Advanced Mathematical Methods for Chemistry
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Module - 07
Lecture - 03
Stability of Critical Points

So in this lecture I am going to discuss a little bit about the nature and the stability of the critical points, this will be a short lecture that will summarize some of the things that we illustrated in the last 2 lectures. So, before I do that I just wanted to mention a few things I had a few things to the last lecture the first thing is if you remember in the last lecture we had done all these we had look at the phase base picture of the pendulum, and we had found that the critical points.

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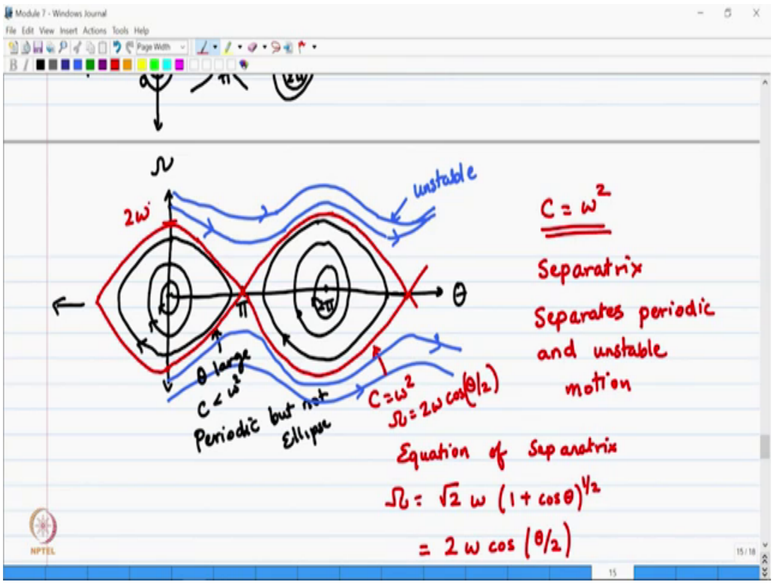

So, they has a critical point at 0 and a critical point at pi, and what we found is that the trajectories near the critical point at pi they actually go in this form ok.

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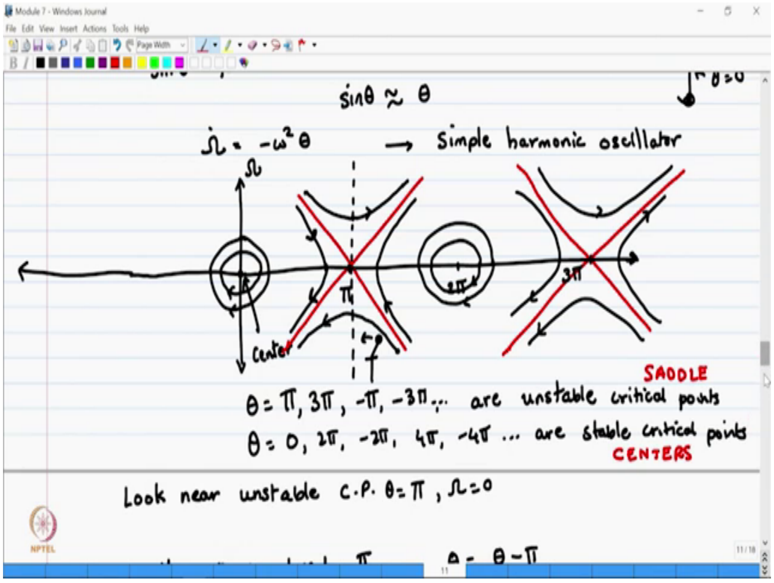

So, they go in this way. So, they do not go to the critical point at pi or 3 pi, but they go away from this. Now there is a some terminology we called these points the 0 2 pi etcetera these we call them as centers these are called center because the trajectories never approach this they are centered there are periodic orbits around this center, whereas these pi 3 pi etcetera. So, these are actually what are refer to as saddles we will see exactly how you defined them. So, this is called a saddle and there is a there is a good reason for calling it a saddle it has to do the Eigen values eigenvectors ok.

Now, the other point I wanted to make was regarding this separatrix. So, in the separatrix you have C equal to omega square. So, you can ask a question what is the equation of this separatrix. So, what is the equation of separatrix? So, the equation of separatrix, you go back our original equation. So, we have this.

