Ultrafast Optics and Spectroscopy
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Lecture - 32
Maxwell's Equations

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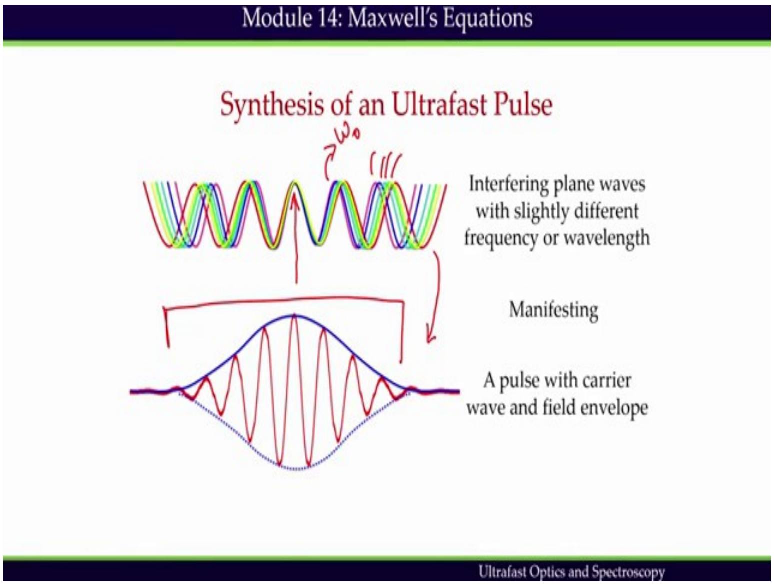

Welcome to module 14 of the course Ultrafast Optics and Spectroscopy. In this module, we will go back to the context which we presented in module 2 which was mathematical representation of ultrafast pulses. In module 2, we have seen we have gone over Maxwell's equation quickly, but in this module we will go over Maxwell equation one more time and try to understand the application of Maxwell's equation in non-linear optics and linear optics.

In module 2, we have seen that an optical pulse originates from the optical interference of slightly different frequency components. Each frequency component represents a plane wave which is shown here. So, each color here red, blue, green, yellow each color is representing slightly different frequencies, they are interfering with each other. And at this point we are having constructive interference, and rest of the parts having to some extent destructive interference and that is why we are localizing electromagnetic energy within this time, and this is the wave we have created the pulse.

So, optical pulse is nothing but the interference of different color components with a particular phase relationship. Now, the plane wave we are talking about plane wave, each plane wave represents a particular frequency omega naught.

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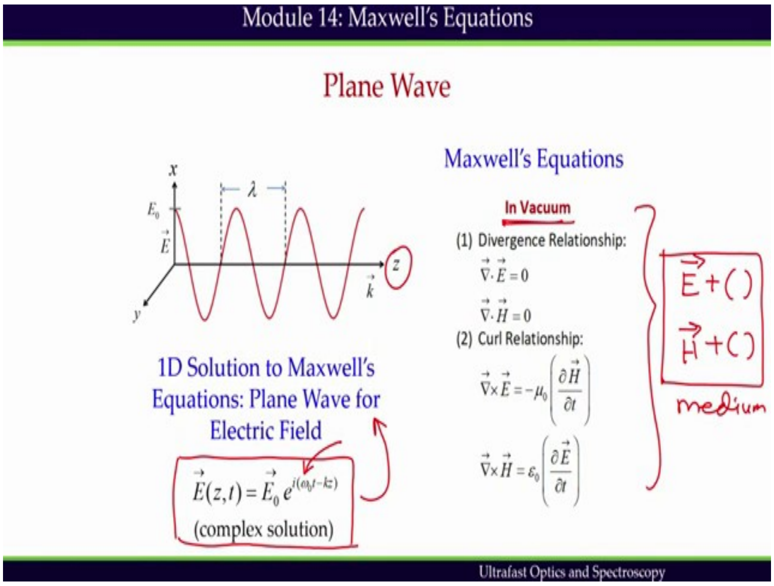

This plane wave we have seen that plane wave is a solution of a fundamental equation of optics, and it is represented the complex notation of a plane wave is represented by this; this is a plane wave propagating along the positive Z-direction. And, we have given a proof for that in module 2. This omega naught[ω_0] is the frequency angular frequency of the plane wave; this is the complex notation of the one-dimensional plane wave. And, this came from the solution of Maxwell's equations. In module 2, we have quickly gone over the Maxwell's equations. These are the Maxwell's equations in vacuum; in medium, it will change.

We mentioned previously then in the medium instead of E, we have a cooperative effect which means field plus medium's response, similarly magnetic field p plus medium's response. So, these are the cooperative effects we have to consider in the medium. But in vacuum we do not have this cooperative effect we have only electric and magnetic fields in the Maxwell's equations. So, in this module, we will just go over the details of this Maxwell's equation.

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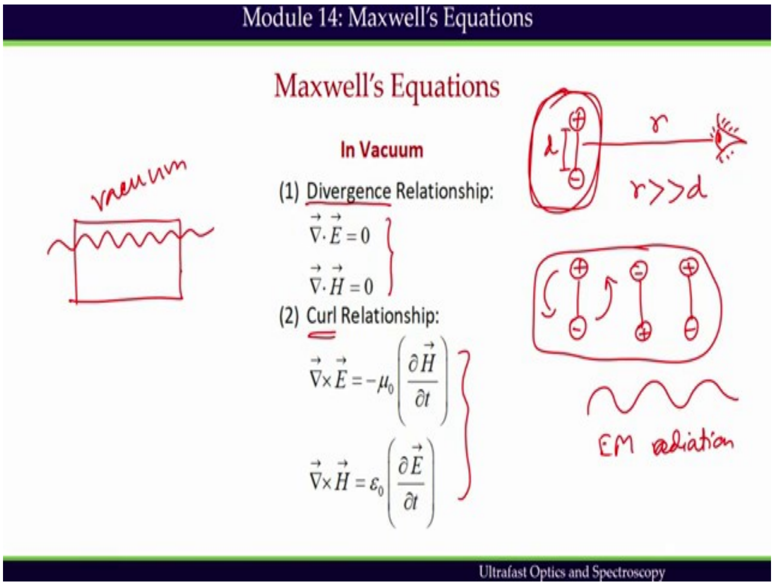

We know that oscillating dipole is a source of electromagnetic radiations. If I have an dipole positive negative. What is dipole by the way? Definition of dipole comes from the fact that if the observer, if I am observing it from a long distance typical distance is let us say r, and the dipole separation distance is d, then I will consider this charge separated entity as dipole when r is much greater than d that is very important to realize; otherwise we will consider is an isolated charge. But r, if the observer distance is very, very long, then I can consider this entity to be dipole. So, the definition of dipole depends on how we are observing the entity.

