Advanced Mathematical Methods for Chemistry
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Module - 09
Lecture - 04
Laplace Transforms

In the fourth lecture of module 9 we are going to introduce a different kind of integral transform known as a Laplace transform. And we will see how the Laplace transform can also be used. A Laplace transform is a especially useful in a certain cases where you have a what are called as initial value problems or you can also have for boundary value problems. So, before we define the Laplace transform I want to say that you know Fourier transforms Laplace transforms are all examples of integral transforms. So, what does a general integral transform look like?

And you might see in literature that you might come across other transforms like Hankel transforms or you know you know there are actually a whole lot of transforms that exist in literature. So, let me just give a general sense of all these integral transforms.

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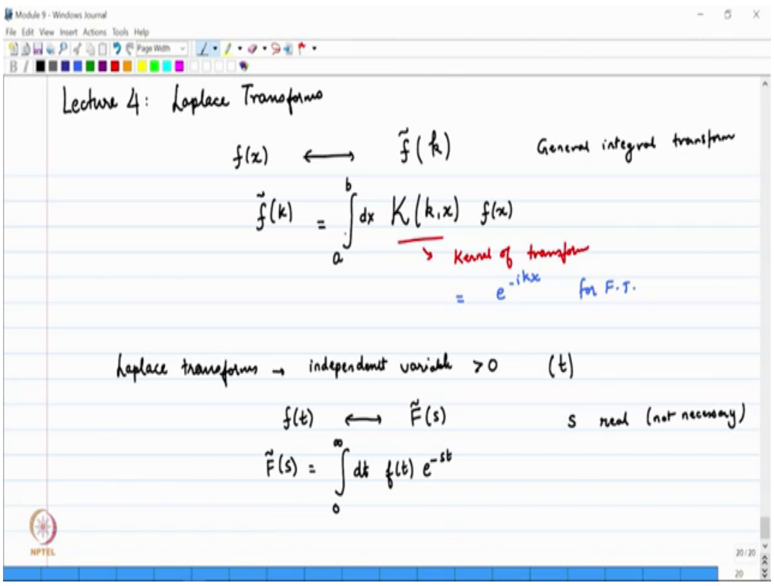

Suppose you have a variable x f of x an integral transform transforms it to some variable let me e call it f tilde and just to emphasize that it need not be the Fourier transform I will just write f tilde of a some variable omega let me call it omega or let me call it k. So, f tilde of k, so, this is a general integral transform.

So, what we are doing is instead of looking at it as a function of x we are looking at it as a as some other function of some variable k and our idea is that whatever information is there in this function is all there in this function. So, anything you wanted to know about f of x you can get that information just by looking at f tilde of k. So, that is the general idea of the integral transforms.

So, now, a general integral transform has this form. So, you integrate over x from whatever the limits of x are I will just say a to b this is the limits of x and what you do is you integrate over f of x and what you have is something called a kernel k of ks this is your kernel of the of transform.

So, for Fourier transforms this is equal to e to the minus I kx or for Fourier transform and of course, a and b are the limits for Fourier transforms they are minus infinity to plus infinity. So, this is a general property. So, now, should be careful with the notation this should be s x. So, now, what happen is when you integrate this, the x will go out you will not have a you will have a function of k. So, what we will have on the left hand side is a function on the fk which we call as the integral transform.

So, now this is a general integral transform now let us look at Laplace transform. So, for Laplace transforms now typically we choose. So, for convenience we take a independent variable greater than 0 and this is denoted by t again this is historic reasons because t is used for time and time is usually greater than 0. So, your function is f of t and your Laplace transform function will be I will write it as capital F tilde f s. So, this is a function of s s is the s is the new variable and s can be real or complex, but we choose it to be real for convenience.

So, this is not necessary, but it will just make some of the discussions a little easier. So, now, what is F tilde of s. So, F tilde of s is equal to integral from 0 to infinity dt f of t e to the minus st. So, we are integrating over time of this function. So, the kernel in this case is just e to the minus st this that is your kernel and the limits are from 0 to infinity again f of t need not be continuous.