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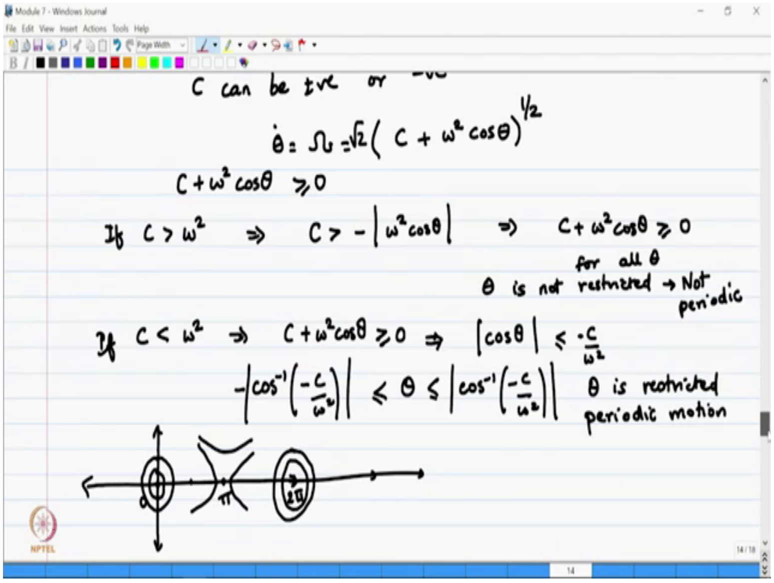

Omega equal to square root of 2 c plus omega square cos square theta now c equal to omega square then I can take I can write omega square plus omega square cos theta and I can take it outside the square root. So, I have omega equal to square root of 2 omega and you have 1 plus cos theta these to half and if you use the trigonometric relation for cos theta, then you can show that cos theta is 2 cos square of theta by 2.

So, I can take the square root of 2 and multiplied by this I will get 2 omega cosine of theta by 2 this is the equation of the separatrix and you can see that it looks like a cosine function with a period of in fact the it is the period is 4 pi not 2 pi because you have theta by 2. So, this is nothing, but this equation of this separatrix is omega equal to 2 cos 2 times this omega which is a constant cos theta by 2. So, the amplitude right here is just 2 omegas, but this is the equation of the separatrix and we can actually find that equation.

Now let us go to the main topic for today that is the stability of the critical points ok.

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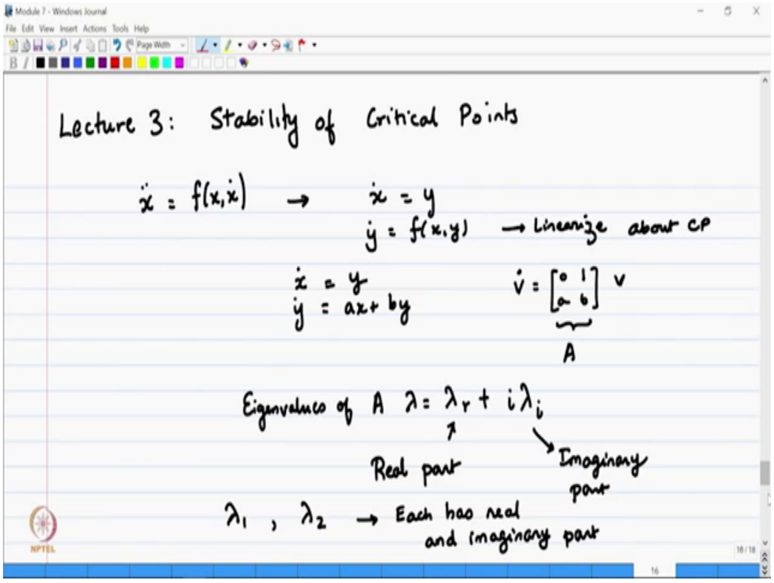

And this will this as I said will be summary of what we have done in the last few last couple of lectures. So, remember the last lecture we started with this differential equation and we said it is a second order differential equation and we can write it in general as f of x x dot, we can write this as x dot equal to y and y dot equal to f of xy. So, this is the general non-linear differential equation and what we said is that you linearize this, linearize this non-linear function about the critical point. So, this is. So, f of xy is basically non-linear. So, it might contain terms like x square or y square or x into y and so on. So, when you linearize about the critical point then your f of xy becomes this form ax plus by a and b are scalars or constants ok.

