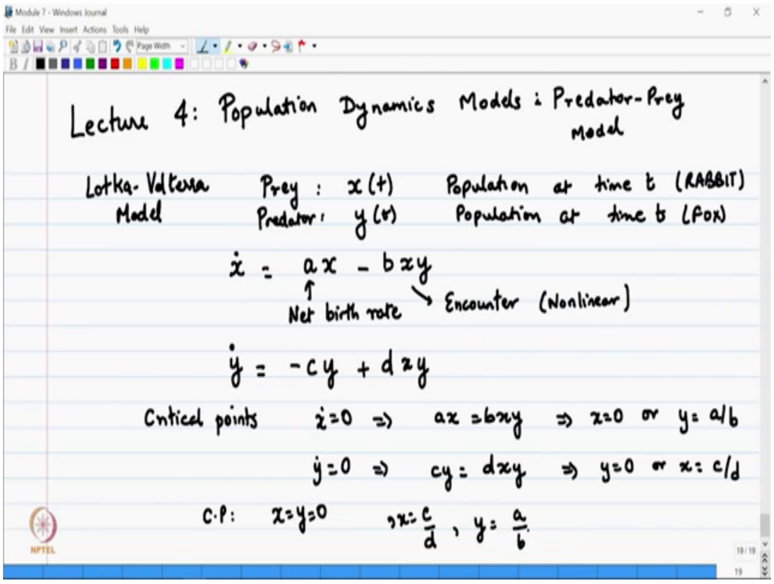
Advanced Mathematical Methods for Chemistry
Prof. Madhav Ranganathan
Department of Chemistry
Indian Institute of Technology, Kanpur

Module – 07
Lecture – 04
Population Dynamics Modules i Predator-Prey Model

(Refer Slide Time: 00:14)



In this lecture, I am going to talk about population dynamics and I will talk about a very well known model called a Predator Prey model. This is a very simple example of a non-linear differential equation and it is an extremely powerful model that has been applied not only to populations of real species like; it can be the population of foxes and rabbits or shark and small fish or you know some Predator and some prey, but it has also been; it is also something that is used in a modeling certain chemical reactions.

So, we will start with this. Let me; these models are referred to as either Predator Prey models or there often referred to as Lotka Volterra model and I will just write the differential equation. So, the idea is that there is some Prey whose population at any time is given by x a x is a function of time x of t is population at time t and there is a Predator which is y at t this is a population at time t. So, you have a Predator and a Prey for example, as I said this can be maybe a rabbit and this can be a fox.

Now the differential equation for the change of population of the Predator is I will write the differential equation and then I will tell you what the idea is a x minus b x y. Now this a is the; I will call it the net birth rate. So, if there was no if there was no Predator if y was is equal to 0 then the Predator population would grow at some rate a that is the net birth rate. So, it is actually the difference of birth and death rate.

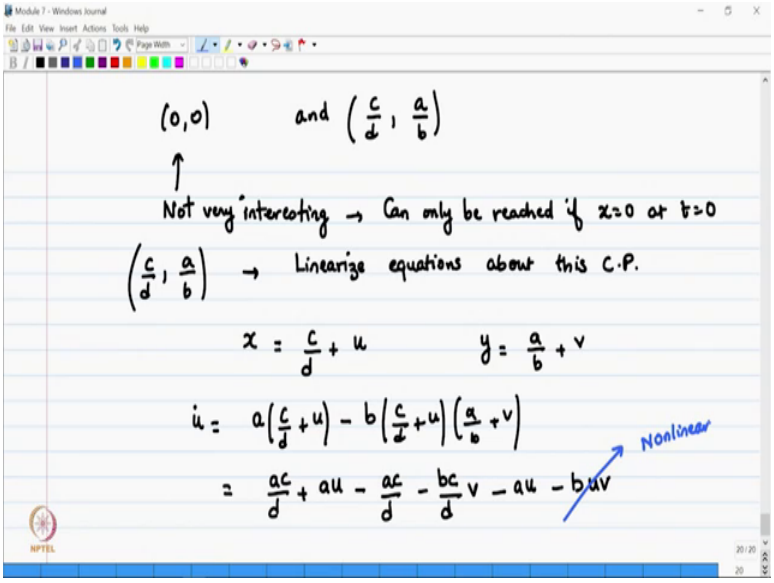
Now, this term is a term. So, x times y x is a population of x times the population of y. This is the rate at which y x and y encounter each other. So, and in an encounter; obviously, the Prey will get eaten. So, the population of Prey will get decree will get reduced due to the encounter. So, this is the encounter term. So, during the encounter between the Prey and Predator, the population will decrease and so and so there is a. So, its population goes down, now this is a non-linear term non-linear term depends on both x and y.

