Advanced Mathematical Methods for Chemistry

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Module - 07
Lecture - 02

Phase Plane of a Pendulum, Linear Stability Analysis

In this lecture I will talk about we look at the phase plane, but for us slightly more complicated problem and in this case we look at a problem where we do not actually solve the problem. So, in the previous case we saw that we could actually solve the problem and we could write and that was because the problem was linear problem, it was a linear ode the either the harmonic oscillator or the damped harmonic oscillator both are linear equations.

Now let us consider a non-linear equation and show how we can analyze the non-linear differential equation. So, the equation that I will be considering is that of a pendulum of arbitrary amplitude.

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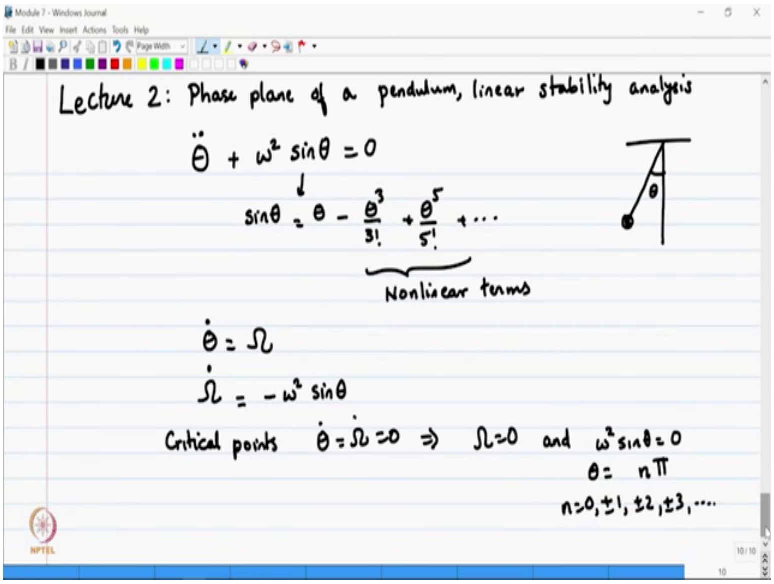

So, the differential equation has this form theta double dot. So, theta is the angle that the pendulum makes. So, if you have a pendulum and this is the vertical. So, the pendulum is oscillating and as it oscillates; it makes an angle theta; so, theta double dot plus omega square sin theta equal to 0.

So, clearly he is a sin theta. So, it is not just theta. So, for small amplitude it becomes linear, but for arbitrary amplitude this is non-linear, sin theta has theta, theta cube etcetera. So, sin theta remember sin theta remember the Taylor series for sin theta sin theta equal to theta minus theta cube by 3 factorial plus theta 5 by 5 factorial plus all these are non-linear terms.

So, this is a non-linear second order ode let us write this in the usual phase plane picture. So, we will say theta dot equal to omega, and omega dot which is theta double dot is equal to minus omega square. This omega is in these this is capital omega this is small omega, square sin theta. Now let us look at the critical points. So, theta dot equal to omega dot equal to 0 implies omega equal to 0, and omega square sin theta equal to 0.

So, now, omega square sin theta equal to 0 implies sin theta equal to 0 or theta equal to plus minus n pi or I will just write it as n pi, where n equal to 0 plus minus 1 plus minus 2 3 and so on. So, these are the critical points. So, critical points omega equal to 0 and theta equal to plus minus n pi. So, it can be 0 plus pi, plus 2 pi, plus 3 pi minus 2 pi minus 3 pi and so on

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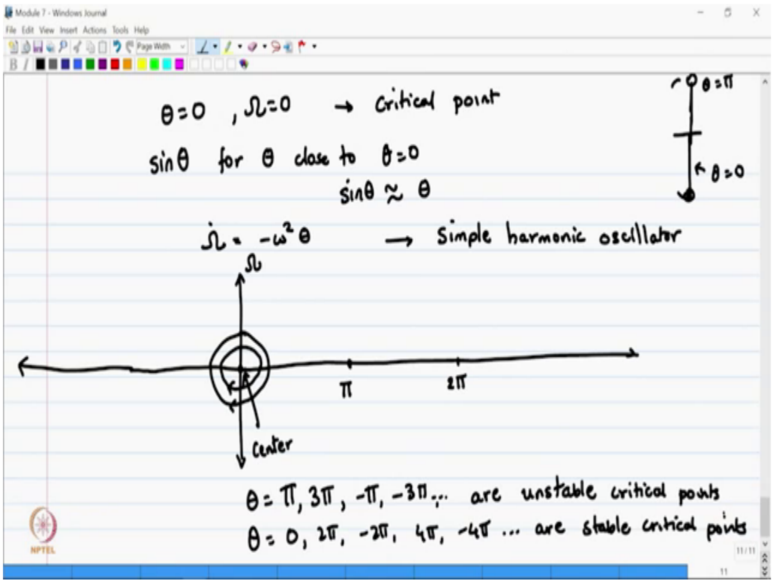