Now, we know that if we have a dipole and it is oscillating. So, after a certain time, I will have polarity changed; again after a certain time I will have a polarity change. And, if this is happening in it is anywhere in space, then definitely this oscillatory dipole will produce electromagnetic radiation, which means that any light we create it is the result of some oscillatory dipole. When light or electromagnetic wave propagates through any medium, electromagnetic state of the medium changes.

So, let us say I have a medium, it could be vacuum let us say vacuum, and electromagnetic radiation is propagating through this medium. If it is propagating through the medium, then electromagnetic state of the medium will change. And what does it mean? Maxwell's laws of electromagnetic state suggest that to find electromagnetic state of a medium, we need to know two characteristics which is divergence, this is the vector characteristics divergence and curl, two different characteristics of the electric and magnetic fields. This electric and magnetic fields are nothing but the light.

In mathematical words, we need to know two del operations on the associated electric and magnetic fields. So, the first operation is dot product and another operation is cross product. So, these are the two operations we need to know. To begin with the simple picture, we have invoked vacuum as medium.

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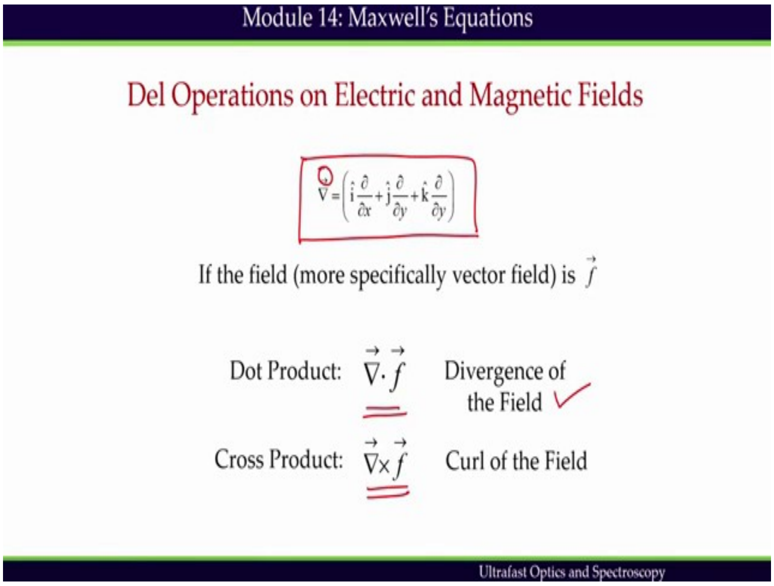

But what is the del operation by the way? A del operator is known as vector differential operator which is given by this. In many text book, we do not use this arrow, but here we are using this arrow just to present that this is a vector dot product with the field is given by this, and cross product with a field is given by this. Dot product features divergence of the field and cross product features the curl of the field.

We will try to understand very qualitatively the physical meaning of these two del operations. First the divergence and curl of a vector are two vector operations which can be understood geometrically by viewing a vector field as a flow of fluid.

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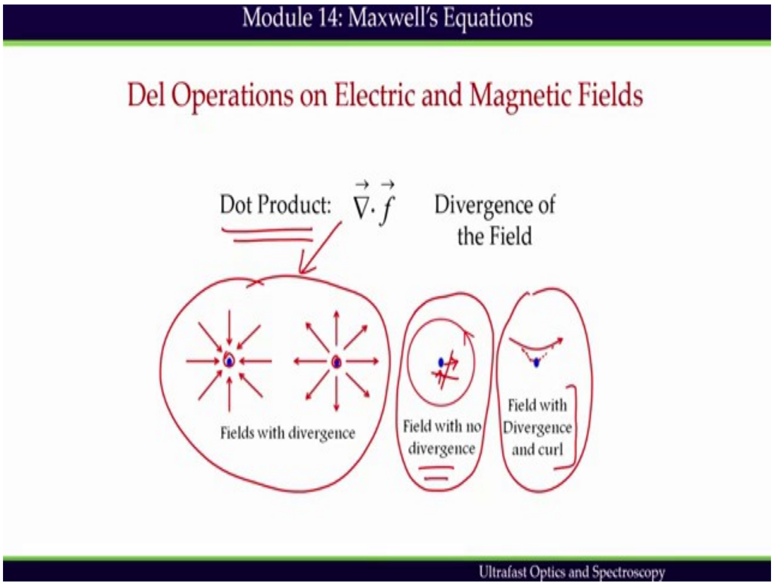

When we have a dot product with a field f, it is a measure of fields tendency to converge towards or diverge from a point. For an example here, if this is representing the field, then it is quite clear that the field either converging towards this point or diverging from this point that means, I should have definitely dot product. So, after having this dot product, I get to know the feature of the field, whether the field is converging to a point or diverging from a point. This is a very qualitative idea, but gives a quick idea of the physical meaning of the dot product.

On the other hand, if we look at this field, this fields does not have any divergence, because it is not converging to this point or it is not diverging from this point that is why this, this kind of field should not have any divergence which means dot product would be 0. On the other hand, if we look at this field qualitatively very qualitatively, then it has certain tendency to converge to this point and diverge from this point that is why this field may have divergence.