Now, we should also have a differential equation for y. So, y dot now what will assume is that is that we all write it as minus c y plus d times x y. So, so this is the net birth rate and the you know the Predator if there is no Prey; the Predator will; its population will decrease. So, if x equal to 0 then y will go to 0 so. So, therefore, it has a negative net birth rate, but if there is Predator then the then it is population will increase and this is the non-linear term. So, due to encounter the population of the Predator increases; due to this factor d.

Now, now these are again; so, these are set of coupled non-linear; non-linear differential equations. Now let us look at the critical points. So, if I put x dot equal to 0 that implies ax equal to b x y that implies x e; x equal to 0 or if x is not equal to 0 y equal to if x is not equal to 0 then y equal to a by b and the other term if I put y dot equal to 0. So, this implies c y equal to d x y. This implies y equal to 0 or x equal to or x equal to c by d.

So, basically you can see that you can see that if x equal to 0 in this case then y has to be 0. Similarly if y equal to 0 then x has to be 0. So, the critical points are basically given by x equal to y equal to 0 that is the first critical point and in the second critical point if y equal to a by b. So, if I substitute y equal to a by b in this case; then you can see if y equal to a by b; that means, y is not equal to 0. So, x has to be c by d. So, so the other critical point is given by x equal to c by d, y equal to a by b.

(Refer Slide Time: 06:22)



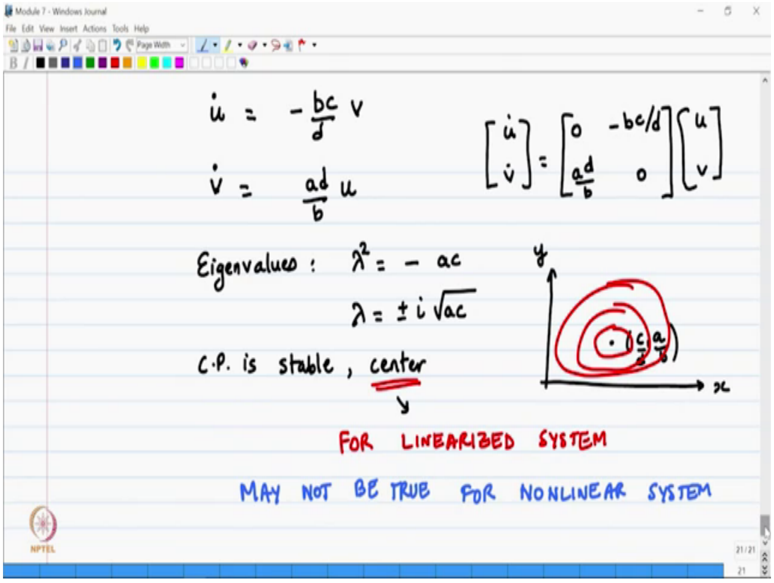
So, we have 2 critical points and. So, so the 2 critical points are 0, 0. I will write it in this form 0, 0 and c by d, a by b. Now the critical point at 0, 0; this critical point is not very interesting. So, this is not very interesting. This can only be reached why is it; why did I say it is not very interesting. So, this can only be reached if x equal to 0 at t equal to 0. So, then what happens if x equal to 0 at t equal to 0; then what happens is y dot equal to minus cy and y will exponentially decay to 0. If any other case you cannot reach that critical point. So, if x is not equal to 0 at t equal to 0 then you cannot reach that case and so this is not very interesting.

So, let us look at the other critical point. So, let us look at c d, a b; this critical point. So, now, let us linearize equations about this critical point. So, what we are going to do. So, take our equations for x and y and linearize about this critical point.

Now, how will you write x y; so, when you linearize about the critical point; then you have a term that looks like. So, so you write x approximately equal to c by d plus let me call it u and y or y equal to equal to a by b plus v. So, so if you think of it in this way then I can write; I can write the first equation. So, x dot is same as u dot. So, I can write I can write u dot equal to. So, now, now the first term I can the a times x I can write as a times u. So, a times c by d plus u plus. Now what you have is d times or b time or minus minus b times c by d plus u a by b plus v now what you can you can write this whole thing in the following way.