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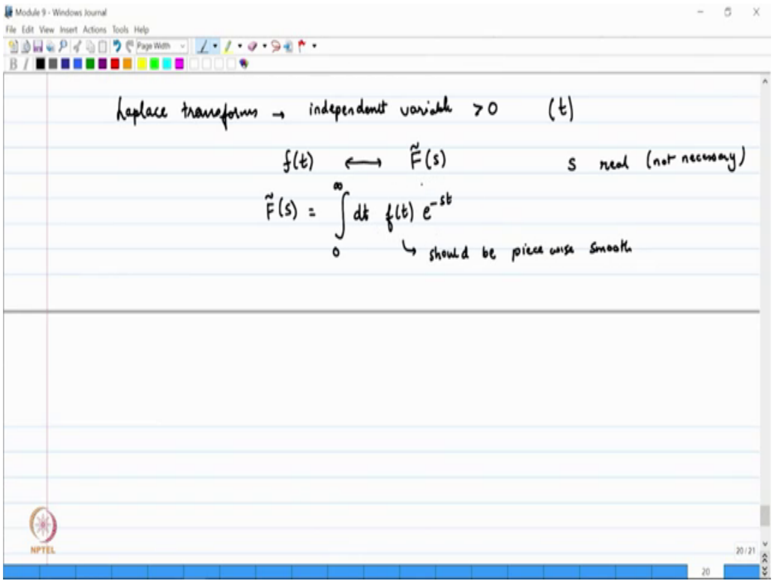

But it should be f of t should be should be piece wise continuous piecewise actually piecewise smooth because all the derivatives should also.

So, this is a condition for it to exist and the needless to say f of c a f of t should not go to as t goes to infinity this e to the minus st will go to 0. So, f of st should not should not overcome that 0 and go to infinity. So, f of ft f of t should not. So, f of t into e to the minus st should go to 0 as t goes to infinity. So, now, let us look at some Laplace transforms.

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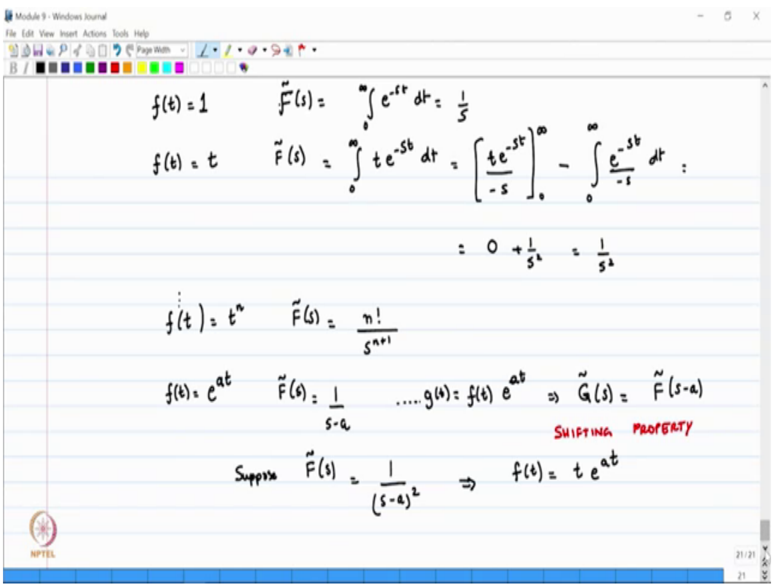

First I will just take f of t equal to 1 then you can easily see that F tilde of s is equal to integral 0 to infinity e to the minus st dt equal to 1 by s that is a Laplace transform of one.

Now, next let me take the Laplace transform of t f of t equal to t then you can see that F tilde of s now what you will have is integral 0 to infinity t times e to the minus st dt now you will do this by parts. So, the first term will be t e to the minus st divided by minus s from 0 to infinity minus integral e to the minus st derivative of t is just one divided by minus s from 0 to infinity dt. So, now, this when you put when you put t equal to infinity then you have te to the minus st. Now e to the minus st goes to 0 and t goes to infinity, but the exponential this is an exponential in t and that goes to 0 much faster than the t goes to infinity.

So, this is actually 0 when t equal to infinity when t equal to 0 then this goes to one, but this goes to 0. So, this integral is actually 0 the value of this at these boundaries is 0 and what you have here now you have minus I can cancel the minus. So, you have e to the minus st divided by s. So, the integral of e to the minus st is 1 by s. So, this is just equal to plus 1 by s square. So, equal to 1 by s square. So, the Laplace transform is 1 by s square.