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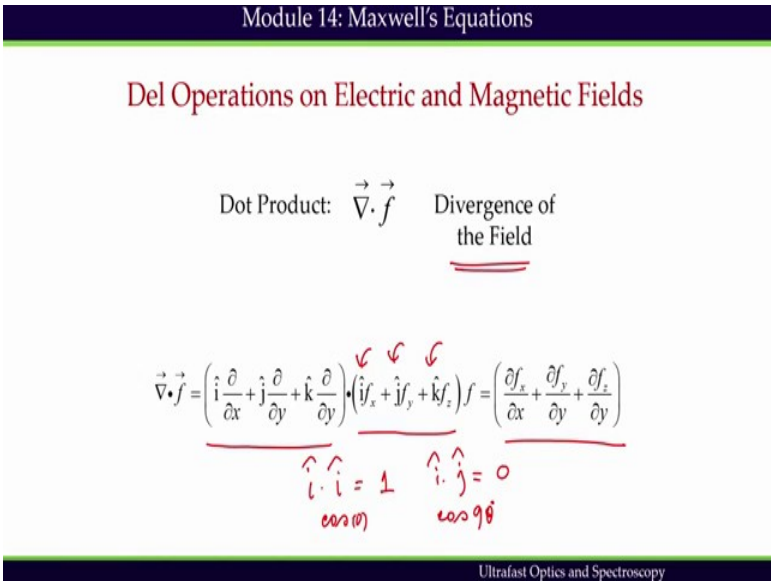

Dot product which represents divergence of the field can be represented mathematically by this which is nothing but finally a scalar quantity. Because, i dot i unit vectors are nothing but 1, but i dot j is zero its because this is becoming cos 90 and this is becoming cos 0. Here i, j and k, these are the vectors along x, y, z axis respectively unit vectors.

On the other hand, cross product of a field represents the curl of the field; in a rotating vector field curl is a measure of torque or - rotation a sphere or rotating object will experience. So, definitely this kind of field will have cross product or curl, and mathematically we can represent this cross product with the help of this determinant.

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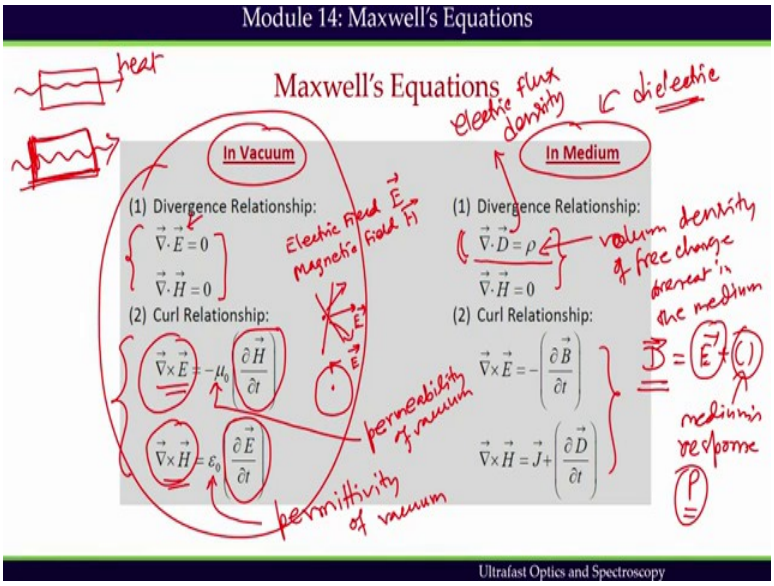

Now, let us look at the Maxwell's equations one more time. This is in vacuum; this is in medium. It could be any medium, but primarily in optics we are interested in dielectric medium, because most of the optics lens mirrors where played they are all dielectric medium. In its simplistic point of view the idea of electromagnetic state can be understood in analogy with thermodynamic state. When heat energy flows through a medium, it changes thermodynamic state. So, let us say I have a medium and heat is flowing through it.

We all know that if heat is flowing through the medium, then mediums thermodynamic state is changing. In a similar sense, when I have a medium and I have electromagnetic radiation going through the medium, it could be vacuum it could be any medium, then we say that electromagnetic state may change in the medium. And how do I know? The current state of the medium the electromagnetic state of a medium is fully described by Maxwell's equations given here.

So, Maxwell's equations are representing a state of the medium where electromagnetic radiation has electromagnetic radiation has passed through it. For now we shall restrict our discussion to the Maxwell's equations in the vacuum only, and then we will go back to the medium. There are many characteristics of vacuum we can understand. This will give us an opportunity to grasp the details of each equation. Later we shall discuss more about the Maxwell's equations in the medium particularly in the dielectric medium.

The Maxwell's equations in vacuum suggest that the electric and magnetic fields represented by E and H. This E is electric field, this is represented by E; and magnetic field is represented by H. What it suggests that in vacuum they are 0. And if they are 0, it means that divergence is 0. Divergence of electric field and magnetic field is 0, it means that it does not converge to the point or it does not diverge from a point in the medium.

However, if you look at the curl, they exist it means that both field's electric and magnetic fields in vacuum are rotating in nature. So, definitely, if I have a point from this point electric fields are not like this, they are not like this, they have something like this because its curl exist. Furthermore, what we understand here the time varying magnetic field this derivative with respect to time is suggesting time varying magnetic field, it is the source of the electric field here or time varying electric field is a source of magnetic field vice versa.

Here epsilon naught, this is this quantity epsilon naught is called the permittivity of vacuum, and mu naught this quantity is permeability of vacuum. Permeability is associated with magnetic field; permittivity is associated with electric field. In the simplistic point of view, the permittivity and permeability of vacuum features some kind of resistance against electric and magnetic fields respectively. This means that vacuum does not bring nothing vacuum has something which exhibits resistance against magnetic and electric fields, and thereby light speed in vacuum is finite not infinite.

