Symmetry and Group Theory
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Lecture -36
Tutorial - 7

So, in this week of lectures, we have been discussing about direct products, which is part of application of group theory into quantum mechanics. So, let us discuss little bit more about direct product, which we have not covered. Some results from direct product applications not applications, but direct product results on to group theory.

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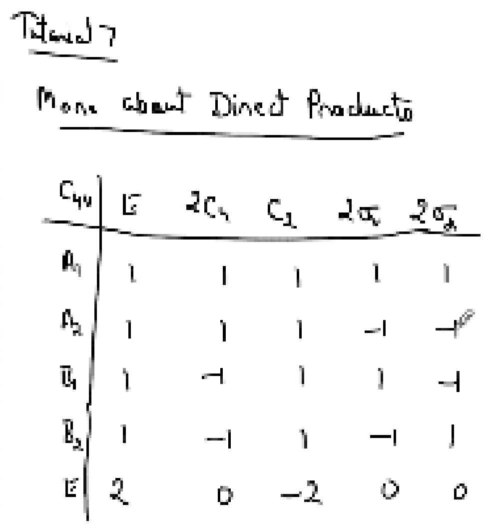

So, let us start with tutorial 7 and we will say more about direct products. So, by now, we all know what are direct products. In case of matrices or in case of traces of matrices, how do we obtain the direct product. So, let us directly go and discuss the results of these. So, when we are discussing these results, let us take an example of let us say C4v point group. So, it has all sorts of 1D and 2D, IR representation.

So, let us take an example of C4v. So, let me write down the character table of this, which is E, 2C4, and C2, 2 sigma v into sigma d. In this side, we have A1, A2, B1, B2 and E. We have 1 this is called as totally symmetric representation, so all one’s. Then we have A2, so A2 will have sigma v cells negative. Then B1 we have -1, 1, 1, -1. For B2 again, you have 1,-1, 1, -1 and 1.

This is a degenerate representation, so this will be 2 dimensional, 0 0.

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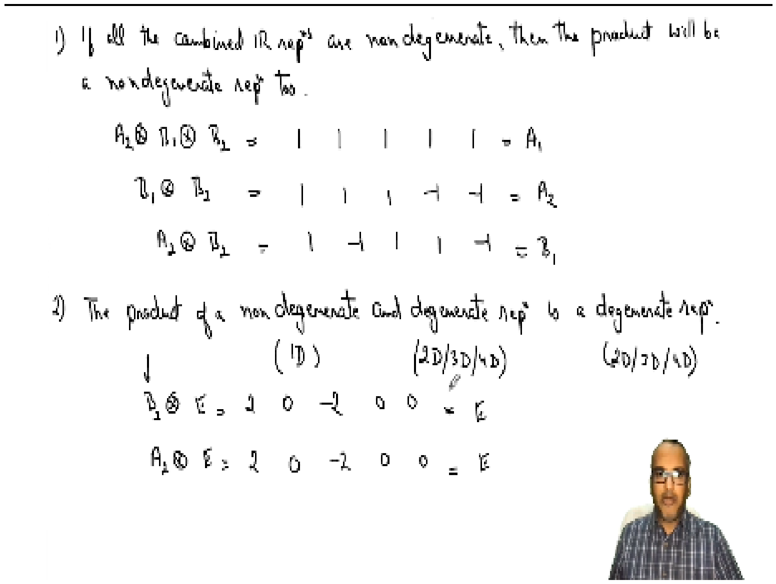

So, let us see the first result of direct product. So, it says, I will first write down the statement and then we will discuss using C4v example. So, it says if, all the combined irreducible representations are non-degenerate, then the product will be a non-degenerate representation too. So, of course if it is a non-degenerate, it will be an IR representation that is irreducible representation.

So, what does it mean? So, it is saying that if all the combined, combined means for which we are calculating the direct product, so if you are calculating the direct product of irreducible representations, which are non-degenerate, so what is non degenerate? So, non degenerate means one dimensional IR representations. So, then the product will be non degenerated representation too. So, let us see an example.

So, let us say if we are making a direct product or if we are combining A2 with B1 with B2. So, what do we have here, so let us try to combine this A2, B1, B2. So, we have 1 x 1 x 1, so that is 1. So, when you have 1 x -1x -1, so that is the 1 again, 1, 1, 1. So, 1 again; -1, 1, -1, so you have 1 again, 1, -1, 1. So, you have 1 again. So, this gives rise to 1-D representation. So, all of these are non-degenerate representation.

So, any combination of non-degenerate or one-dimensional irreducible representation will give you a non-degenerate representation. So, this gives me A1 directly. All of these are so, so let us do one more calculation, so let us say this term will combine only B1 and B2, what do we get? So, you have 1, 1, 1, -1, -1, what are we getting here? 1, 1, 1 -1 so just gives you A2. So, again it is a one-dimensional representation.

