{x} + K x = U(t)$ considering this heaviside step function. Now, $U(t)$ represent a unit load, instead of unit load if a load F is applied what we should write, $F U(t)$ is that clear. So, if there is a unit load, we express that function in terms of $U(t)$ instead of unit load if there is a magnitude of force F , then mathematically we can write it as f times that function.

And then we can use the initial conditions, as usual we do that is x at t equals to 0 is x_0 naught, and velocity \dot{x} and t equals to 0 is \dot{x}_0 naught. Now, let us see what are the different cases we can solve from this situation, so first let me solve case 1 that is a

constant load constant load F naught is applied to the system. So, what does it mean, the variation of that applied dynamic load F of t with respect to time is something like this, so for this forced vibration, what should be the equation of motion.

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Initial Conditions, $x_0 = 0$
 $\dot{x}_0 = 0$

$$m\ddot{x} + c\dot{x} + kx = F_0$$

or, $\ddot{x} + 2\eta\omega\dot{x} + \omega^2x = \frac{F_0}{m}$

$$x(t) = x(t)_{c.f.} + x(t)_{p.I.}$$

$$x(t)_{c.f.} = e^{-\eta\omega t} (A\cos\omega_d t + B\sin\omega_d t)$$

$$\omega_d = \omega\sqrt{1-\eta^2} \quad (\eta < 1)$$

And let us consider for this system, the initial conditions given to us are, let us say initial conditions given to us that x dot is, x naught is 0 and x naught dot is 0, that is the system is starting from absolute rest condition it is possible, because we are using the forced vibration, so that is why it can vibrate, so no problem. And the equation of motion will now change to m x double dot plus c x dot plus k x equals to F naught simple equation, which we can simplify now, as x double dot plus 2η ω x dot plus ω square x equals to F naught by m , we have divided both the sides by m .

Now, we have to solve this equation, so solution is same like we have done earlier x of t complementary function plus x of t particular integral; and complementary function we know, if we equate it with respect to 0, we will get complementary functions. So, x of t complementary functions is already known to us, which is e to the power minus η ω t . A cosine ω d t plus B sign ω d t where ω d is nothing but ω t times d root over 1 minus η square and of course, η less than 1, the damper case we are considering, so this solution we know.

We need to find out this particular integral, for this case with the constant value obtaining particular integral is very easy, if we apply our concept of basic mathematics, we will get the solution for particular integral x of t .

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$$x(t)_{\text{P.I.}} = \frac{F_0}{m\omega^2}$$

$$x(t) = e^{-\eta\omega t} (A\cos\omega_d t + B\sin\omega_d t) + \frac{F_0}{m\omega^2}$$

$$x_0 = \dot{x}_0 = 0$$

$$\dot{x}(t) = -\eta\omega e^{-\eta\omega t} (A\cos\omega_d t + B\sin\omega_d t) + e^{-\eta\omega t} (-A\omega_d \sin\omega_d t + B\omega_d \cos\omega_d t) + 0$$

Particular integral is nothing but that constant divided by m omega square that is it, so what is the complete solution, the x of t complete solution is given by e to the power minus eta omega t A cosine omega d t plus B sign omega d t plus this particular integral F naught by m omega square. And on this total solution we have to now apply the initial condition, which is given to us that these all are 0 .

So, what we can do now, let us differentiate this total solution to get the expression for the velocity, what we will get, I am differentiating this first, it gives me minus eta omega e to the power minus eta omega t , it remains as it is A cosine of omega d t plus B sign of omega d t plus. Next I am differentiating this component, so e to the power minus eta omega t minus A omega d sign omega d t plus B omega d cos omega d t .