The Maxwell's equations of electromagnetic state in vacuum show that both electric and magnetic fields are rotating in nature, and that we get from these equations we have curl exist the divergent divergence they do not exist. However, in medium if we look at the mediums response, it is not 0, which means divergence exist and also curl exist in the medium. So, it poses a sum extent of divergence as well in the medium. The extent of divergence of electric field depends on the volume density of the free charge rho; this is the volume density of free charge present in the medium. So, extent of divergence will depends on this volume density and D is the cooperative effect, D is nothing but E plus some medium response. This is called medium response.

So, in vacuum, we have only E electric field. But the moment light propagates through the medium, we get an cooperative effect which is D which is nothing but E vacuum contribution plus medium contribution. What does it mean by medium contribution, we will see, we will see that this is going to be nothing but polarization. One of the primary differences between vacuum and the medium is that if the electric field propagates through the medium, it introduces additional effects such as polarization. This additional effect in turn can affect the propagating electric field itself.

So, in medium instead of electric field, we invoke a cooperative field which is called electric flux density represented by this D. This D is electric for flux density. It is generated due to joint action of electric field and mediums response such as polarization. This is what we have seen already in non-linear effects encountered by ultrafast pulses while propagating through the dielectric medium. Here, however we will focus on the Maxwell's equations in vacuum only to develop a few important basic concepts on light propagation.

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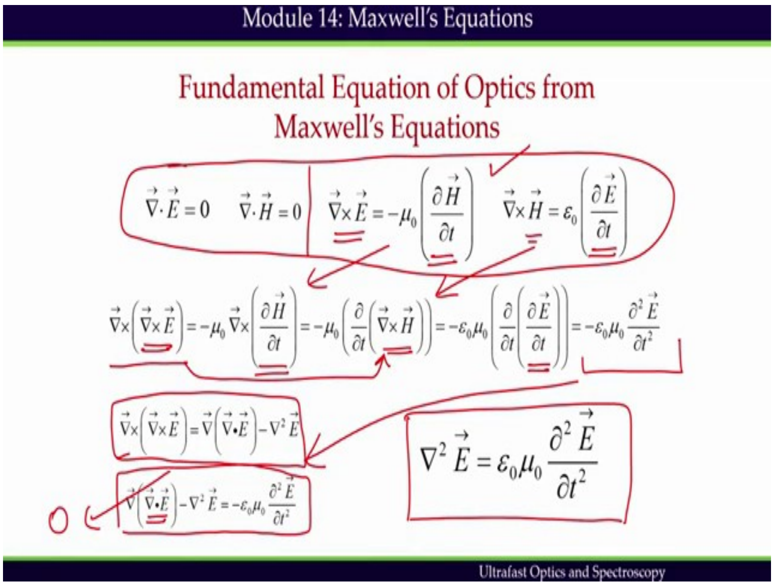

Let us rewrite Maxwell's equations in vacuum one more time. These are the four equations we have we see that divergence is zero, no divergence we see that curl exist and time varying magnetic field is a source of electric field and time varying electric field is a source of magnetic field. We can rewrite this equation. We can take triple product, if we take the triple product then we know this value from Maxwell's equation, we insert that value here, and then we get this triple product is converted to this product. And now we know that this product can be written as this from Maxwell's equation again. So, here we have inserted this equation and here we have inserted this equation.

So, finally, we get this form second derivative of the electric field. From vector identity, triple product vector identity, we can write down this. So, we will be able to write down this from here. And we know that the divergence is 0. So, this term becomes 0, and we get finally a simple equation like this. This is a three-dimensional equation, for simplification to understand the idea we can definitely reduce it to one-dimensional problem.

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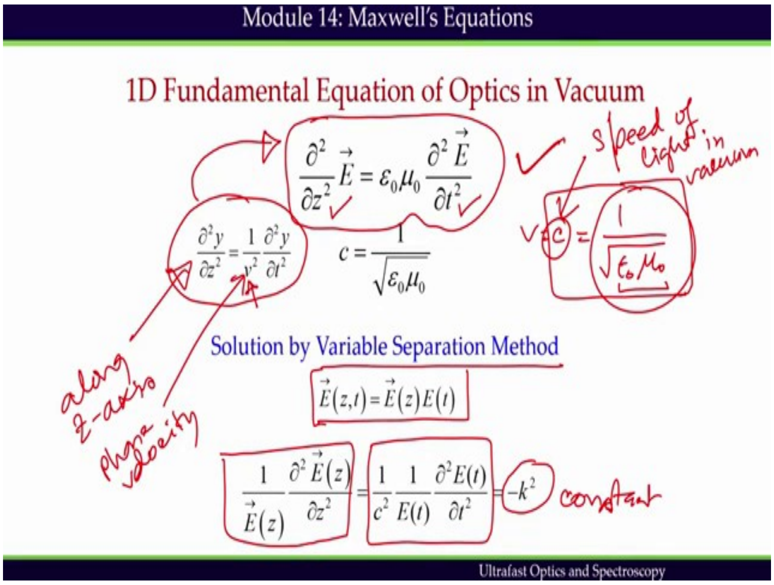

And one dimension and here dimension we have taken to be Z-direction. So, this is the one-dimensional equation of optics in vacuum. Every electromagnetic state in vacuum can be explained with the help of this equation. This equation has a general form of wave equation which can be written like this. So, this wave equation and the one-dimensional equation of optics in vacuum, they are comparable. If they are comparable then this is v the velocity, because it is vacuum, v should be c that is the velocity of light in vacuum which is nothing but can be represented as epsilon naught mu naught.

We can solve this equation now. So, now we have one expression connecting speed of light in vacuum and free space permittivity and permeability. We get the solution by variable separation method which is very common method in this kind of problems, because if we look at this equation we have time dependent part here space dependent part here we have made it one-dimensional that is why we can under variable separation method we can separate the variables. And the moment we separate the variables what I see is that this part is purely space dependent, this part is purely t dependent. Space and t, they are not correlated they are independent variables.