So, now, this becomes a linear system and I can write this as V dot equal to equal to 0 a 1 b into v where v is basically xy. So, when I linearize about the critical point remember linearizing involves a Taylor expansion some sort of expansion. So, that is you have to keep that in mind. Now once you have a linear system then you know that the solutions have this form. So, solutions look like e to the look like e to the lambda t, a where lambda is an Eigenvalue of a. So, in general this Eigenvalue of a is written as a real part plus an imaginary part. So, in general it is complex and since it is complex I can write it as lambda real plus I lambda times lambda imaginary. So, this a imaginary part and this is the real part and I is the unit imaginary number and now you and additionally you have 2 Eigenvalues. So, lambda 1 and lambda 2 and each has real and imaginary part each has a real and an imaginary part it is possible that the real part might be 0, possible that the imaginary part might be 0, but in general each has a real and an imaginary part.

Now, let us look at some cases now if lambda r is less than 0.

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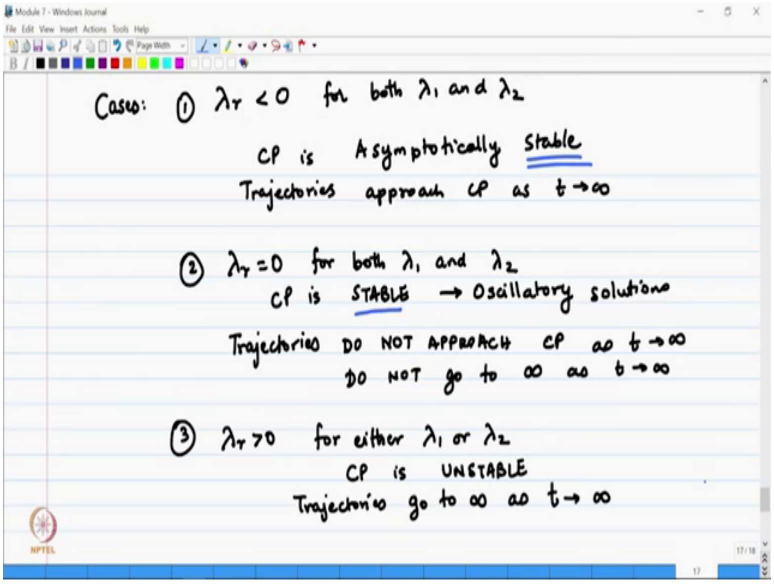

For both lambda 1 and lambda 2 lambda r is less than 0 for both lambda 1 and lambda 2 then than we say that the critical point is asymptotically stable; that means, the trajectories they approach the critical point as t tends to infinity. So, as t goes to infinity the trajectories will go towards the critical point.

So, such a critical point is said to be asymptotically stable. Now there are several cases which we will also look into in more detail, but generally if the real part is less than 0 for both lambda 1 and lambda 2, then the important thing is it is stable; that means, your trajectories will tend towards the critical point; that means, if you go to very large time then you will end up at the critical point. Now if on the other hand if the real part is 0 if the real part is 0 and if the real part is 0 for lambda 1 it has to be 0 for lambda 2 also.

That means the both lambda 1 and lambda 2 are purely imaginary, if they are purely imaginary then we say that the critical point is stable and the solution is oscillatory solution is purely oscillatory. So, you will just get an oscillating solution, but the critical point is stable in the sense that you do not have any you know trajectories do not approach critical point as t tends to infinity, but they do not go to infinity the trajectories do not diverge away and go all the way to infinity as t tends to infinity. So, this is the case what we did in the. So, this for example, all the all these 0 to pi these are examples of stable critical points where the trajectories are periodic around it the Eigenvalues are purely imaginary.