Now theta equal to 0 omega equal to 0 this is your critical point. So, this is one critical point. So, when theta becomes 0 then you can clearly see omega is 0, omega dot is 0 everything is 0, one kind of critical point and we will see soon that this point has a certain feature whereas, the other ones will have certain other features. Let us take this point sin theta for theta close to theta equal to 0. So, close to the critical point I can write sin theta as sin theta approximately equal to theta. I get omega dot equal to minus omega square theta.

And this is basically a simple harmonic oscillator and it is a simple harmonic oscillator. So, now, if I look at this omega theta plane the phase plane, and I will deliberately show it a little long and the theta axis. So, theta equal to 0 is one critical point, and around this critical point your trajectories look like they ellipses. This is what the trajectories look like about this point; this is theta equal to 0, then you have another point where let us say theta equal to pi, this is another critical point let us look at what the trajectories look like around theta equal to pi.

So, in this case your trajectories look like periodic solutions, this is a center theta equal to omega equal to 0 is the center and around this critical point the trajectories look like ellipses. What about theta equal to pi? So, now, in this case we will also look at 2 pi just to just for completeness; what happens is that we have to look at what the solutions look like when you make small displacements around theta equal to pi, theta equal to pi if you look at your pendulum that is this; theta equal to 0. So, this is theta equal to 0 and theta equal to pi is the pendulum pointing straight up.

So, clearly if you make a small displacement around this, if you make any small displacement you will you can easily show that it will keep going away and away. If you make a small displacement around this the solution is oscillatory, if you make a small displacement around this the solution is not oscillatory, it is it will completely go away from that. So, clearly theta equal to pi, 3 pi, minus pi, minus 3 pi. So, pi is same as 3 pi same as 5 pi etcetera.

So, are unstable critical points whereas, theta equal to 0, 2 pi minus 2 pi, 4 pi, minus 4pi etcetera are stable critical points.

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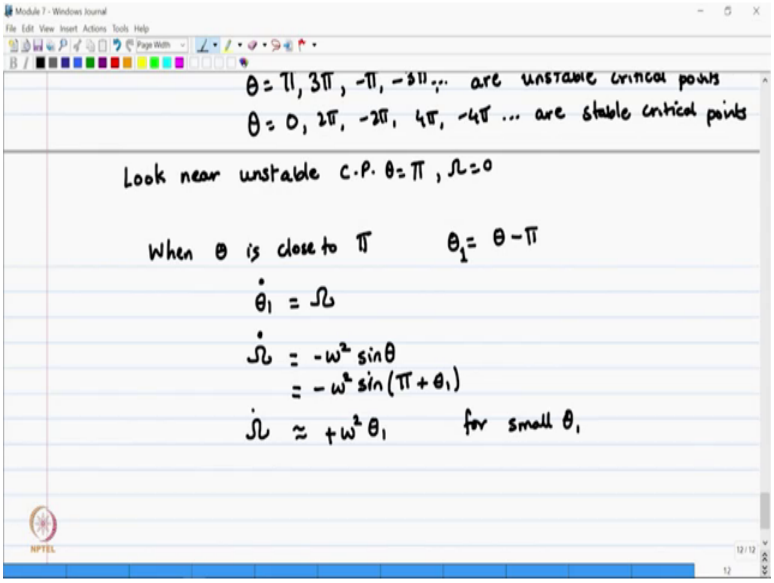

So, now, let us look near unstable critical point pi theta equal to pi omega equal to 0. So, when theta is close to pi we define theta equal to, theta are theta 1 equal to theta minus pi. So, then I will just. So, see theta 1 is a difference of theta from pi, and now I can write my equations as d theta 1 by dt or theta 1 dot equal to omega C; theta 1 dot is same as theta dot which is show omega omega.