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$$x(t=0) = 0 = A + \frac{F_0}{m\omega^2}$$
$$\therefore A = -\frac{F_0}{m\omega^2}$$
$$\dot{x}(t=0) = 0 = -\eta\omega A + B\omega_d$$
$$\therefore B = \frac{\eta\omega A}{\omega_d} = -\frac{\eta\omega F_0}{\omega_d m\omega^2}$$
$$\therefore B = -\frac{\eta F_0}{\omega_d m\omega} = -\frac{F_0}{k} \cdot \frac{\eta}{\sqrt{1-\eta^2}}$$

And this component if we differentiate of course, it will give us 0 plus 0 with this now let us put the boundary conditions of x at t equals to 0 is 0. Let me put the equation here, so it will be easy for us to follow. So, I am putting t is equal to 0, in this expression what I am getting here this gives us 1, A , this vanishes, this remains, so it gives us A plus F naught by m omega square therefore, A is minus F naught by m omega square.

And now, so compute B let us take the second equation, this equation let me put it here, so that we can follow x dot at t equals to 0 is 0 given to us by putting it t equals to 0, we are getting how much minus η omega A plus from this how much we are getting this remains, so B omega d . Therefore, B works out to be η omega a by omega d , now how much is A η omega, A is minus F naught by m omega square, so let us put minus here F naught by omega d m omega square.

So, this gets cancelled. Therefore, B is minus η F naught by omega d m omega, so if we simplify what we will get minus F naught by k η by root over 1 minus η square. Because, omega d is nothing but omega times root over 1 minus η square, so it becomes m omega square, m omega square means nothing but k . So, this is the expression for B , and this is the expression for A , so we get the complete solution now.

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$$x(t) = \frac{F_0}{k} \left[1 - e^{-\eta\omega t} \left(\cos\omega_d t + \frac{\eta}{\sqrt{1-\eta^2}} \sin\omega_d t \right) \right]$$

For $\eta = 0$ [$\omega_d = \omega\sqrt{1-\eta^2}$]

Solution, $x(t) = \frac{F_0}{k} (1 - \cos\omega t)$

As x of t equals to F naught by k , hence $1 - e^{-\eta\omega t} \left(\cos\omega_d t + \frac{\eta}{\sqrt{1-\eta^2}} \sin\omega_d t \right)$. Let us look at the solution once again. So, that it will be easy for us to follow, this was our complete solution, we got the value of A as $-\frac{F_0}{m\omega^2}$, and B we got as $\frac{F_0}{k} - \frac{F_0}{k\eta} \sqrt{1-\eta^2}$.

So, this was our B component $\frac{F_0}{k}$, $\frac{F_0}{m\omega^2}$ is nothing but $\frac{F_0}{k}$ that is nothing but our A and this part $\frac{F_0}{\omega^2}$ is like this, so clear the complete solution. Now, if it is known to us the complete solution of the system, how the response should be like let us look like it, let us first take undamped case, for the case η equals to 0 . What is the solution for undamped case, the solution for x of t is nothing but $\frac{F_0}{k}$ how much we are getting, $1 - \eta$ is 0 .

So, it becomes 1 , this becomes ω_d is nothing but ω is nothing but ω times $\sqrt{1-\eta^2}$, so it becomes ωt . So, $\cos\omega t$ here η is 0 , so everything vanishes, so that is our solution, so for undamped system subjected to a forced vibration with a constant force F_0 , the solution is like this. And if we want to plot the response of it with respect to time, how it should look like, so we are plotting the response with respect to time t , the total response x of t , when t equals to 0 where it should start, at t equals to 0 it is 1 , so value is 0 .