Now, the third case is if the real part is greater than 0 for either lambda 1 or lambda 2 see if the real part of either lambda 1 or lambda 2 is greater than 0. So, this could happen in 2 ways it could be that both lambda 1 and are real. So, both are real, but one of them is greater than 0 one of them is less than 0 that is one possibility. The other possibility is that lambda 1 and lambda 2 are complex if they are complex then they have to be complex conjugates of each other, but there real part is greater than 0. In this case the critical point is unstable the trajectories go to infinity as t tends to infinity, this is the I mean we are just looking at the in the way in the vicinity of the critical point and that would be this is this pi and 2 pi are examples of such critical point; because you see what happens to the trajectories they go they go away from this and they diverge all the way at least in the linear limit they go they diverge away from that critical point ok.

So, this is the these are the nature of the critical points. So, what we have said is we have basically said whether they are stable or unstable. This is a very important thing to know when you are analyzing non-linear differential equations you want to know which are the stable and unstable critical points. There is a more you know there is a broader classification of critical points and I will just mention this terminology and you know you can actually work out examples of each of these we will do it during the practice problems.

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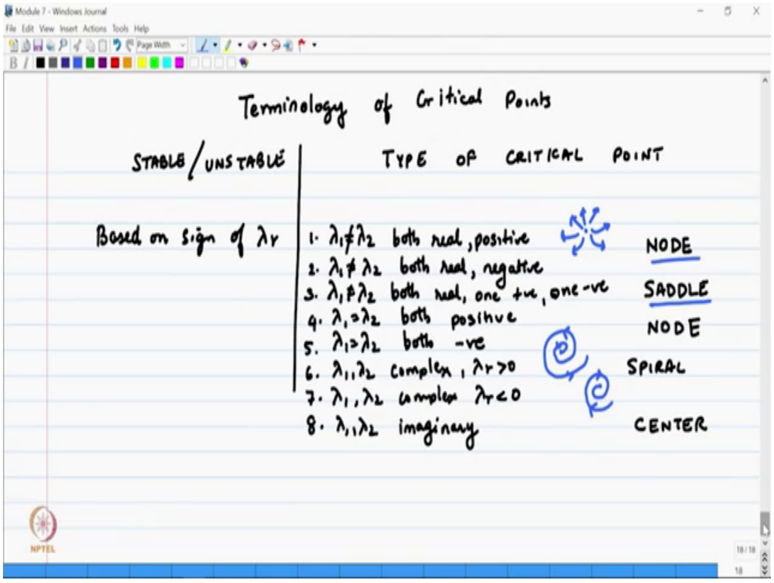

So, the terminology of critical points you can refer to them as stable or unstable this is based on the sign of the real part. So, that is just what we saw here, we saw the real part is either less than 0 then it is stable equal to 0 then it is stable and if it is greater than 0 then it is unstable. So, that is one part the second part is you know we talked about we talked about saddle and spiral and center we saw these kind of critical points and if you remember the spiral point ok.

I can show you from the last lecture. So, what was the spiral point yeah. So, this is an example of the spiral point which we saw in the case of the damped harmonic oscillator.

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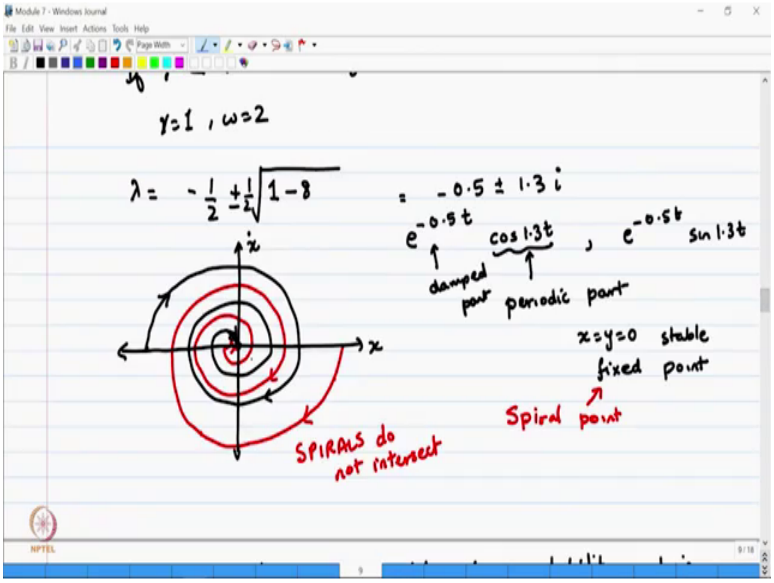

So, this critical point is a spiral point what is important is that each of these can be characterized now this is what we called a node a stable node.