But what I will get is that omega dot now omega dot was basically given as was given by minus omega square sin theta this is. So, close to this. So, this is approximately equal to minus omega square, it is equal to sin of pi plus theta 1 which is close to for small theta 1, so, this goes as plus omega square theta 1; so, for small for small theta 1. So, theta 1 is the difference of theta from pi and is since this is small, I can do this sort of expression and now I can write my now I have a linear equation.

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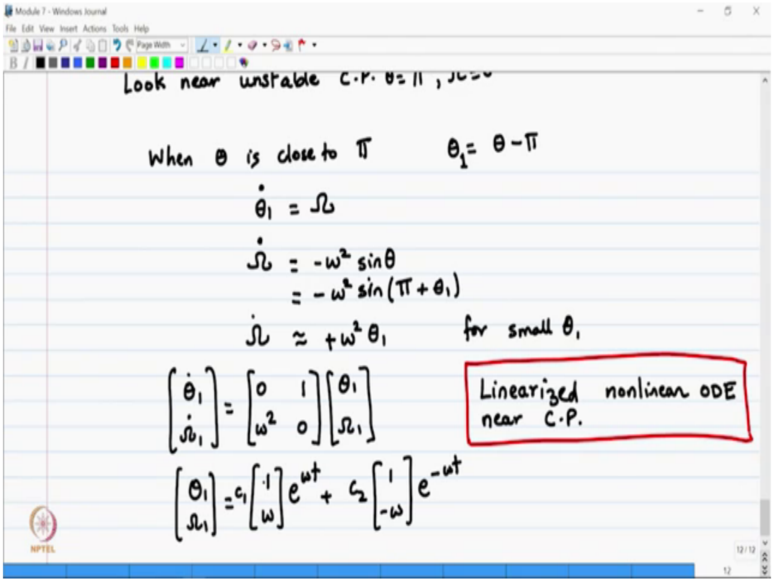

So, I have theta 1 dot omega 1 dot is equal to 0 1 omega square 0, theta 1 omega 1. So, this procedure is called linearizing this equation. So, what we did is we linearized non-linear ODE near critical point. So, near the critical point we linearized and we wrote it in this form; this is the very important step. So, this step linearization or of a non-linear ode near the critical point is a very crucial step in analyzing of non-linear odes.

So, what we did was first we found critical points. So, we had a non-linear ode. So, first we found the critical points and then we linearize the ode. So, we identified the stable critical point. So, we know what the solution looks near the stable critical point. Now we are going to linearize allow around the unstable critical point, and we know what the solution looks like. Now this you can easily write the solution theta 1 omega 1 this is a linear ode only thing now it is plus omega square.

So, this will have exponential solutions. So, it look like C 1 plus C 2 1 minus omega, e to the minus omega t. You can do the analysis of this around this critical point what you will find if you analyze the trajectories, what you will find is that you have lines of slope omega and minus omega. So, you have 2 lines; so, one of slope omega and 1 a slope minus omega.

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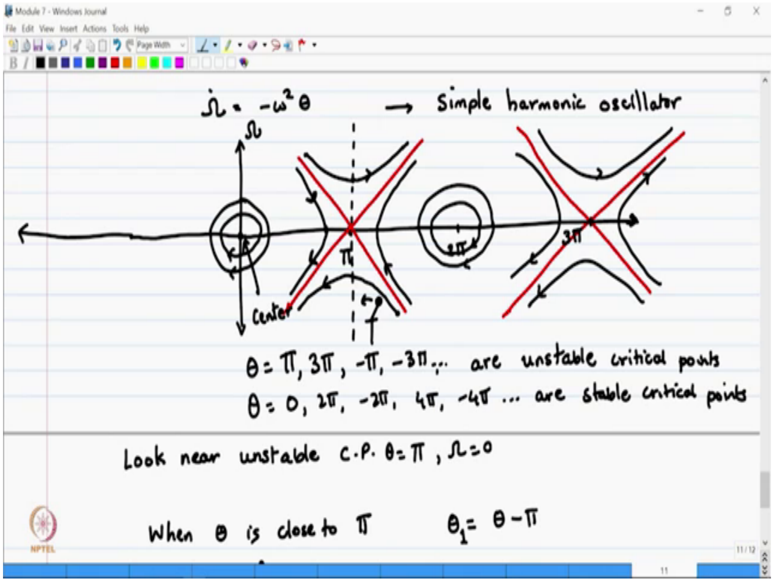

So, let me show it in this figure. So, you have a line of slope omega and a line of slope minus omega.