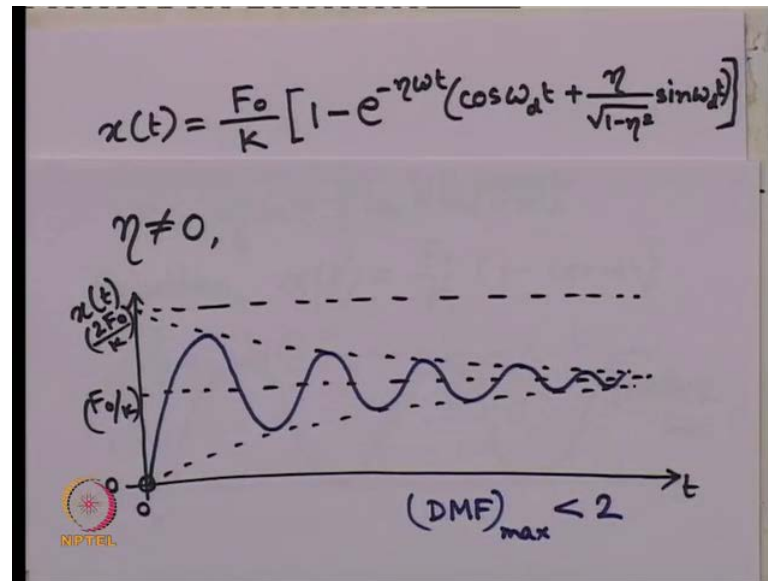
So, it should start from this point 0 , when t equals to, let us see the variation $1 - \cos \omega t$. How it should vary a \cos function starting from this 0 a harmonic function, it is oscillating about which line, what is the maximum value, and what is the minimum value, minimum value is about this 0 that we have obtained. Maximum value will be $2F/k$, because maximum it can go up to 1 , then it gives 0 , minimum it can go to 0 on minus 1 when it becomes, then it becomes $2F/k$, and when it is 1 then we are getting 0 .

So, three about this axis of F/k , let me draw two parallel lines, this is F/k by k , this will be $2F/k$, when it is giving minimum value of minus 1 , we are getting $2F/k$. So, it is now a harmonic function oscillating like this, it is undamped case, so there is no damping it is keep on oscillating like this about this line, it is oscillating not about the base line of x t equals to 0 .

But, oscillating about the central line of magnitude F/k , and what is our F/k , it is nothing but the static displacement; so F/k is static displacement, and maximum it can reach up to $2F/k$. So, what we can say what is the DMF, for this case Dynamic Magnification Factor, maximum value of the dynamic magnification factor for this case is 2 . What is the definition of dynamic magnification factor once again, it is the the steady state or the dynamic displacement divided by the static displacement.

And static displacement is F/k , and the maximum value of the dynamic displacement is $2F/k$, so that is why the maximum value DMF is 2 . This is the response of this solution, which is pretty simple for us to draw for undamped case, now if it a damped case, how if the response should look like, let us see.

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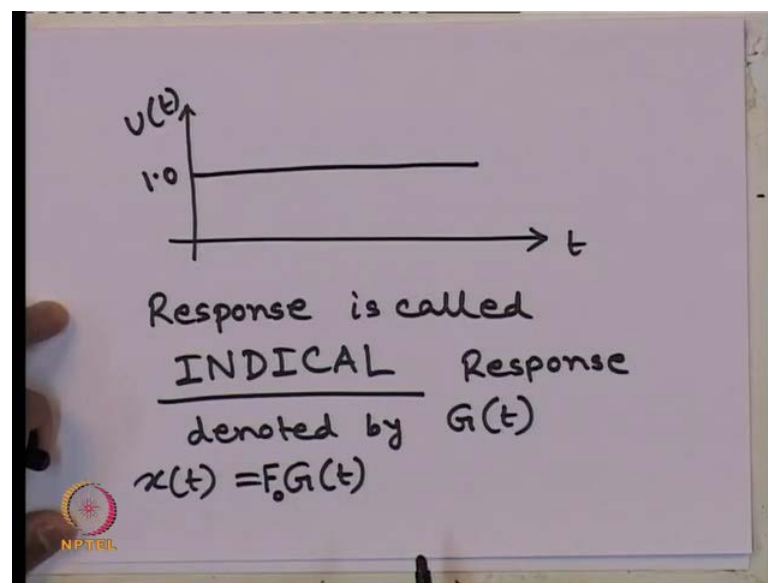
So, for a damped case, when η is not equal to 0, in that case the solution will already we have, so let us take the solution once again here, and let me plot it the response x of t with respect to the variation of t . We have seen from the previous solution, it oscillates about on the central line which is having a displacement of equal to the static displacement, F_0 by k . And in this case, if we look at the solution, it is a harmonic function an exponential decay function, and a constant term; so here also it should oscillate about this value of static displacement.