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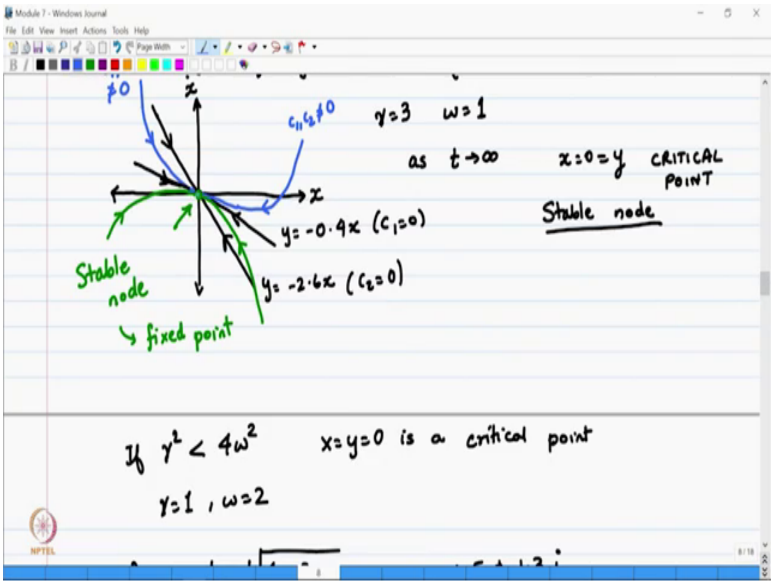

So, this is the node because all trajectories go towards that node. So, we also saw this other terminology where you have nodes and spirals and you know we called this is a center this is a saddle. So, there are center saddles nodes and spirals, now these are actually related to the sign of the real and imaginary parts.

So, if you have critical points there are various possibilities; you can have lambda 1 the Eigenvalues lambda 1 and lambda 2 are not equal to each other, but both are real and both are positives, then you will get a node and you know you can easily tell that it will be unstable, but the point is this is a node and a node remember in a node what happens is that all the trajectories either go to the node if it is stable or they go away from the node if it is unstable. So, if they are positive then your critical point and all the trajectories go away from the node and they can go in various ways, they can go they can go either straight or they can go in curves.

But the point is they go away from the node and in this case lambda 1 and lambda 2 are not equal to each other both are real both are negative now that is a that is also a node and in this case it is a stable node. So, since you are negative it is a stable node. If they are both real one positive and one negative if lambda 1 and lambda 2 are both real, but one is positive and one is negative then you get what is then you get the saddle point. If lambda 1 is equal to lambda 2 and both are positive then or if lambda 1 is equal to lambda 2 and both are negative both these cases you get a node, and it is actually a slightly different kind of node this is what is called a proper node.

But again it can be either stable or unstable. So, in this case if both are negative then it will be a stable in this case it will be unstable. Now if lambda 1 and lambda 2 are complex such that the real part is greater than 0 then you get a spiral, if they are complex where the real part is less than 0 you also get a spiral and if lambda 1 and lambda 2 are purely imaginary then you get a center again the spiral can be either stable or unstable. So, if it is a you could have a spiral where trajectories go into the into this they go to the node or they can go away from the node. So, you could have a spiral where the trajectories they go they start at the node and they go outs away from the node or you could have a spiral in which case the trajectories you could have a stable spiral where the trajectories go and into the node.

So, they go from outside and they approach the node both these are possible and again the stability or instability depends on the sign of lambda r. And the last thing is if lambda 1 and lambda 2 are both imaginary then it is a center and center is considered a stable point even though the trajectories do not approach that point. So, these are the various kinds of critical points and what we will do is to look through examples how we use this how we identify the behavior of the system by identifying these various critical points. So, in the next lecture we will look at some of these critical points.

Thank you.