So, similarly you can show that f of t raised to n f if f of t equal to t raised to n then F tilde of s is equal to n factorial divided by s raised to n plus one. So, this is a general relation. So, any power can be calculated in this way next what if f of t equal to e raised to a times t then F tilde of s is equal to 1 over s minus a. So, you have. So, you have e raised to minus s minus at. So, you get 1 over s minus a. So, in general if f of if let us say g of t is equal to f of t into e raised to at g of s is equal to F of s minus a.

So, if your function is multiplied by an exponential then the Laplace transform is shifted by the exponent this is a shifting property of Laplace transforms and this is extremely useful to calculate inverses of Laplace transforms. So, suppose you have suppose you have something like. So, suppose F tilde of s is equal to 1 over s minus a square then immediately this implies this implies f of t is equal to. So, 1 over, so if it was just 1 over s square then it will been t. So, it is t into e to the at. So, you can immediately write this. So, it is t multiplied by e to the at. So, that will be 1 over s minus a square.

So, this shifting property is of Laplace transforms is extremely useful a few more Laplace transforms which are which are fairly useful. So, once you know the Laplace transform of the exponential suppose you had an exponential of an imaginary quantity then you will have s you will have you will have 1 over s plus some imaginary quantity. So, that can be used to get the Laplace transform of sin of omega t.

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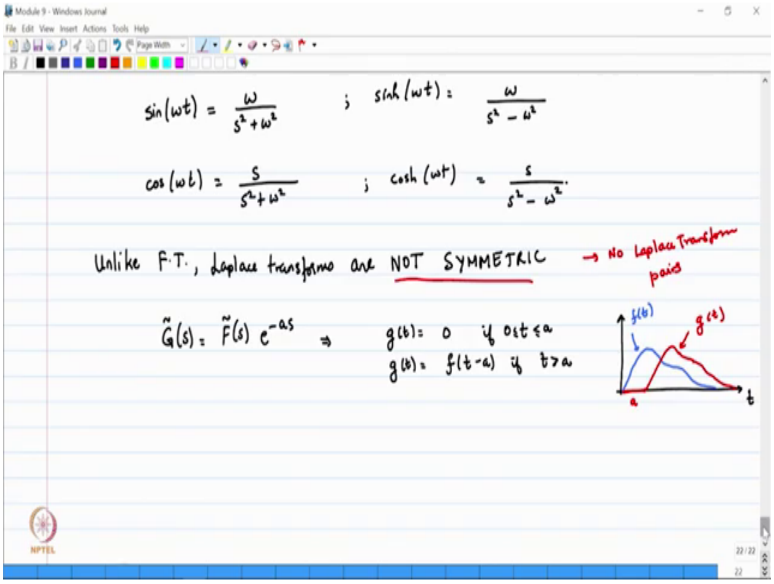

And you can show that the Laplace transform of a sin of omega t is omega divided by s square plus omega square.

Now, if on the other hand you had sin hyperbolic. So, the hyperbolic sin of omega t then that would look like omega divided by s square minus omega square if you had a cosine of omega t then in this case instead of omega you have an s over s square plus omega square and similarly if you had cosine hyperbolic of omega t this would look like s over square minus omega square. So, these can be easily derived all these relations can be easily derived.

Now, let us mention one last thing. So, we mentioned the shifting property of Laplace transforms now Laplace transforms unlike Fourier transforms Laplace transforms are not symmetric are not symmetric. So, you cannot have things like what you had in. So, we had Fourier transform pairs we cannot have Laplace transform pairs. So, no Laplace transform pairs.

So, then you have to actually evaluate both the I mean you have to do then you have to explicitly do the inverse Laplace transform now one of the things is you can ask a question suppose I had suppose I had suppose you had let us say let us say you had g tilde of s is equal to F tilde of s times e to the minus as then what does the Laplace transform what does the inverse transform of this look likes. So, this implies g of t equal to 0 if t is less than a if 0 less than equal to t less than equal to a and g of t is equal to f of t minus a if t greater than a.

So, how g of t and f of t are related is the following. So, suppose you had t now if f of t f of t had let us say this is what f of t is f of t now if I take a function which is shifted by a and it exactly follows. So, we take a new function that is 0 for less than a and it is exactly f of t minus a for. So, this is my g of t and g of s will be. So, if I take a function and just shift it to the right then the Laplace transform will be multiplied by an exponent by e to the minus as.