So, let us draw that line which is equal to f_0 by k , and let us draw another line here, earlier for undamped case which we got $2 F_0$ by k is the maximum value, this is the 0 line, and this is the t equals to 0. When t equals to 0, it should start from again this origin 0 0 value, if we put we can get it, how the response should look like, it will be a harmonic function, oscillatory function, but with the decay function, and it will decay after it decays when time tends to infinity, the value of x t approaches to F_0 by k .

Because, these goes to zero but this component remains, so that why the oscillation I said will occur about the static displacement point. So, if we want to draw the response, earlier whatever we have drawn the similar response only change is the baseline of oscillation has now changed, so instead of the 0 baseline, it is now oscillating about the static displacement. So, the final response curve is like this, this is the final response of the system, with t tends to infinity which is the static displacement.

So, in this case the DMF max, Dynamic Magnification Factor maximum what should be its value, we can compute it, it will be in the first cycle here, but it will be always less than 2, that we can comment directly from the looking at the figure. It will never reach this line, because there is decay function, so this envelop is nothing but the decay envelop. So, this oscillatory motion is for now clear to us, now there is some mathematical expression or the terminology that is used for this kind of response.

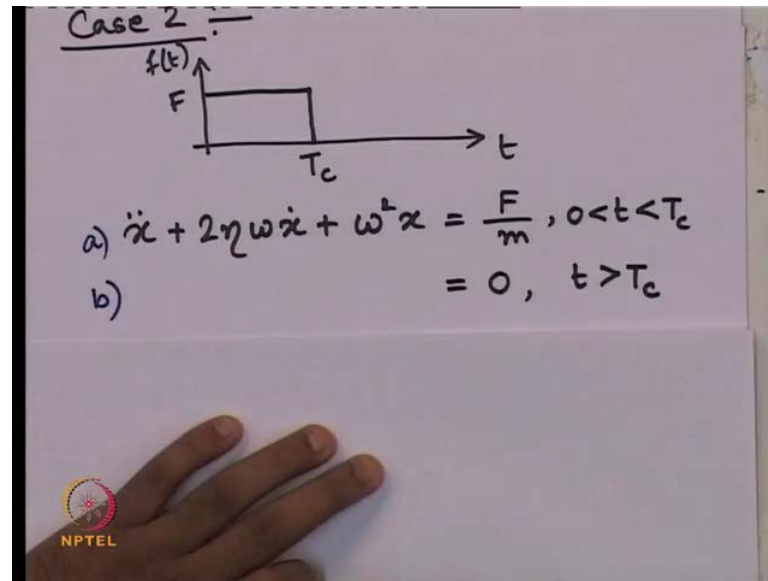
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And let me now give you that, if we have U of t which is nothing but a unit load acting for constantly with respect to time that we have expressed as heaviside step function U of t . In that case the response what we get, response is known as, response is called INDICAL response, and mathematically it is denoted by G of t . What does it mean that if a single degree of freedom system is subjected to a heaviside step function of U of t , in that case x of t is called INDICAL response function and denoted by G of t function.

So, for any force F , as we have used a constant forces of F naught in that case the total response should be F naught times that INDICAL response, INDICAL response is for a unit load. So, if there is F naught load is applied in that case we call it as F naught times F t F naught times the INDICAL response, this is just a terminology. So, you should know this is the terminology used for a single degree of freedom system, subjected to a forced vibration in terms of heaviside step function the response is known as INDICAL response.