So, in order to become equal, they both have to be equal to they both have to be equal to this constant. This is a very common way of doing variable separation method. We see this kind of variable separation method in quantum mechanics as well because we have time and space dependent components. Here this wave I said that this is the wave equation. This wave propagates along z-axis either positive z or negative z. And v is the phase velocity, we have given the definition in module 2, phase velocity of the wave. And the phase velocity in vacuum is nothing but c speed of light in vacuum which can be expressed by this.

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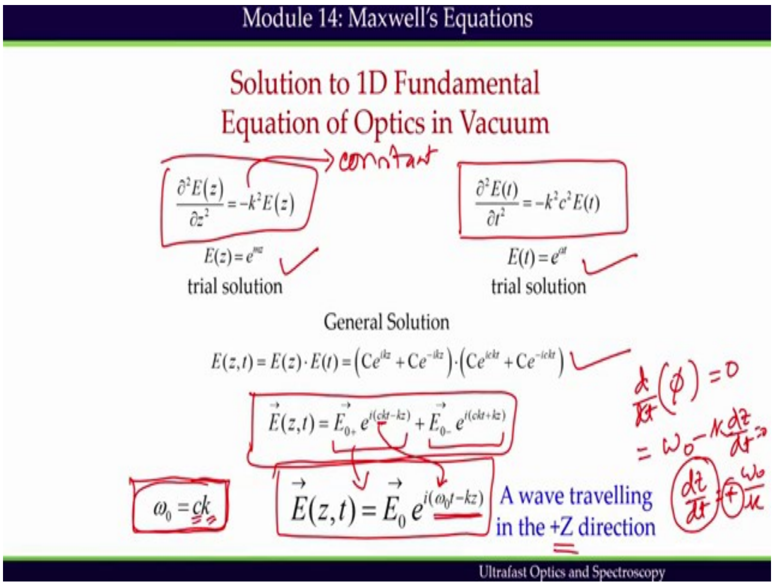

So, finally, what we get here is that we get two different equation; one is time dependent, another one is space, space dependent. We take the trial solution of this kind. And finally, we get a general solution of this kind which can be rewritten as this. Now, there are two components in this expression of electric field. Similarly, magnetic field can be also expressed, but we are not expressing magnetic field.

So, in order to express magnetic field, we have to do the vector triple product one more time and we have to express Maxwell's equations in terms of magnetic field and we get the similar equation. But mostly we are interested in electric field because the field strength of electric field is much higher than magnetic field.

So, to realize magnetic field effect, we have to increase the field strength extremely high. In general, in spectroscopy lab, we use low field strength and that is why only electric field becomes important. So, in electric field, the final solution what we get has two components, one component is like this another components like this. These two components are nothing but wave propagating along positive Z-direction and negative Z-direction. This expression is representing a wave propagating along positive Z-direction that we know

How do we know that? The phase this is the total phase, if we look at the phase, if we take the first derivative of the phase, then we get d dt of this total phase and we say that the constant phase front the velocity of the constant phase front is nothing but the phase velocity. So, to make this constant, it has to be 0 which means I get omega naught minus k dz dt equals 0, which is nothing but dz dt is up is a positive quantity positive which is not which can be expressed as omega naught by k. So, as this is a positive quantity, then this represents a wave propagating along the positive direction. <u>(Please look at the slides for mathematical expressions)</u>

Here we must note that omega naught[ω_0] comes from this c k; c k this term has been represented as omega naught. So, now this expression is suggesting that omega naught is connected to is related to the speed of light, and k is the wave vector. What is the meaning of k, we will immediately find out, but during this solution k was nothing but a constant what is that constant mean we will see very soon. It is constant so far.

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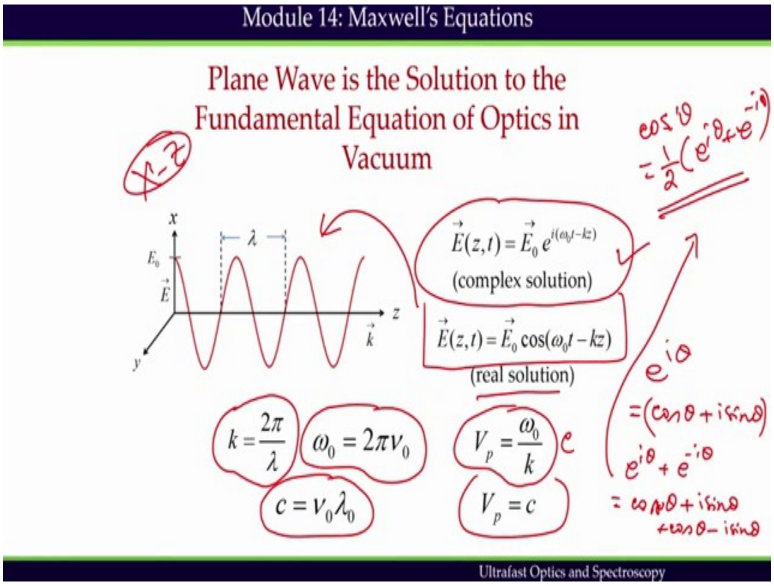

Here we should note that the actual electric field is the real part of the solution. So, although we have got this complex solution, generally we use this kind of complex solution to make the generally we use this kind of complex solution to make the mathematics simple. But we have to in then remember that actual electric field is a real part of the solution which can be expressed by either this or one can express also like this way, e to the power I theta is nothing but cos theta plus i sine theta. (Please look at the slides for mathematical expressions)

So, I can express e to the power i theta plus e to the power minus i theta is nothing but cos theta plus i sine theta plus cos theta minus i sine theta, which suggest that the real part cos theta can be expressed that is the real part can be expressed as half of e to the power i theta plus e to the power minus i theta. So, there are that different representation of the real solution we have, and we can express different expression for the real solution as we need that. (Please look at the slides for mathematical expressions)