So, you can take any combination. Let us say if you do A2 into B2, so again, we are combining two one dimensional representations, so now this is 1 into 1, 1 into -1, so -1 1 into 1,-1 into-1 so that is 1 and -1. So, this is second, fourth, fifth place as -1, so you have B1. So, if you keep on combining different one-dimensional representations you will always be getting one-dimensional representations back, which is obvious.

Because your matrix sizes not increasing if you are combining one dimensional matrix multiplying one-dimensional matrix with one-dimensional matrix, the product will be one dimensional matrix, so it will have to be one dimensional or non-degenerate irreducible representation. So, that is first rule, first result of direct product. So, then second is again, let me first write down the statement and then we will explain.

The product of a non-degenerate and a degenerate representation is a degenerate representation. So, now we are combining a one-dimensional representation, non-degenerate and more than one dimensional, a degenerate representation. So, if you combine one-dimensional and let us say in this case, it is two-dimensional but you can also call it as nD so this is 1D, this can be 2D, 3D, 4D anything.

So, if I am doing this then the result is corresponding 2D, 3D, or 4D. Now again, let us take an example and see so let us say if we do B2 into E. So, what do we have? So, 1 into 2, we have 2 - 1 into 0, so you have 0, 1 to -2, so you have -2 and then 0, 0. So, because C4v point group has only one degenerate representation and you are bound to get a degenerate representation, so you will no matter what one-dimensional representation you choose here, you will always get E.

So, let us take another example here. Let us say if we have A2 direct product with E, so A2 is 1 into 2, so we have to again 1 into 0, 1 into - 2, -2 and -1 into 0 -1 into 0, so again, you are getting E. So, irrespective of what you have at this position, because there is only one degenerate representation you are going to get E degenerate representation back, because the product of a non-degenerate and degenerate is supposed to give you a degenerate representation only.

Of course, if you have multiple regenerative representations, then you can get any other, let us say if you have in some point group if you have E1 and E2. So, if you now combine something with E1, you might end up getting E1 or E2, because now there are two options. So, depending on what is the result, what is a product, you can either get one of the two degenerate presentations. I hope this is clear.

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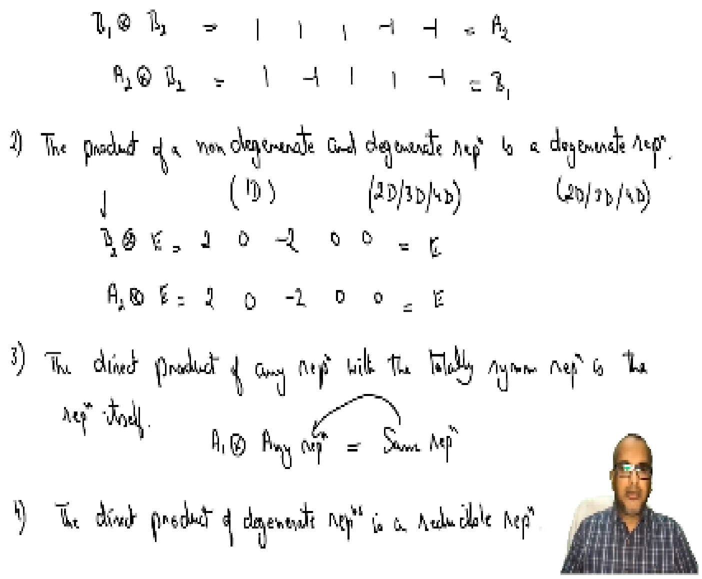

So, you can keep on taking any example and see if all the rules are verified. So, third is the direct product of any representation, with the totally symmetric representation, is the representation itself. That means if you now combine any of this representation, let us say A2, B1, B2, E with totally symmetric representation. There is only one totally symmetric representation in each point group and there is always one totally symmetric representation in every point.

So, if you combine any or if you take a direct product of any irreducible representation, it can be degenerate or non-degenerate. If you take direct product of any representation with totally symmetric representation will get that representation back. That is very simple to see because let us say if your multiplying A2 into A1, so all the characters are multiplied by 1. So, that is not going to change anything.

So, A2 into 1 will give you A2, B 1 into A1 will give you B1, B2 into A1 will give B2, E into A1 will give you E and so on, so forth. So, that is very easy to see. So, let me just write down this so A1 into any representation is equal to the same representation. So, this is also very simple to see. Now let us say the fourth one: The direct product of degenerate representations is a reducible representation.

So, so far our direct products were not giving any reducible representation, they were giving either 1D representation or degenerate representation, but from the character table itself. We do not have to reduce that. But if you take a direct product of any two degenerate representations, then this will give rise to a reducible representation. So, that means in this particular case we have only one degenerate representation.