So, these 2 lines are there, now the trajectories will what will they do. So, in this case suppose you start some value of theta and some value of omega. So, theta 1 is this difference from here. So, what you notice is that you have these 2 lines where whether are basically asymptote. For example, if both C 1 and C 2 are not equal to 0, then you will be starting somewhere let us say you start here. Then what will happen to the trajectory is that as time increases in this case what you will see is that as time increases this trajectory will go I will cross 0, and then it will go in the other direction and it will keep it will keep going away. So, theta 1 will keep increasing in the. So, in this case theta 1 is negative.

So, we started with a negative theta 1 and if theta 1 is negative then omega 1 will keep decreasing. So, it will go in this direction. So, omega is decreasing. So, omega omega keeps decreasing, theta 1 is negative now as omega keeps decreasing theta becomes less negative. So, it follows. So, theta becomes less negative it goes closer to pi, when omega goes to 0 then your theta 1 dot becomes 0. So, theta 1; so, the slope of theta 1 goes to 0. So, in other sense that when you are omega goes to 0, essentially what you are what you are having is your theta 1 just crossing this your theta 1 is essentially constant. Your theta is not changing right at this point where omega goes to 0.

So, the line becomes parallel to the omega axis and then and then it goes away. Now what is also important I did not show this correctly here. So, this will asymptotically approach these graphs. So, this graph will asymptotically approach these lines maybe I will show these lines in a different color. So, let me show them in a slightly different color just to make it clear. So, you have this line, you have this line, and it will asymptotically approach this. Now if you started with a point somewhere here then what would happen; well let us do the easier part let us do this first. So, if we start with somewhere here you will go like this.

So, and in this case you we will have it moving in the other direction. If you start somewhere here then what will happen is that in this case your omega will never go to 0. So, what we started was in important this case your theta 1 will go to 0. So, you will get something like this. So, this is a line where theta 1 equal to 0. So, it will cross this and it will go like this and again it will asymptotically go to these, and the last one is if you start here you will end up like this.

So, this is the asymptotic behavior around. So, asymptotically all the graphs will either approach e to the it will it will either go to this one or to this one because theta is greater than 0 and as t goes to infinity then it will approach this curve in one of these ways. We have a picture of what the trajectories look like around the unstable critical point and this will be true for theta equal to 3 pi. Now theta equal to 2 pi is a stable critical point. So, around this the trajectories will again look like this will look like ellipses. So, now, what you can do is you can make the overall picture. So, you have your trajectories look like ellipses and then you have these asymptotes. So, what is happening is that your trajectories are now they look like ellipses was small.

Then as you go away they will start looking like this, asymptotically they will start approaching this line, and in this direction it will look like this. Now similarly the next critical point 3 pi will also have the same feature. So, 3 pi will also have the same feature. So, that will have that will show you will have these 2 lines, and will have graphs that look like this. Now let us look what is happening here. So, suppose you take any one trajectory. So, this is where your theta is actually greater than pi and you are starting with omega that is negative. So, you are starting you are starting at a value of theta that is greater than pi and a value of omega that is negative, and what you will find is that. So, you started somewhere up here and your omega is negative; that means, you are moving in this direction.

So, then what you are imagining is that you are starting in this case with your pendulum something like this, and your omega being negative. So, omega is pushing it in this direction. What will happen is it will go past theta equal to 0. So, you are sending it with a all the way to theta equal to 0, it will cross that and then it will come this way exponentially towards this.

So, that is what is happening. So, this is the force this is the phase portrait of this simple pendulum and now there is a very special curve. So, we saw that you have the simple harmonic oscillator solutions and now if you look at the simple harmonic oscillator solutions.

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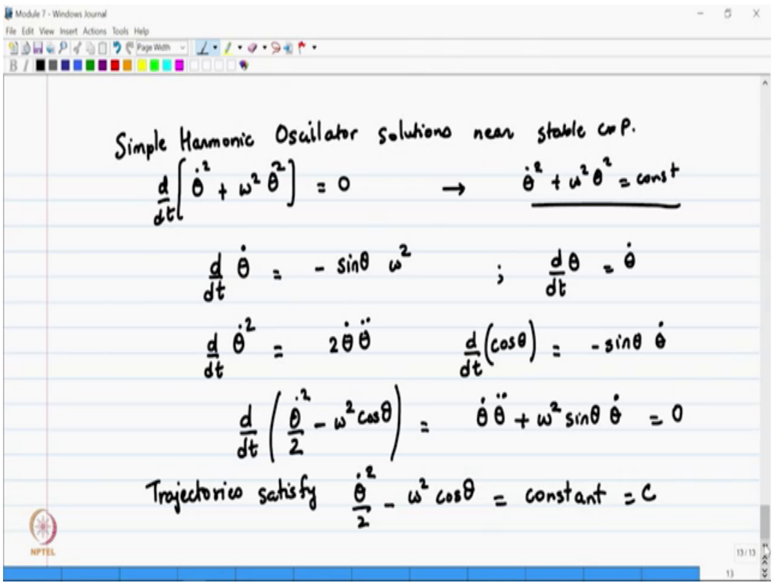