So, what you will get if you if you expand this out. So, so let us do this. So, this is a c by d plus a u lets multiply this out. So, so you will have b times c; c by d into a by b. So, it is; so, the b will cancel. So, will get a c by d then the next term you will have c by d into v. So, so I will write it as minus b c by d v then you will have a term that looks like u into a by b. So, you into a by b into b is minus a u and you have a term minus b u v. So, this is a non-linear term. So, this is thrown away. So, this is a non-linear term. This is non-linear. Since it involves both u and v, it involves a product of u and v and what you can also see is that is that a u term will cancel a c by d will cancel.

(Refer Slide Time: 10:59)

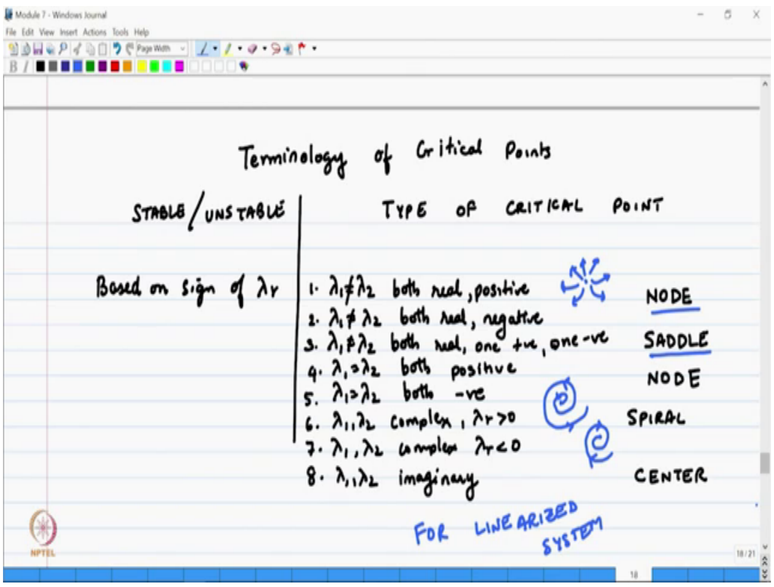


So, finally, what you get is u dot is equal to minus b c by d into v. Similarly you can show that v dot is equal to a d by b into u. So, now, the linear equations look like looks like u dot v dot is equal to 0 minus b c by d a d by b 0 u v. So, what does this mean? So, so if you if you calculate Eigen values; you can easily see that lambda square is equal to minus. So, the b will cancel the d will cancel. So, we will get minus a c. So, implies lambda is equal to I plus minus I square root of a c. So, this is a; this critical point is stable and it is a center and if you look at trajectories. So, what will happen is that is that if you put certain values of a b and c then you will find that the trajectories.

So, if you look at if you look at y versus x and we are only interested in the case where x and y is greater than 0 and this point which we took this critical point which was given by c by d; a by b. So, this is critical point is c by d a by b. So, what happens? So, around this critical point your trajectories will be basically some sort of curves around this. So, will be some sort of curves we do not know what exactly the shapes of the curves are, but your trajectories will be some sort of curves around this and remember they cannot go they cannot they cannot cross 0. So, x and y are strictly greater than 0. So, this is one case and in this case your critical point is a center. So, the critical point is stable or a center. Now, there is a small point that that we have to emphasize. So, this critical point is a stable center this is for the linearized system.

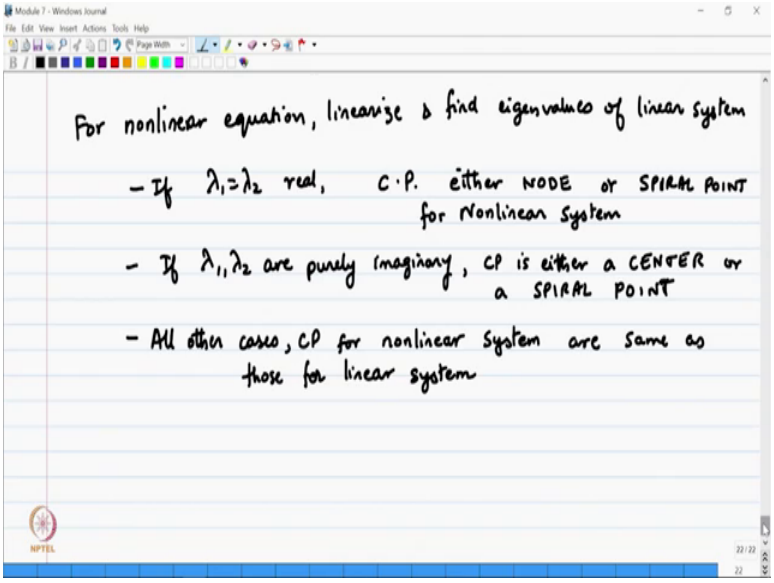
Now; however, this may not be true for the non-linear system. This is a very subtle part that the nature of the critical point what we what we said if we go back to what we discussed in the last class we discussed the type of critical points.