So, the plane wave solution is representing a - oscillatory electric field, this is a plane wave because the electric field oscillates on the XZ plane, this is the x z plane on this plane only we have this oscillation going on. This is a solution to the fundamental equation of optics in vacuum. In module 2 we have seen that these quantities are related to each other. So, now, k is related to lambda, which is wave length of the wave, omega naught is represented to the optical frequency mu naught, c is related to wave length, V p which is phase velocity which is nothing but c in vacuum. So, these are the relationship we have seen in module 2. (Please look at the slides for mathematical expressions)

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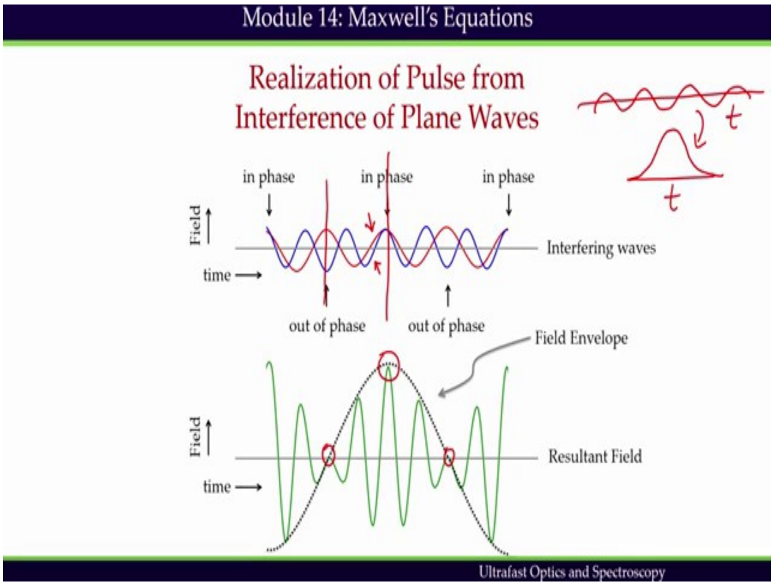

Furthermore, in module 2 we have seen that a pulse is realized only by the theory of optical interference single frequency component would never create a pulse. So, if I have a plane wave, then this plane will this plane wave will never create a pulse which means in time I would not be able to localize the energy, it is impossible because it is spreading over the whole time. So, it is impossible to get a pulse from us plane wave in order to realize a pulse I need to interfere at least two waves of slightly different frequency.

What happens here I am representing two waves plane waves, one red, another one is blue, they are interfering with each other these are the places, where we have constructive interference these are the places we have destructive interference and that is why we have intensified the field here and destroyed the field here and this is nothing but localization of electromagnetic energy. We need to interference many frequency components with constant phase relationship to obtain an optical pulse. This is also shown in module 2.

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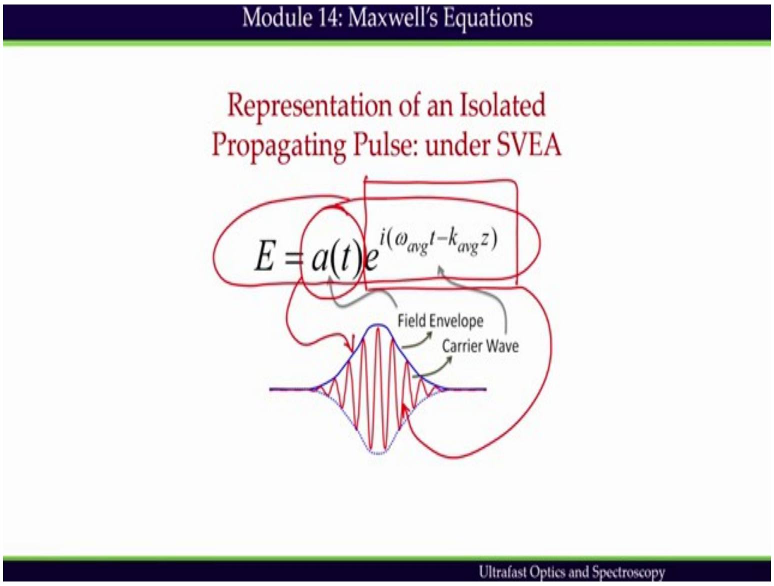

In addition, in module 2, we have seen that under the slowly varying envelope approximation a pulse can be represented by this expression. This is slowly varying envelope approximation we have seen. This part of the pulse representing the carrier wave as represented by this oscillation, and this part is representing the field envelope represented by this blue field envelope. So, there are two components in the pulse which can be expressed.

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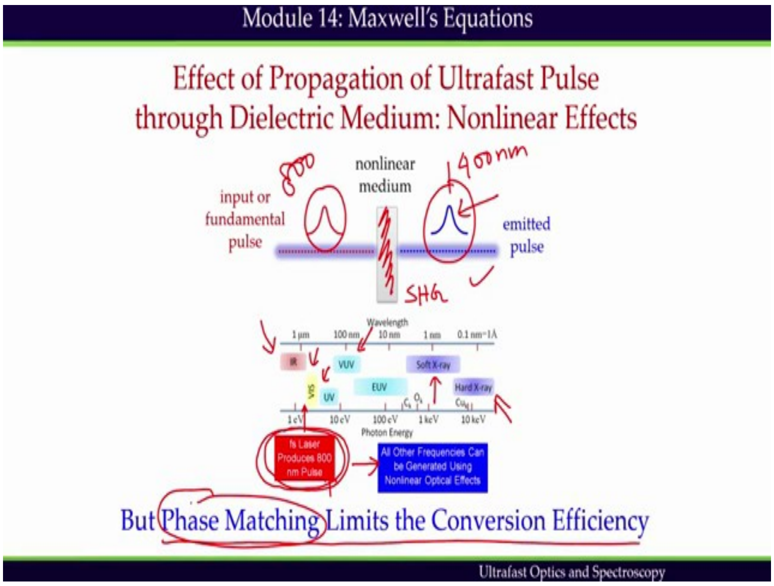

With this understanding, in module 3, we discussed propagation of ultrafast pulse through dialectic medium and its non-linear consequences. What are the consequences we have seen, if this is a non-linear medium, then if the fundamental pulse eight hundred nanometer propagates through the medium, let us say due to second harmonic generation effect is SHG effect, we can create another pulse which is centered at 400 nanometer light. So, this is a consequence of non-linear optical effect. Due to the very high peak incent intensity of an ultrafast pulse, we see non-linear optical effects in dielectric medium when and ultrafast pulse propagates through the medium.