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Now, if we have finite time duration applied load, now we are considering case 2, the case 1 was the load was acting for infinite time constant load. But, in this case now what I am telling the variation of our F of t is something like this, up to say T_c it is occurring which is quite possible, let a load of constant magnitude F is acting for a finite time T_c on the system, and after that the load is removed and not acting on the system. Then how we write down the equation of motion for this kind of loading system, which is pretty easy x double dot plus $2\eta\omega$ x dot plus ω^2x equals to F by m , if t is between this.

But, this equals to 0, if t is greater than T_c that is what the equation of motion takes the form, it is a forced vibration case till the time T_c , but beyond time T_c is a free vibration case. So, how the solution of this type of loading to the system we can get and finally, we are interested about the response looks like or how the displacement function looks like with the respect to time. Let us see how we can do, this is nothing but composed of two parts as we have already given here, let me say this is our past equation is part a, and the second equation is part b.

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Solution of (a),
$$x(t) = \frac{F}{k} \left[1 - e^{-\eta\omega t} \left(\cos\omega_d t + \frac{\eta}{\sqrt{1-\eta^2}} \sin\omega_d t \right) \right]$$

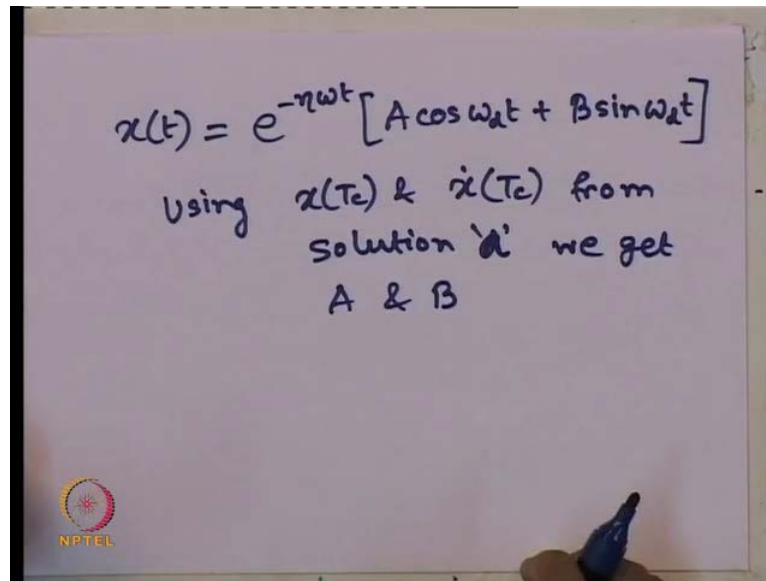
at $t = T_c$,
$$\boxed{x(T_c) \text{ \& \dot{x}(T_c)}$$

↓
Initial Conditions
for part (b)

So, the solution for part a is obtained just now, so solution of part a what was the solution that complete solution x of t is equals to F by k into 1 minus e to the power η ω t cosine ω d t plus η by 1 minus η square under root sign off ω d t , that was the complete solution for part a. Now, how to go to part b, part a solution and response is known to us, to go to part b what we have to do, at t equals to T_c we can find out $x(T_c)$ and the velocity $\dot{x}(T_c)$, from this expression. If we put the time T_c up to which the load is acting, we can get the displacement and velocity.

Now, this is, these two are the initial conditions for part b, so these are the initial conditions for part b. Because, look at the system here, when the time reached T_c whatever is the displacement and velocity at that point is nothing but the initial condition for the second part when it is vibrating freely.

