Advanced Mathematical Methods for Chemistry
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Module - 09
Lecture - 02
Properties of Fourier Transform; Shifting, Derivatives, Convolutions

Now that we have learnt about Fourier transforms let us look at certain properties of Fourier transforms.

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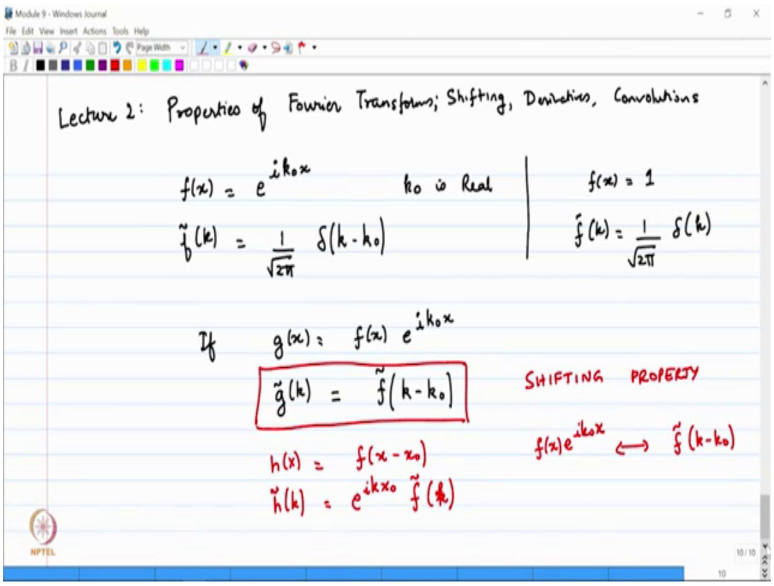

And the properties that I will be talking about are related to 2 things something called shifting, and it is also related to Fourier transforms of derivatives and another concept called convolutions.

So, let us start with shifting. Now we ask a question suppose I take f of x is equal to e to the i k 0 x, again assume that k is real and greater than 0, assume that k is real k 0 is real. Now if I take f tilde of k then the what I will get you can you can work this out, you will get you will get 1 over square root of 2 pi, you will get delta of k minus k 0. Now remember if I had if I had f of x is equal to 1 then I would I had f tilde of k 1 over square root of 2 pi delta of k, this was what we did. So, if we had a f of x being a constant then it is Fourier transform is just delta function.

Now, what I see is that if I take if I multiply it by e to the i k 0 x, then all I am doing is I am getting a delta function, but located at a different value of k at k 0. So, there is a general theorem that that suppose I had so, if g of x equal to f of x into e types i k 0 x. So, what I did was I took a function f of x multiplied by e to the i k 0 x and called it g of x. Then you can show that g tilde of k is equal to the Fourier transform of f evaluated at k minus k 0. So, this property is referred to as shifting property. And obviously, if you can you could also do suppose, suppose I had h of x is equal to f of x minus f, f of x minus x 0.

So, if the function evaluated at some different point, then you can immediately see that h tilde of k is equal to e to the i k x 0 times f tilde of k. So, if I had a shifted function, then the Fourier transform would be multiplied by e to the ik 0 x, because these form of Fourier transform pair. So, this is also something that is that is very clear. So, the Fourier transform pair is f of x e to the i k 0 x and f tilde of k minus k 0, and this is a Fourier transform pair. So, the shifting property is one very useful property of Fourier transforms and it is and we will see that this property is extremely useful in solving various differential equations.

The next the next significant property of Fourier transforms is related to derivative. So, if g of x is equal to d f of x by dx or f prime of x.

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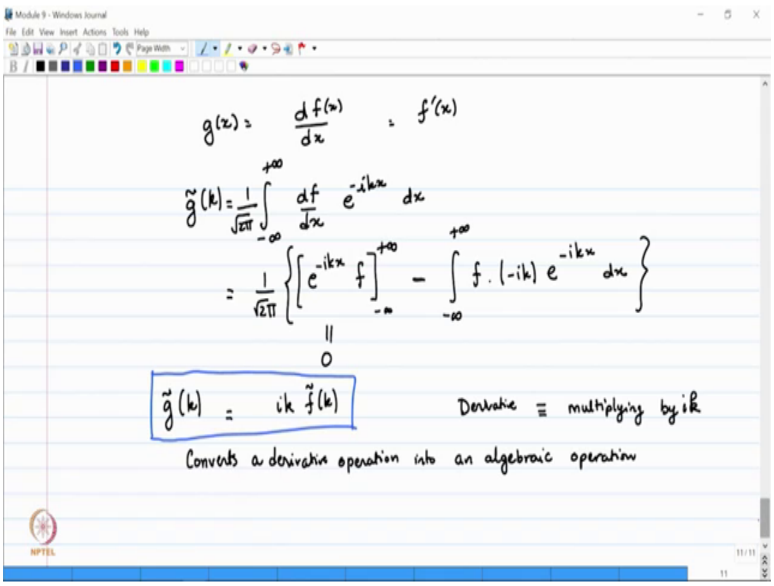

Then what can you say about g tilde of k? G tilde of k is equal to integral minus infinity to plus infinity 1 over square root of 2 pi, and what I have is df by dx e to the ikx e to the minus ikx dx. So, I can integrate this by parts. So, when you integrate by parts what you will get is 1 over square root of 2 pi, you have the first term, we will just have we will just have e to the minus i k x into f evaluated at minus infinity to plus infinity, the second term is integral f, now we have to take the derivative of e to the ikx, that is minus i k e to the minus i k x from minus infinity to plus infinity.