Non-linear effects are very useful, because one can use non-linear effects to generate new light, most of the laser system femtosecond laser system amplifier system. They produce light at 800 nanometer; the pulse is centered at 800 nanometer. And in many experiments we demand to have frequencies in visible range in the UV range in the VUV range, UV, soft X-ray, hard X-ray in this range. How do I, and then also in the higher range as well. Question is how do I convert this 800 nanometer which is produced with the help of Ti-sapphire crystal, how can I convert this pulses to any other frequency components and answer is the non-linear frequency conversion, conversion process.

What we have seen in module 3 that we can convert this, but only catch is that we have to have phase matching condition fulfilled. So, the conversion efficiency or whether we will be able to achieve some intensity in a different frequency that depends on purely depends on phase matching. So, this is something which we have studied in module 3.

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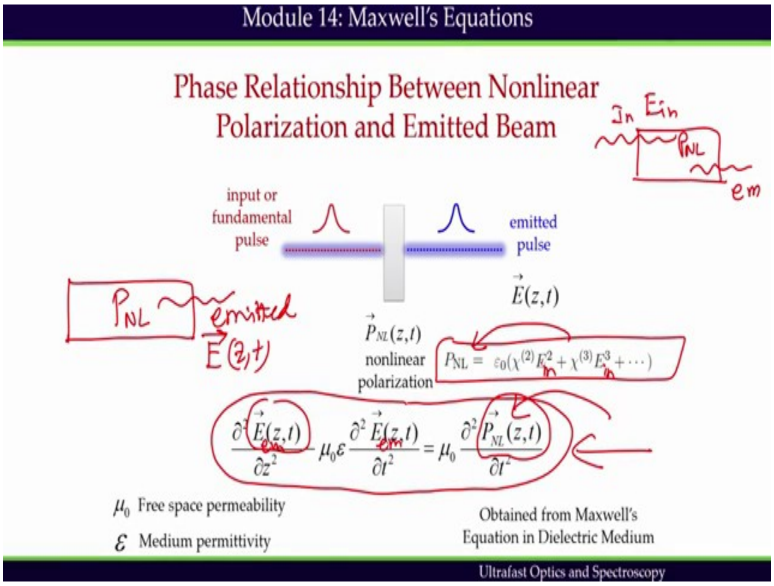

We have also mentioned in module 3 that we have solved fundamental equation of optics in non-linear medium to understand phase matching. So, what we said that we directly used this equation in module 3, and this equation suggests that non-linear polarization in the medium creates this field E field. So, this equation, in this equation, we have to remember that I have a medium this medium can produce non-linear polarization. This non-linear polarization can create an emitted beam, create a new beam. This emitted beam intensity E is z, t. So, this is non-linear polarization created by the input beam.

How it is related to input beam that is related to input beam by this expression, non-linear linear polarization depends on the input beam. So, there are three effects going on simultaneously, input beam, this is input beam fundamental beam. Input beam is creating E input creating pull non-linear polarization, in turn non-linear polarization creating new field which is called emitted beam. This E field is emitted field that is related to non-linear polarization, and these E field are input beam that is creating non-linear polarization.

So, these are the two important effects we should remember and we have gone over these details in module 3. Previously we have shown neither the derivation of this equation, this equation was not derived in the in the in module 3, this is why in this module we will discuss Maxwell's equations in details in the medium, and we shall derive this equation from Maxwell's equation and solve it to achieve phase matching condition. So, we will try to derive this equation here.

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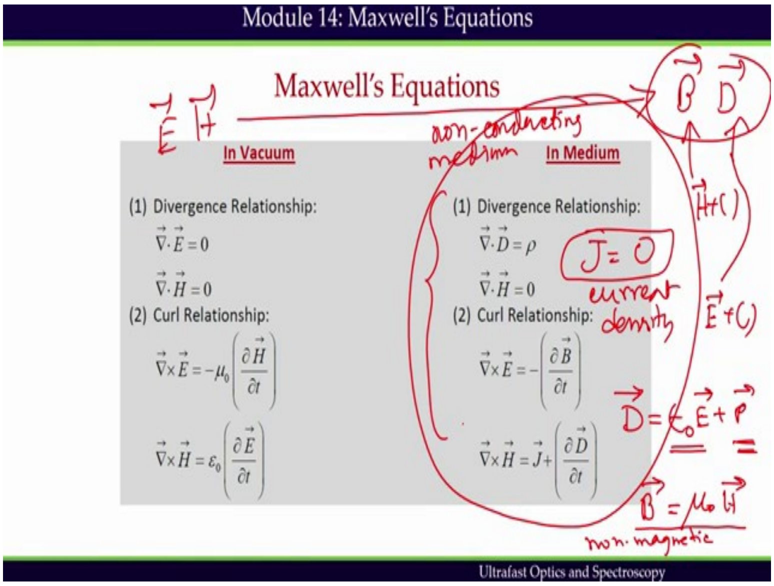

We will one more look at Maxwell's equations of the medium. In the medium, we have already pointed out that we should consider cooperative fields in the medium, cooperative fields such as D and B. So, in vacuum what is my E and H, it is converted they are converted to B and D. This is related to electric field, this is related to magnetic field. But the includes the cooperative effect which means this is including H plus its medium response, and this is including E, E plus its medium response instead of electric and magnetic fields.