So, these are the important properties of Laplace transforms now, now with these properties you can you can calculate Laplace transforms and actually you can also do inverse we will do a few examples of a how to do the inverse. In fact, the most important property for doing the inverse is this is this shifting property. So, using the shifting properties and Laplace transform of simple algebraic and exponential and sine and cosine functions we can calculate Laplace transforms of we can easily invert Laplace transforms.

Now, you know one of the applications of Laplace transforms is to solve differential equations now. So, then you need to know the Laplace transform of a derivative. So, suppose and let me do this in the next page. So, Laplace transforms of derivatives.

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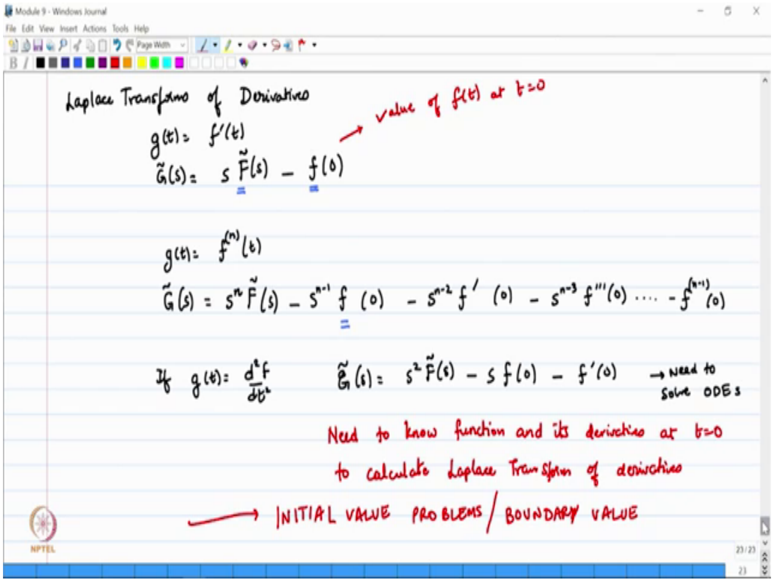

So, suppose you take g of t equal to f prime of t then you can show that a g tilde of s is equal to s times F tilde of s minus f at t equal to 0 this is value of function at t equal to 0 and you can easily show this you can easily show this by integration by parts that you will get such a result.

So, and what is the other interest. So, if you want to calculate the Laplace transform of a derivative it is not just equal to multiplying by s it is actually multiplying by s and then you have to subtract the value of the function at t equal to 0 notice this is f, this is little f and this is capital F. So, these 2 are important. So, this is little f this is capital f now you can extend this and you can show that that if g of t equal to n th derivative of t then g tilde of s is equal to now you will get s raised to n F tilde of s minus now what happens is its not only the value that is involved you also have you also have each of the each of the derivatives at t equal to 0 that are involved.

So, what you have is f s raised to n minus 1 now you have the n minus 1 th derivative evaluated at t equal to 0. So, again notice that this is a function of time and then not only that you have minus s raised to n minus 2 the n minus 2 th derivative evaluated at 0 sorry the other way around its s raised to n minus 1 f evaluated at 0 s raised to n minus 2 f prime evaluated at 0 and you have minus s raised to n minus 3 f triple prime evaluated at 0 all the way up to up to; now you have a when you have s raised to n minus n then will be s raised to 0. So, that time you have just f raised to n minus 1 evaluated at 0.

So, these are the various terms that you have to subtract now using this. So, for example, if g of t is equal to d square f by dt square that is the second derivative then you can write g tilde of s is equal to s square F tilde of s minus s f at 0 minus f prime at 0. So, you can write in this form and this is extremely useful to evaluate this is extremely useful for differentially this is what you need to solve odes.

Now notice that when you take the second derivative the value of value should be known at 0 and on the derivatives should be known at 0. So, you need to know need to know values and derivatives at t equal to 0 function and its derivatives at t equal to 0 to calculate Laplace transform of derivatives. So, you need to know the function and its derivatives at t equal to 0 only then you can calculate the Laplace transform and. So, what this means is that Laplace transform in if you want to use Laplace transform to solve a differential equations you need that boundary condition or that initial condition.

So, it is valid for what are called as initial value problems sometimes you can also call them as boundary value if your boundary looks like looks like a something like x equal to 0 or So,, this is the use of Laplace transforms and especially the kind of boundary value problems we are talking about you know when 1 boundary is fixed and the other side you do not have any boundary it goes all the way to infinity those are the cases where Laplace transform is useful to solve differential equations.