So, what is the simple harmonic oscillator solution? So, simple harmonic oscillator solutions near stable critical points, so, this has the feature that theta. So, we can write this as half theta dot square minus omega square. So, what we remember is that we had something like this, theta square was equal to 0 or d by dt of this; this was the was equal to 0 or we got this was constant. So, we got this was constant.

Now, in this case in; so, this led to theta dot square plus, plus omega square theta square equal to constant. So, this was the simple harmonic oscillator solution that is the simple pendulum solutions. Now in this case you will get a slightly different equation what you will get is. So, let us get back to our equation. So, we have d by dt of theta dot is equal to minus sin theta into omega square. So, now if I look at d by dt of theta dot square and in the other the equation was d by dt of theta equal to theta dot.

So, if I look at d by dt of theta dot square. So, this looks like 2 theta dot theta double dot and if I look at d by d by dt of let us say cos theta . So, this is minus sin theta theta dot. So, now, what you can see is that these 2 are related these 2 are related to each other. So, suppose I take d by dt of suppose I take of theta dot square by 2 plus cos plus omega square cos theta, theta or maybe yeah.

So, if I take minus omega square cos theta. So, if I look at this quantity. So, this is equal to theta dot theta double dot plus omega square sin theta, theta dot and this is equal to 0. So, the trajectories satisfy theta dot square by 2 minus omega square cos theta equal to constant equal to C. Now C can be either positive or negative. So, C can be either positive or negative and what we can get from this is that, what I can write is that I can write theta dot equal to omega is equal to 2 C plus omega square cos theta raise to half.

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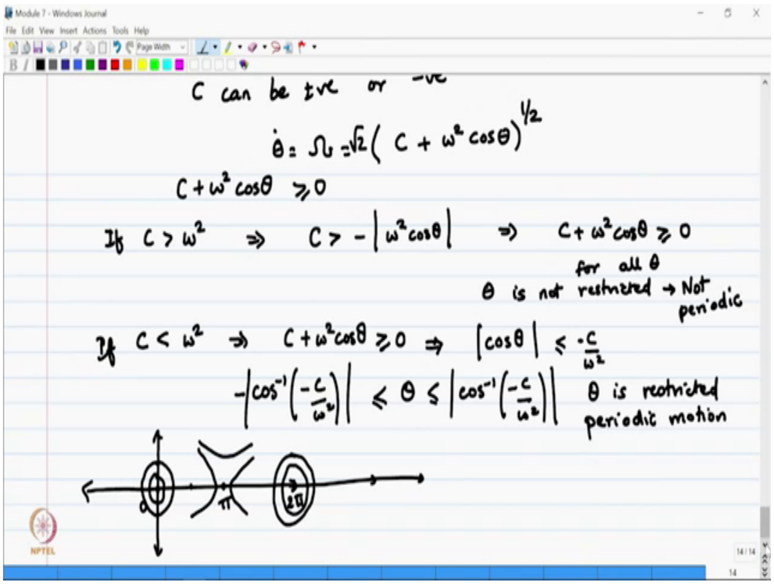

So, we have this expression for omega, now omega is supposed to be real. So, you should have the C plus omega square cos theta has to be greater than equal to 0. Now if C greater than omega square then you can have then omega square cos theta will always be implies C as will be greater than minus of absolute value of omega square cos theta;

and what it implies is that C plus omega square cos theta greater than equal to 0 for all theta.

So, theta is under is not restricted. If C is less than omega square then what you can see is that C plus omega square cos theta is greater than equal to 0 implies cos theta should be strictly such that. So, absolute value of cos theta should be less than equal to C by omega square so; that means, theta is. So, cos inverse of minus C by omega square, and you know I will just put a minus absolute value of cos inverse of this, should be less than theta should be less than equal to cos inverse minus C by omega square.

So, what is important is that your theta is restricted. So, in this case, your theta can take only certain values. So, this is actually your periodic motion. So, and this is basically this is non periodic motion this is not periodic. So, what is the implication of this; so, based on the value of C and omega square. So, what your equation will. So, if you take I will just take a few points just to illustrate. So, 0 pi.