We use B and D in the medium, the flux density D is represented by epsilon naught E plus P polarization. So, this is coming from vacuum contribution, this is coming from medium contribution, and I am getting flux density D effective field electric field in the in the medium. Similarly, magnetic flux density B can be expressed it as mu naught H, this is true only for non-magnetic material for non-magnetic material.

For magnetic material, you have to include the magnetization part also that will exclude we are thinking that the medium is not magnetic that is why B can be expressed like this. So, medium response is neglected. Due to magnetic field, if there is a magnetization in the medium, then only we get the cooperative effect. But, for polarization any directing medium can be polarized, but any medium cannot be magnetized unless there is a magnetization in the medium, and also we are considering j that is the current density is 0. We are considering current density 0, because this is also non-conducting medium, we are considering non-conducting if the medium is not conducting then j current density current will be 0, and under that condition I can write all this equations.

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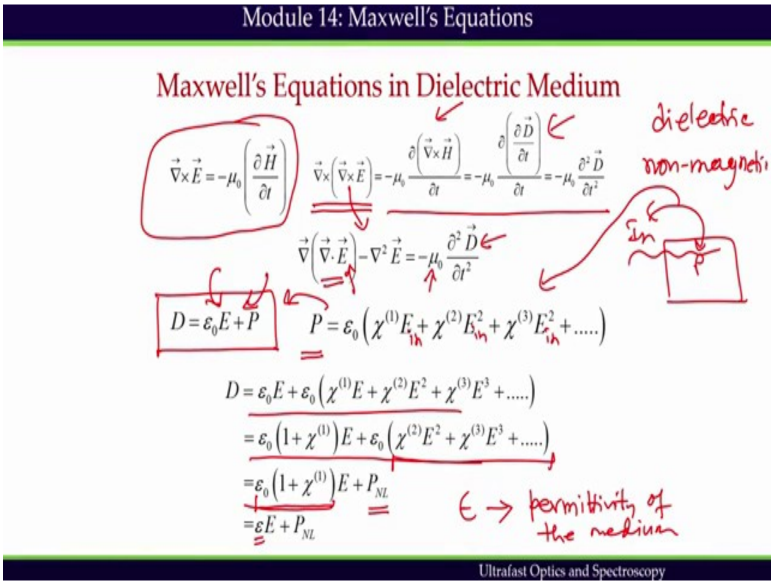

Therefore for dielectric, dielectric non-magnetic dielectric non-magnetic medium, Maxwell's equations becomes this. And we have this expression now. What we do? We do, we take triple product vector triple product and if we take that to break the triple product then from Maxwell's equations. We can write down this, this is one Maxwell's equation we are inserting, then another Maxwell's equations we are inserting, and we are getting this.

So, finally, this triple product can be decomposed like this way. Now, question is where the divergence of E is 0 or not. In the medium, in the medium, E is the electric field, mu naught is the magnetic permeability of free space, D is the flux density which is which includes cooperative effects, here both E and D are function of space and time. If we use now scalar notations instead of vector notations you can use scalar notation, we can write down expression for D. D having contribution from vacuum, electric field from vacuum and its medium response which is polarization.

And we have seen that polarization is created by input field. This is the polarization which you have created. So, if I have a medium input beam is propagating through the medium this is input then I create a polarization, and how this input field and polarization they are related, this is the Taylor series expansion we use. So, we can plug that in here and we get this expression simplify this expression, and finally, I can write down all non-linear components as P_{NL}. So, P_{NL} does not have linear contribution or polarization, it is only including non-linear contribution of the polarization.

And this epsilon naught multiplied by 1 plus chi 1[$\chi^{(1)}$] is expressed as epsilon; epsilon is the permittivity of the medium. Epsilon is considered to be permittivity of the medium. This is nothing but a resistance against the electric field in that dialectic medium as we have seen permittivity. And, permeability is representing some kind of resistance against the respective fields. And, P_{NL} the polarization non-linear polarization is the polarization component which is responsible for the non-linear effects.

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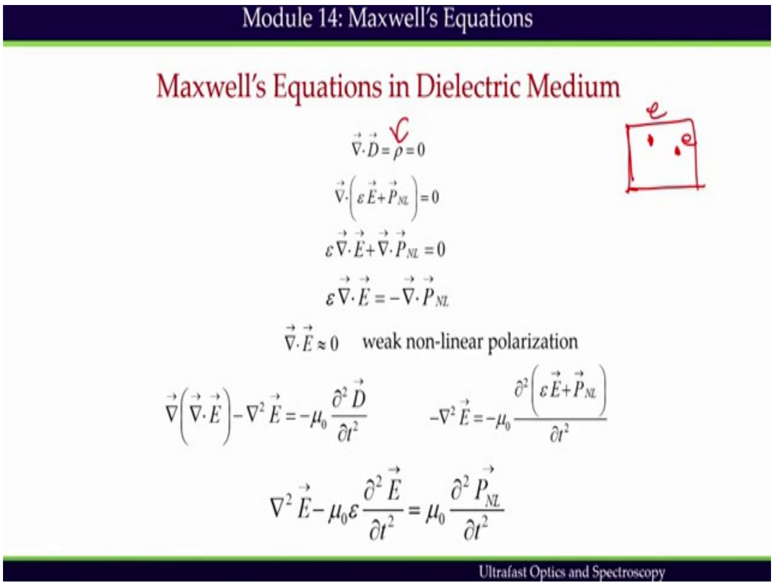

Now, if you consider a source free medium which means I do not have any charge localized i medium can have suddenly localized charge electrons, here may be charged electrons. If I forget these charging effects we say that medium does not have any volume density for charge, so then rho will be 0. In that case, if it is a source free medium there is no charge localizing the medium which is a legitimate approximation for any optics we did not get rho to be 0.

We will stop here and we will continue this module in the next session.