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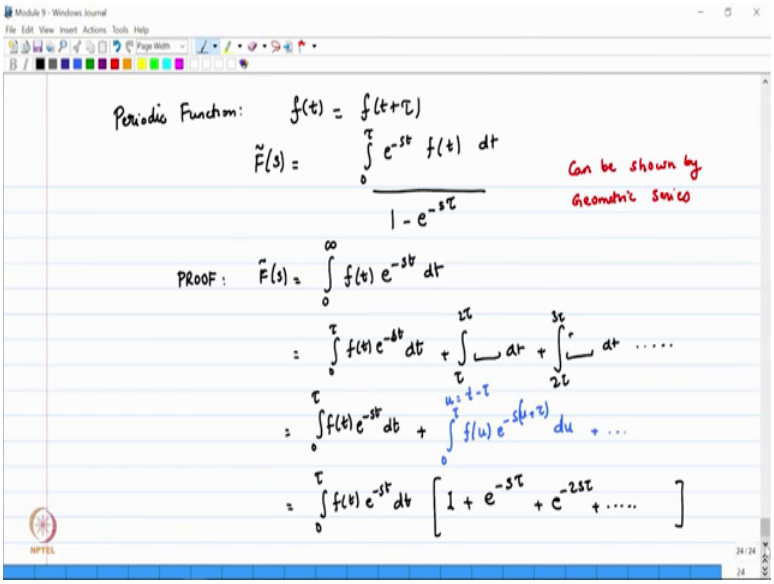

Now, the other thing I want to mention is that is that suppose you have a periodic function f of t is equal to f of t plus tau then what you can do is you can show that f tilde fs can be written as now. Now instead of doing integral from 0 to infinity you do an integral over only one period 0 to tau e to the minus st f of t dt and then you have a factor of 1 minus e raised to minus s tau you can be this can be shown by geometric series. So, just to just to illustrate I will just show this very quickly.

So, I will just write the proof here the I am showing the proof because this illustrates you know some general mathematical techniques that you that you that you need to call often in when you are when you are solving various problems. So, F tilde of s by definition is integral 0 to infinity f of t e raised to minus st dt.

Now I can write this as integral 0 to tau f of t e raised to minus st dt plus integral tau to 2 tau of the same thing plus integral 2 tau to three tau of the same thing and so on all the way up to infinity. So, now, what I can do is in the. So, the first term I will leave as it is integral 0 to tau ft e to the minus st dt plus now in this what I will do a change of variables.

So, I will say u. So, only in the second term I will put u is equal to t minus tau now du is same as dt. So, this becomes integral now this goes from 0 to tau and what you have is now you have f of t minus tau f of u. So,rry you have f of t now f of t is same as f of t plus tau. So, I can just write fu e to the minus and now I have s times s times now I am going to replace t by t minus tau I am going to replace it by t by u plus tau su u plus tau du. So, I can write it in this form plus the other terms.

So, what I can show is that now e to the minus s tau can be taken outside and then what I have is exactly the same as this integral. So, what I can write. So, I can combine the first 2 terms and write ft e to the minus st dt and then I can write one plus e to the minus s tau in the second case I will get plus e to the minus 2 s tau plus. So, on and using the geometric series this goes all the way to infinity. So, I can replace it by this. So, a periodic function; so, when you have a periodic function you do not need to integrate from 0 to infinity you just integrate over one period and you divided by this factor.

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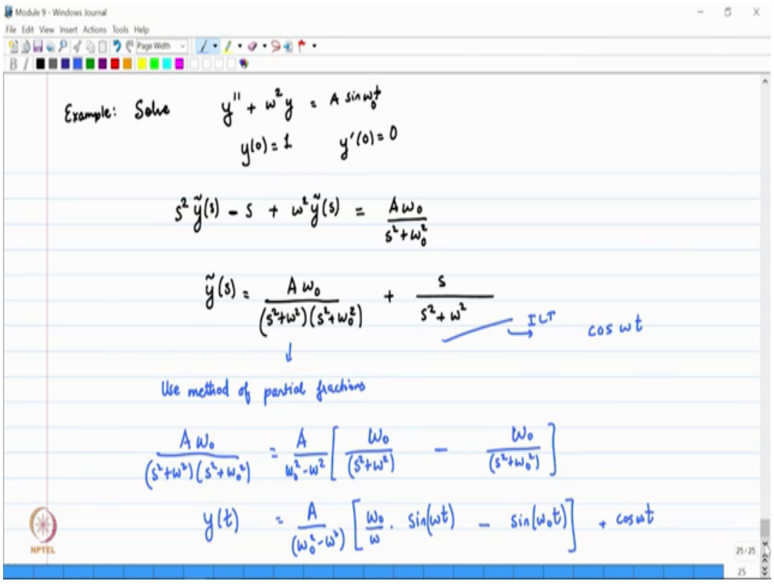