(Refer Slide Time: 14:48)



This is a type of critical point, this is only for linear system; this is for the linearized system; however, now for the non-linear system; then what is the nature of critical points I will not go into this in too much detail. So, I will just mention.

(Refer Slide Time: 15:17)

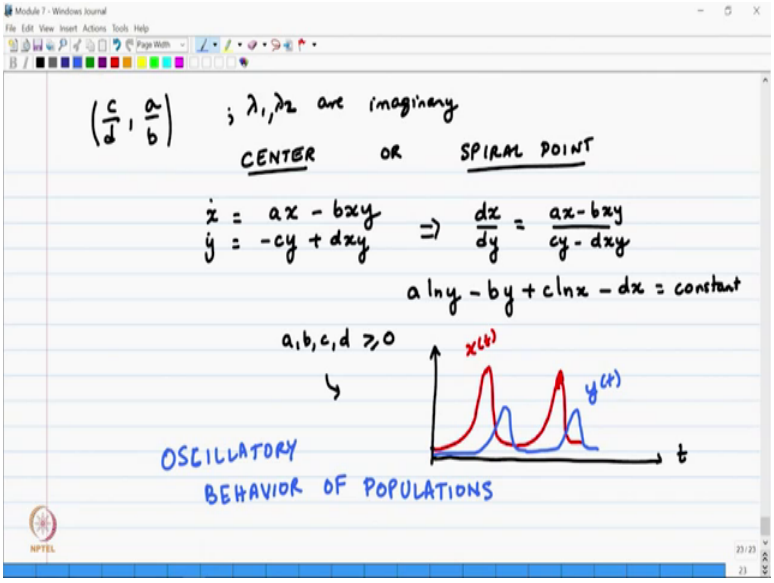


So, for non-linear system, then the nature of critical point for non-linear equation you can still linearize; so, we linearize and find Eigen values of linear system. So, you can only find Eigen values of the linear system because only in the linear system you have the matrix.

Now if lambda 1 equal to lambda 2 and it is real; lambda 1 equal to lambda 2 and it is real then the critical point of the of the non-linear system is either node or a spiral point. So, for the non-linear system; so, either node; this is for non-linear for non-linear system. The next thing is if lambda 1, lambda 2 are purely imaginary. So, so if they are purely imaginary then the critical point is either a center or a spiral point. So, for the linear case critical point was purely a center, but for the non-linear case, if these 2 Eigen values are purely imaginary then it can be either a center or a spiral point and the and the third case is. So, so all other cases; critical points for non-linear system are same as those for linear system.

So, what; that means, coming back to coming that back to what we what we discussed here. So, for the linear system this critical point this critical point that we saw where x was equal to c by d and y was equal to a by b that is a center and it is stable, but for the non-linear system. it may be a center or a spiral point. So, what; that means, is that your point c by d, a by b Eigen Values are purely imaginary; may be a center or spiral point.

(Refer Slide Time: 18:25)



So, it may be either a center or a spiral point and actually both these cases are fairly interesting. So, what we can do is suppose we take this equation. So, in this particular case, so it may be either a center or a spiral point keep this c by d a, a by b. So, if it is a spiral point then again you know based on certain conditions, you will get either stable or unstable, but this is definitely a possibility. So, what happens to the species as time evolves?

Now, in this particular case, where we had x dot is equal to a x minus b x y and y dot is equal to minus c y plus d x y we can do we can we can divide them and we can and we can write d x by d y is equal to ax minus b x y divided by c y minus d x y and this you can use our usual method of you know you can you can do the usual method of solving first order differential equations like using integrating factor you can you can convert these 2 an exact differential and then and then use the integrating factor to get the solution.

So, the solution of this is given by a lny minus b y plus c lnx minus d x equal to constant. You can easily verify that this satisfies this equation. So, now you can think of these this is constant. So, for various values of constant if you plot this you will get a various you will get a families of curves. Now we said that a, b, c, d are all greater than equal to 0 and you can analyze this behavior for various values of a, b, c and d.

So, you can get basically you can plot let us say suppose I plot versus time; now you could have you could have let us say starting from some if you if you start at some point you might you might you might have you know one species go like this might be x of t. So, so what is happening is that initially the population is small and then and then it keeps increasing.