Now, the first term again if we assume that k has a if we assume that k has a positive imaginary part then this goes to 0. And so I can write this as i k and then what I have here is exactly what I have left over with the square root of 1 over 2 pi is just f, f tilde of k. So, this is a very again a very interesting property, that if you take the derivative of a function then the Fourier transform is nothing but i k multiplied by the Fourier transform of the function. So, derivative we corresponds to multiplying by k by i k.

So, this is another very interesting property of Fourier transforms that that convert this converts a derivative operation into an algebraic operation. So, this is again I am extremely important property of Fourier transforms. So, before I go to convolution. So, I want to emphasize one point about Fourier transforms. I will just mention it here, I will just I will just write this and I am writing this in red just to emphasize. So, this is beware, so there are several definitions of Fourier transforms.

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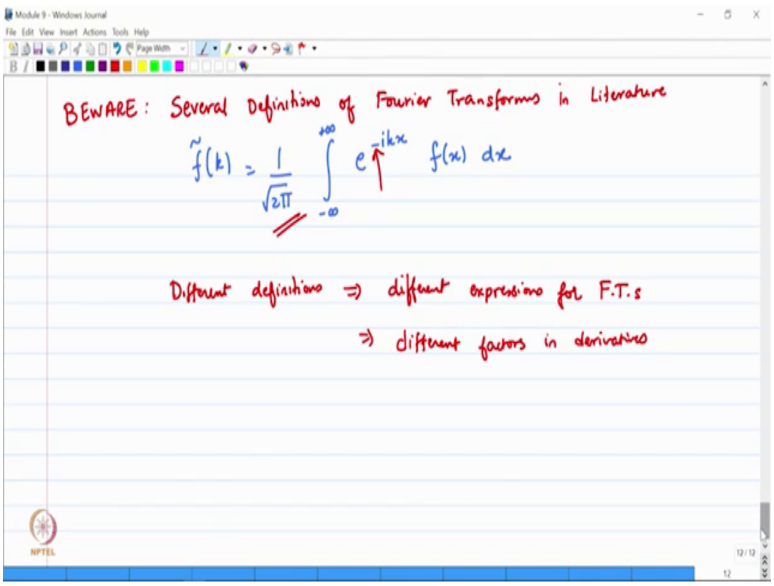

And what I want to say is that is that so, we used f tilde of k is equal to 1 by square root of 2 pi integral e to the minus i k x f of x dx.

So, now there are books which might use which might not have this factor of 2 pi, there are some places this factor of 2 pi is not included. Square root of 2 pi is not included. Some places this might be a positive sign. So, sometimes even the I is taken in front. So, there are several definitions of Fourier transforms in literature. So, always these different definitions, we will lead to we will lead to different expressions for Fourier transforms and typically they usually they have different factors in derivatives ok.

So, so when we took the Fourier transform of the derivative we had a factor of i k. So, this i k might be different if you are if you use it if a different definition of Fourier transform is used. And you will regularly find this in books they might they might define a some books will define a Fourier transform without this factor of square root of 2 pi. So, it is always important that when you are reading literature look at the definition that that particular book has used for the Fourier transforms, it may not be exactly the same as the as the definition that that we have used here, but nevertheless most of the expressions will be very similar to what we have got.

Now, the next interesting property of Fourier transform is a following.

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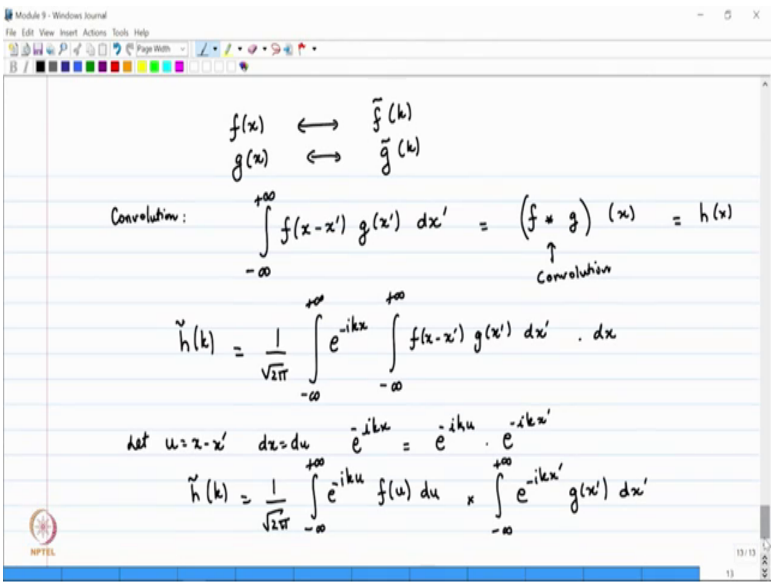

Suppose I have f of x it is Fourier transform is f tilde of k, and g of x has a Fourier transform of g tilde of k. Now suppose I consider define a convolution. So, I am going to define this convolution in the following way, I will say integral f of x minus x prime g of x prime dx prime from minus infinity to plus infinity. So, this is I call this as f convoluted with g this is the function of x. So, this function the name of the function I have denoted by this. So, this is a convolution ok.