So, if we take, if we combine E into E, if we take direct product of E and E, then whatever result you will get will be a definitely it has to be a reducible representation. That means now you can reduce that representation into component irreducible representations using reduction formula.

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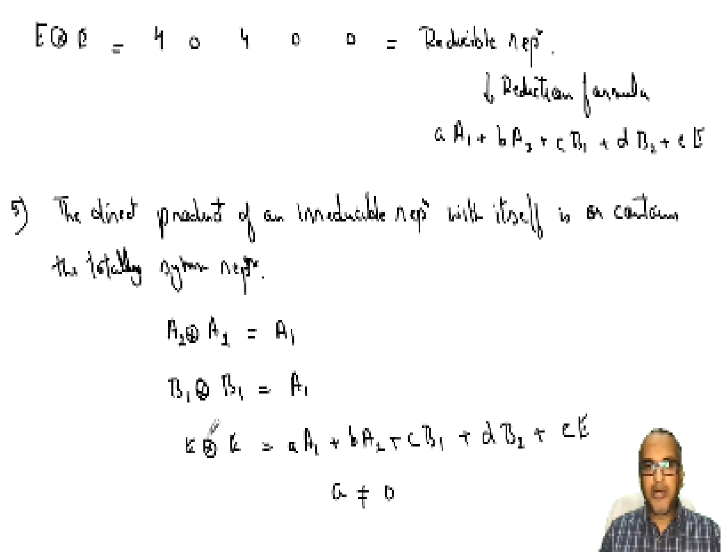

So, this means if I take E cross E, then I will get 4 0 4 0 0 now this is a reducible representation. So, using reduction formula, you can now get, a times A1 + b times A2 + c times B1+ d times B2 + e times E. So, of course you have to calculate these coefficients a, b, c, d, e some of them will be 0, some of them will be some other number, integral numbers. But it will give rise to combination of A1, A2, B1, B2, and E.

So, that can be easily done that we have seen in last tutorial also how to do that and we have learnt in lectures. So, that is fourth. Fifth one is the direct product of an irreducible representation, this will be of already seen, but I am just listing it out for completeness, we will not explain it, with itself is or contains the totally symmetric representation. So, if you are taking direct product of an irreducible representation with itself, it might result in two or totally symmetric representation or it will contain; When I say contain means if you end up getting a reducible representation it must contain A1 if and only if the direct product is of two irreducible representations which are same. That means direct product of a reducible representation with itself will contain totally symmetric representation. This case we have already discussed in last lectures, we have also derived an expression for this.

So, let us not go into those details but just for completeness I will write that if you take A2 into A2, you can just close your eyes and write it down as A1 without calculating. B1 into B1 will be A1. See why am I saying that? Because B1 into B1 both are two one dimensional representation it is bound to give you a one-dimensional representation and the rules also says that if I am combining two same irreducible representation, it is also bound to give me a totally symmetric representation.

So, no matter what you do, you will always get A1 if you combine two same IRs. So, similarly, now if I do even one E into E, my coefficient a here will definitely be non-zero. So, that means, A + bA 2 + cB 1 + dB 2 + e times E. So, in this case I can see that a will definitely be non-zero because it must always contain A1 because you are combining two irreducible representations, which are same that we have seen earlier.

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i) Only the direct product of a rep with itself is or contains the totally symm rep^n.

$$A_2 \otimes B_1 \neq A_1$$

$$B_1 \otimes B_2 \neq A_1$$

Home exercise: Pick up a character table → O_h and show that above six properties of Direct Product holds true.

And the last one also is a corollary of this fifth one only, only the direct product of a representation with itself is or contains the totally symmetric representation. So, what do I mean here? So, that means if I am taking our representation, which is not the same, so for example, they can say that A2 into B1 will never be equal to A1. Similarly, B1 into B2 will never be equal to A1. So, totally symmetric representation will only come as a result if the direct product is of a representation with itself otherwise it will never come.

So, basically it is opposite of this 5th and 6th, the same things put in different words. So, that makes the calculation of direct products little easier. So, I hope these results will be clear and I leave it to you as a home exercise. Pick up a character table. Let us say Oh so that it is a big one and you have almost everything in there. So, pick up a character table, Oh and show that above six properties of direct product hold true. So, that can be easily done.

So, just go find a character table. If you do not have any book, you can just Google character table octahedral point group and you will see that Google will spit out the character table and just use that character table to see if all the above six properties are verified using any of this character table. So, that is your home exercise and next week's lectures will be looking at direct product applications. So, before that, please do practice these properties so the application part will become very easy. All right, that is all for today, thank you very much.