So, the case where C and omega square are this is the rotary case, where the motion is given by rotations and 2 pi. So, and what you have is here you have these motions showed it. So, when you approach this you will end with something like this, and you have you also have things like this. Similarly at this next point you will have the same thing.

So, what will happen is that whenever you are C is less than omega square you will have rotatory motion, when C is not restricted then you will have this motion that is not periodic. Now let me show this in the next page let us just put all the all of this together.

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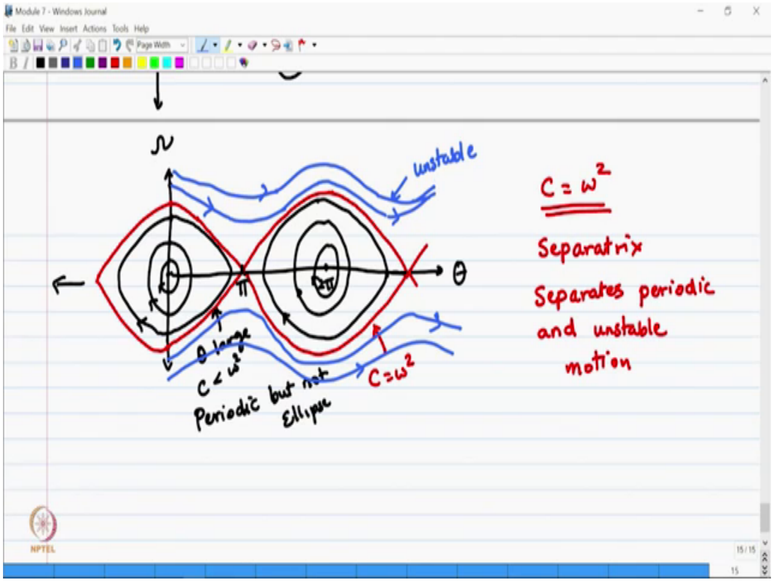

So, what the motion looks like? For certain values of C you will just have motion like this. So, let us consider around this critical point.

Now when C becomes greater than certain value, then your theta is allowed to vary theta can go to any values there is no restriction on theta. So, you have this periodic. So, you might for larger theta you might get something that is not exactly elliptical, you might get something like this. Now when this is theta large, but C is less than omega square, so, C is less than omega square. So, theta is still restricted and theta is large. So, I am showing it as not exactly an ellipse it is say this elliptical when it is small.

So, it is periodic, but not ellipse now. So, this is what it looks like and then you have pi and what you have is if C is greater than omega square then theta can take any values and you will get solutions that look like this. So, this is the next. So, let me show this in a different color. So, this is the unstable parts, this is the next critical point that is next stable critical point where you will have again the same kind of solutions you have periodic then you have things like this.

So, what is important is that if you start with something like this then you will just keep going your trajectory will just your pendulum. So, here you started with a very large amplitude and it just keeps going spinning and going like this. So, your theta can just keep your theta can go all the way to infinity it can just keep going round and round . So, now there is a boundary the boundary of these trajectories when C equal to omega square this is called a separatrix. Now when C equal to omega square then the pendulum what it will do is wherever you start off you will exactly go up to the up to pi. So, you start with some motion and theta will just go all the way to pi and then it will not go beyond pi or it is right at this point which separates these 2. These 2 stable and unstable modes this graph where C equal to omega square is called the separatrix.

So, this C equal to omega square this separates periodic and we will call this as unstable motion and unstable motion. So, what I want. So, this curve is what is called the separatrix this red curve is called the separatrix, and what I what we what I want to show in this is how you can take an equation that is completely non-linear. So, the equation that we start off with it does not look like a linear equation where we started off with a highly non-linear equation. I say highly non-linear because it contains terms of all orders of all powers in theta, and we took it and we do the stability analysis around the critical points and then just by using ideas that the curves should be smooth, we could draw the entire phase plane picture.

So, this is a very powerful way of looking at this where this problem I can I should also emphasize that you could also have solutions that look like this in this direction, you could have more other graphs also you could have more graphs also like this and so this picture is extremely powerful to get an idea of what the pendulum motion is, and we got all this ideas without actually solving the equation, but there is one important thing to keep in mind you should know the solution of the corresponding linear equation only then you can apply these methods. So, in the next class we look at some of these critical points in more detail.

Thank you.