So, so what I have done is I have taken 2 functions, but I have constructed a third function of x by not by directly multiplying them, but by multiplying them in this following way with this integral. Now the convolution is something that that plays a very interesting role in Fourier transforms. Now what is what is So, let me call this h of x ok.

So, now what is h tilde of k, now if you just go ahead and do the integral what you will get is 1 over square root of 2 pi and what you have is integral now you have integral of h of x, h of x into e to the minus i k x. Now for h of x I have f star of g and for s star of g I will put this other I will put this I will put this integral. So, I have integral minus infinity to plus infinity f of x minus x prime g of x prime dx prime, and I have this whole thing integrated over dx.

So, now, this is my Fourier transform of the convolution. Now what you can do is you can write this. So, you have integral over 2 variables. So, x prime is an independent x prime is independent variable. So, now the second variable I let u equal to x minus x prime. So, u and x prime are in dependent variables. And you can immediately see that d x equal to d u e to the ikx equal to e to the i k u into e to the i k x prime. And what I can do is I can write h tilde of k as 1 over square root of 2 pi and I can write this as 2 integrals minus infinity to plus infinity e to the minus i k u or let me put a minus here minus i k u f of u du multiplied by integral minus infinity to plus infinity e to the minus i k x prime g of x prime dx prime. So, that is h tilde of k and so and so, now, you can see that this is this is nothing but Fourier transform of f.

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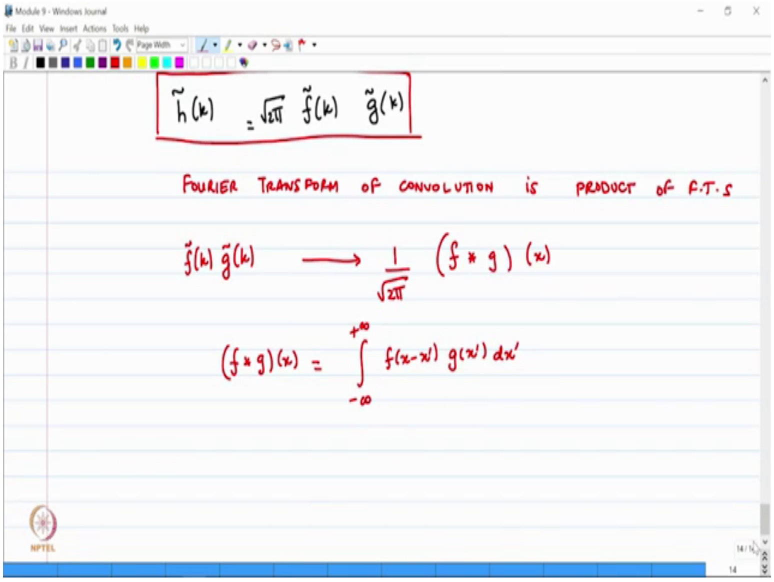

So, this is f tilde of k this is this with the factor of 1 by root 2 pi will be Fourier transform of g.

So, I have a g of g tilde of k into square root of 2 pi. So, what you get is that h tilde of k is equal to square root of 2 pi into f tilde of k g tilde of k. So, again this is a very, very important result. So, Fourier transform of convolution of convolution is product. So, taking a convolution of 2 functions is like taking a product of 2 functions in Fourier space ok.

Now, now this is again something that are extremely useful. So, the other way suppose you had f tilde of k g tilde of k it is inverse Fourier transform if you take the inverse Fourier transform of this then, you will get you will get f star g of x multiplied by 1 by square root of 2 pi. So, both these relations are extremely useful these are the same relation, but basically you can you know taking these convolutions is something that you will get used to when you are dealing with Fourier transforms. Again let us emphasize that definition of convolution f dot g of x is equal to integral minus infinity to plus infinity f of x minus x prime g of x prime dx prime. I could all, this is this is the definition of the convolution and this plays an important role in Fourier transforms.

So, I will conclude this lecture. So, far we have seened in the in the first 2 lectures we have seen what are Fourier transforms, how to take Fourier transforms. And in this lecture we have seen some import some interesting properties of Fourier transforms. So, in the next lecture what I want to do is to try to apply these apply Fourier transforms to solve partial differential equations. And what we will see is that the 2 properties that we or the various properties like shifting convolution etcetera are extremely useful in the in solving partial differential equations. I will also emphasize this that even though we are looking at partial differential equations we will use techniques that are very synonymous to ordinary differential equations. So, ah in that sense it would not be it would not be very advanced partial differential equations. We will have a module separately on partial differential equations ok.

Thank you.