So, that is the Laplace transform of a periodic function. Now, let us take an example solve y double prime plus omega square y is equal to a sin omega t subject to boundary conditions y of 0 equal to 1 y prime of 0 equal to 0. So, this is the boundary this is the initial condition. So, the value at t equal to 0 is one derivative at t equal to 0 is 0. Now you know how to solve this I mean this is a this is a straightforward non homogeneous equation and you know the solution this is will be will be related to sins and cosines.

But let us solve it by Laplace transforms. So, if you take Laplace transforms on both sides a Laplace transform of the second derivative is s square y tilde of s and then you have minus. Now what you have is s times the value at 0. So, value at 0 is 1 and then you have derivative at 0 which is 0. So, that is the. So, this is the Laplace transform of y double prime then the for omega square y you get omega square y tilde of s and what you have is a sin omega t. Now if you have sin omega t you go back to your table you go back to the list of Laplace transform. So, sin omega t is omega divided by s square plus omega square.

So, I will get a omega divided by s square plus omega square now what you will do is I will I will take the s to the right I will write y tilde of s is equal to a omega divided by now what I have is s square plus omega let me make this omega 0 just to just to distinguish it from the omega I will make it omega 0 just to distinguish it from this omega.

So, this omega and this omega 0 can be different. So, I will just make this omega 0. So, a omega 0 divided by now I have s square plus omega square. So, s square plus omega square and I have s square plus omega 0 square. So, that is the first term and then I have plus s divided by s square plus omega s square plus omega square.

Now, if I want to invert if I want to take the inverse Laplace transform of this now s appears here s appears here and s appears in these 2 now what you know you know that you know the inverse of this. So, the inverse of, so if I want to calculate y of t then I know the inverse of this the inverse Laplace transform is nothing, but cosine of omega t. So, the inverse Laplace transform of this quantity is just cosine of omega t what about the inverse Laplace transforms of this quantity.

So, how do you take the inverse Laplace transform of this? So, to take inverse Laplace transform of this you use method of partial fractions now I will just illustrate this. So, what you will do is you will write a omega 0 divided by s square plus omega square s square plus omega 0 square this is equal to; now, what I want to write it as something divided by s square plus omega square and something divided by s square plus omega 0 square ok.

. So, now, you can easily see that I mean. So, you need you need to find out what these quantities are you can again you can easily see that since the s has to cancel you should have a minus sign here. So, you should have a minus sign here. So, then you will get s square plus omega square and you will get a s square plus omega 0 square and s square plus omega square. So, if I just divide this whole thing just multiply this whole thing by a omega 0 divided by omega 0 square minus omega square and I put a one here.

So, then I will get exactly this will be equal to this now what I will do is I will just take this omega 0 and I will take it inside. So, here I will just put omega 0 and I will put an omega 0 here and the reason for that will become obvious, now when you taking the inverse Laplace transform of this then what this will give me this will just give me. So, I can I can write this if I had an omega instead of omega 0 here then they then this would look like cosine let us go back to our table. So, cosine has an s over s square plus omega square sine has an omega over s square plus omega square.

So, if I had an omega here this would look like sine. So, I can I can write this as simply as a over omega 0 square minus omega square this is this will go outside this is independent of s now the first term I can write. So, let me just write y of t a or now the first term I want an omega here. So, what I will do is multiply. So, I will I will write this as omega 0 divided by omega into you will have omega over s square plus omega square the inverse Laplace transform of that is just sin omega t and the inverse Laplace transform of this is just sin omega 0 t and the last term is called cosine of omega t.

So, this is the inverse Laplace this is the solution of this and we have used this method of partial fractions to write this expression. So, I will conclude this lecture here. So, I will conclude the discussion on Laplace transforms. Now in the next lecture we will work out some practice problems, that will show you the usage of Fourier and Laplace transforms.

Thank you.