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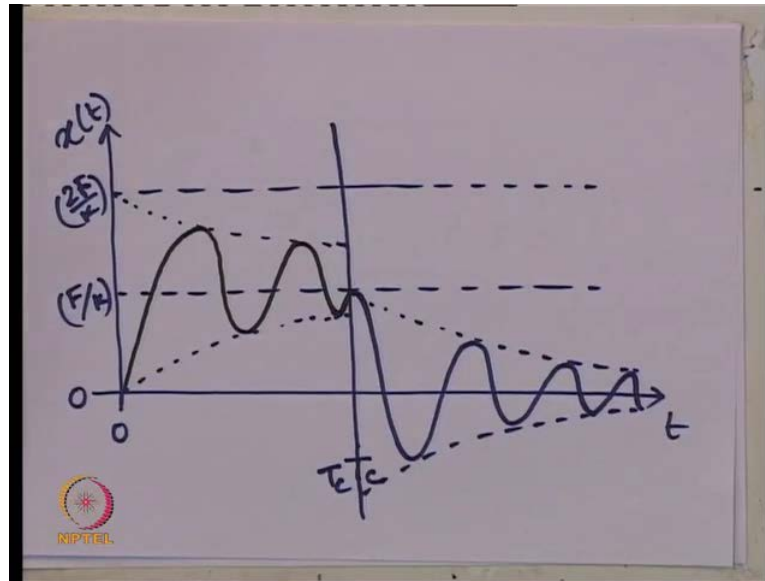
The image shows a whiteboard with handwritten text. At the top, the equation $x(t) = e^{-\eta\omega t} [A \cos \omega_d t + B \sin \omega_d t]$ is written. Below it, the text reads: "Using $x(T_c)$ & $\dot{x}(T_c)$ from solution 'a' we get A & B". In the bottom left corner, there is a small circular logo with a star and the text "NPTEL". A blue marker is visible in the bottom right corner of the whiteboard.

$$x(t) = e^{-\eta\omega t} [A \cos \omega_d t + B \sin \omega_d t]$$

Using $x(T_c)$ & $\dot{x}(T_c)$ from solution 'a' we get A & B

Now, for the second part, we know what is the solution x of t the complete solution here we do not have it is a free vibration case, so no need of particular integral the complete solution we already have minus $\eta \omega t$ $A \cos \omega_d t$ plus $B \sin \omega_d t$. Now, we can easily get this using the initial conditions of $x(T_c)$ and $\dot{x}(T_c)$ from solution a, a solution a we get A and B, so that finally, gives us the total solution, so numerically or analytical we can compute easily, but putting these values here; so analytical solution complete solution is available to us. Now, if I want to find out what is the response that is how displacement function, varies with respect to time for the complete system, how it should look like, let me plot it.

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This is variation of x of t I am trying to plot this is our 0 line, these two parallel lines I am drawing for F by k , and this is $2F$ by k , initially what we have seen it will oscillate about this static displacement till the time T_c . So, let us draw the exponential decay, let us say this is our time T_c , so the function will move like this, now what happens here when it reaches T_c at this point, we have some displacement it can stop here, it can stop here, it can stop here, within this range right, I have stopped somewhere for example, here. So, it is having this as initial displacement, and whatever be the slope is initial velocity for this second part that is beyond time t greater than T_c .

Now, how it should vibrate or oscillate it should vibrate about again a decay function, but its baseline will now change, why because the solution for free vibration is this one, which oscillates about the baseline of x of t equals to 0 naught x of t equals to F by k right. So, now from this point it will start oscillating about this baseline not about this baseline, so what will happen just emetic diagram I am drawing, now it will follow another decay function from this point onwards, and from here it will simply drop to this baseline. From this baseline it will drop to this baseline, and follow that harmonic pattern with the decay function, that is the combined response of the system.

Till time T_c when the constant force was acting it was vibrating about its static displacement, about this line and beyond T_c when it is removed and the case is free vibration, it is vibrating or oscillating about this 0 baseline clear, why the response

should look like this. see if we have understood this one, we can now solve for any kind of heave sides stave function whatever I have given that any kind of constant load acting for any finite duration of time. Whether, from the beginning of the time or in between or starting from a finite time, and remaining constant throughout for any case, we can solve the problem right, with this we will end this lecture.