Now, let me draw the curve in black also just to just to show what. So, so this is x of t now let me draw y of t. So, y of t will look something like this and. So, on and there and it will keep going. So, this is y of t. So, so what is happening here is the; is that initially the population of y is very low x grows at a much faster rate and since x is also small y is just increasing very very slowly. So, y is increasing very slowly then once x becomes large enough then y starts increasing very rapidly because y is because of the dita and you know and what happens is when x is maximum then y is really growing at a fast rate and that causes x to go down because this x because y becomes large this term minus b x y becomes substantial. So, that causes x to go down this causes a reduction in x, but now as x goes down the d term also going down. So, so at one point after that the population of y starts decreasing with because d becomes smaller then y y starts going down.

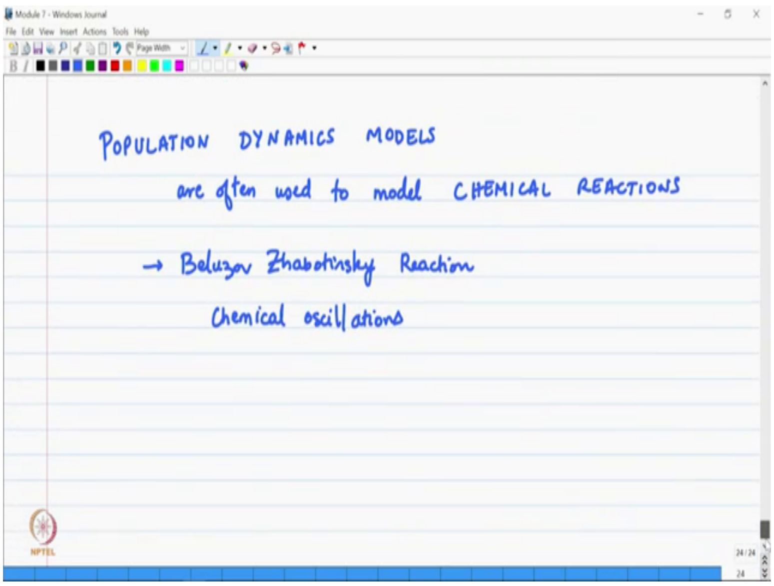
Now y goes down it keeps going down because x is also going down y is also going down in this range and then and then and then once y becomes very low then x starts increasing because the b x y term is small a x y term is greater x starts increasing and then and then it goes in this sort of periodic cycles. So, what you have is that the population of each of the species shows oscillatory behavior; so, oscillatory behavior of population. So, this is something that has been observed you know fairly often and in these systems.

Now the nature of these oscillations might change, you might it might be shifted or in some case you might have different kind of behavior, but nevertheless what I want to show through this is that this non; this rather simple looking non-linear differential equation can show oscillatory behavior and the reason for this oscillatory behavior is due to this due to this critical point. So, so if this critical point is a center then it will show oscillatory behavior of course, if it becomes a spiral point. So, for certain choices of a a and b this can actually become a spiral point in which case in which case what can happen is that the you know the species will you know one of the species will get might get exterminated or so on.

So, or actually it goes you know the population of the species changes and at one point it reaches a point where the population of both the species do not change. So, the rate at which rate at which the species rabbits are born is equal to the rate at which they are they encounter in that in they in they get killed in encounters similarly the rate at which the foxes die is equal to the rate at which they are produced due to encounters.

So, in that case you will have a in that case the both these will go to 0; the rates will go to 0, but there I mean I mean there population is not 0. So, that would be the case of a stable spiral point I want I want discuss that in detail, but let me emphasize one thing that population models.

(Refer Slide Time: 27:01)



Population dynamics models are often used to model chemical equations chemical reactions and in fact one of the most celebrated chemical reaction is called the Beluzov Zhabotinsky reaction and this is a chemical reaction that shows oscillating chemical oscillations and this was; this is again modeled using these population dynamics models using equations that look very much like this.

So, this could be a concentration x could be a concentration of one reactant and that is that is you know increases by this factor. So, this is a rate of change of concentration of one reaction and it is it might react with the with some other reactant y. So, it might be consumed in this way and it might be produced due to some other process. Similarly y might be consumed due to one process and produced to another and you can have the important thing is there is non-linear term that involves product of concentration of reactants.

So, these kind of chemical oscillations have been observed in systems and they can be analyzed again exactly using these population dynamics models. So, in this module, I have trying to show you how you can use techniques of you can use some techniques to analyze non-linear differential equations and you know this is a very powerful and very I would I say it is extremely interesting because you know you can see by doing this analysis how you can get more information about a differential equation without actually solving it.

So, in the last lecture of this module I will do a few practice problems. So, I will try to analyze nature of certain non-linear differential equations.

Thank you.