Advanced Mathematical Methods for Chemistry

Prof. Madhav Ranganathan
Department of Chemistry
Indian Institute of Technology, Kanpur

Module – 05
Lecture – 03
Power Series Method of Solving ODEs – Power Series, Singular Points

So far we have seen second order differential equations and we have looked at homogeneous second order differential equations and non homogeneous second order differential equations. We saw how we can, if we know the solution of the homogenous second order differential equation how you can get the solution of the non homogeneous second order differential equation and then we also saw that if you know one solution of the homogeneous equation how you can find another linearly independent solution.

So what I want to do is now take the homogeneous equation and see; what is a general method to solve the homogeneous differential equation. And the method that I am going to talk, talking about in the next 2 lectures is known as the power series method. I will this is a very general method and can be applied to many equations both homogeneous and non homogeneous, but I will just take the example of the homogeneous differential equation and show how this is applied.

So, before we go into the power series method of solving different ODEs, I want to talk a little bit about power series what is meant by power series and; what are the kind of issues that we should keep in mind while doing power series?

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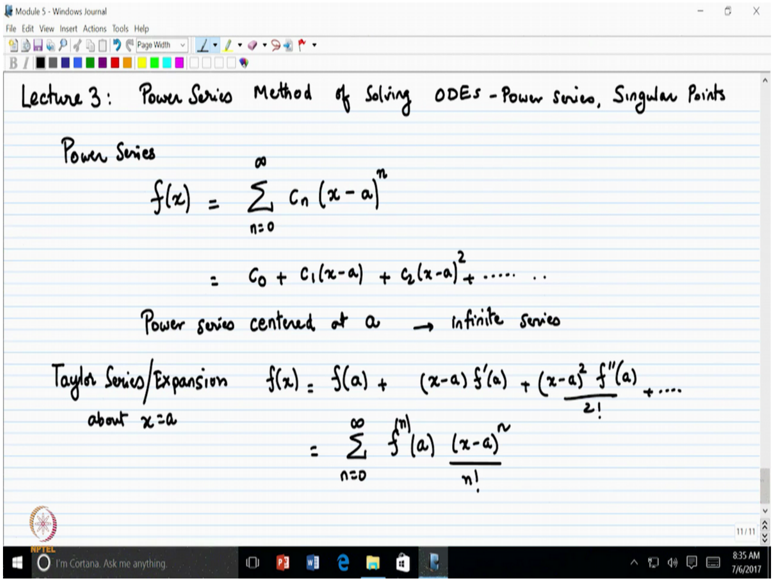

So, let us get to the topic for today, let us first discuss; what is the power series. So, suppose you have a function f of x is written as sum over infinity c n x minus a raised to n. So, this is an infinite series the sum goes from 0 to infinity and this is the I can write this as c 0 plus c 1 times x minus a plus c 2 times x minus a square plus so on.

This is a power series it is a power series because it has different powers of x minus a, each term this has the x minus a to the 0th power x minus a to the first power x minus k to the second power and so on, and this series has infinite terms. So, what we say is that it is a power series that is centered at x equal to a, or you can just say center at a, center at a and you know and this has only positive terms and it is and infinite series.

Now, you might be familiar that there is one form of power series that we use, that you might have used quite often this is called the Taylor series or Taylor expansion in which in which what is done is you take some function and you expand it in the power series. So, you write it as, if I write Taylor expansion about x equal to a then you will write this as f of a plus x minus a times f prime of a plus x minus a square times f double prime of a by 2 factorial plus and so on.

So, we write this as sum over n equal to 0 to infinity f n, n refers to the derivative. So, this is the nth derivative of a times x minus a raise to n divided by n factorial. So, the Taylor series or Taylor expansion of a function is one very popular form of the power series or it is actually it is nothing, but a power series.

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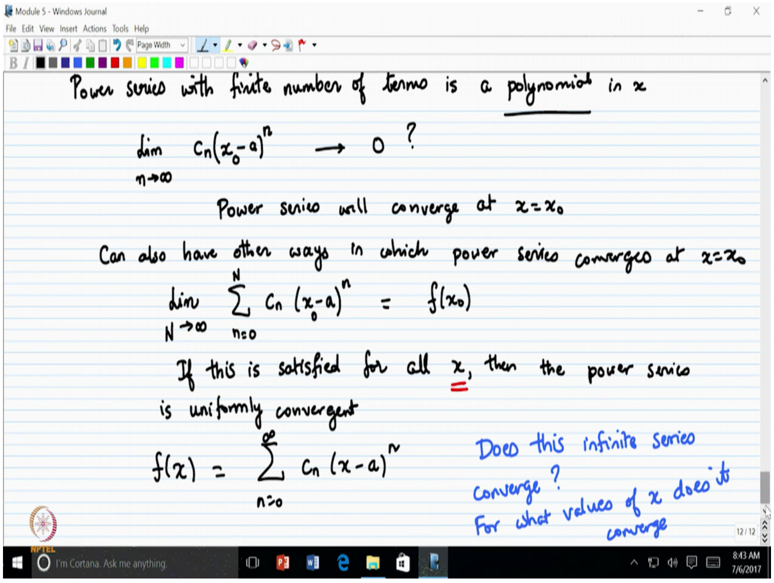

Now, suppose this power series instead of having infinite terms if it has a power series with finite terms, finite number of terms is a polynomial. So, if you have just a finite number of terms then it is nothing, but a polynomial in x.

So, if it does not go all the way to infinity you just call it a polynomial. Let us get back to this power series that we wrote and we can ask the question, as a series, so if you take any value of x, for any value of x any particular value of x you have an infinite series. So, for any given any value of x you take you will have an infinite series of terms. Now the question is as you go to higher and higher as you go as you take more and more terms does this go to some fixed number or does it go to infinity.

So, the question is limit as n tends to infinity and you can ask in particular about the coefficient n times x minus a raise to n. So, this is the term does this go to 0, if as you go to larger and larger terms this term becomes smaller and far smaller and eventually goes to 0, then you can be sure that that the power series will converge let me put an x 0 here, power series will converge at x equal to x 0. So, at some point x 0, you look at the; you look at this power series and you when you see what is happening I do these terms as you go to larger and larger terms either it could go to 0. So, this is one way in which the power series could converge or it could actually cancel some of the other terms or there could be some sort of cancellation. So, can also have other ways in which power series converges at x equal to x 0, so, basically what you want is that you want limit as n tends to infinity sum over (Refer Time: 07:49) capital N tends to infinity sum over n equal to 0 to n up to N, c n x minus a raise to n is equal to f of x 0.

So, we take the limit as n tends to infinity. So, it goes to f of x 0. So, if this for all x then the power series is uniformly convergent. So, now, what we said is that if this is satisfied for all x then the power series is said to be uniformly convergent, we actually jumped ahead a little bit because what you really want to ask is the following question. So, suppose you, suppose this I mean if you think of f of x written in the as an infinite series, you think of this as an infinite series, so c n x minus a raise to n n equal to 0 to infinity.

So, if you think of this as an infinite series now this is a function of x therefore, for any value of x you will have an infinite series and what you can ask is does it converge for that value of x. So, the important question is does this infinite series converge and if so for what values of x does it converge, x does it converge. Now an infinite series converging means that you are always tending to that function you do not go to some, you do not go to some undefined value.

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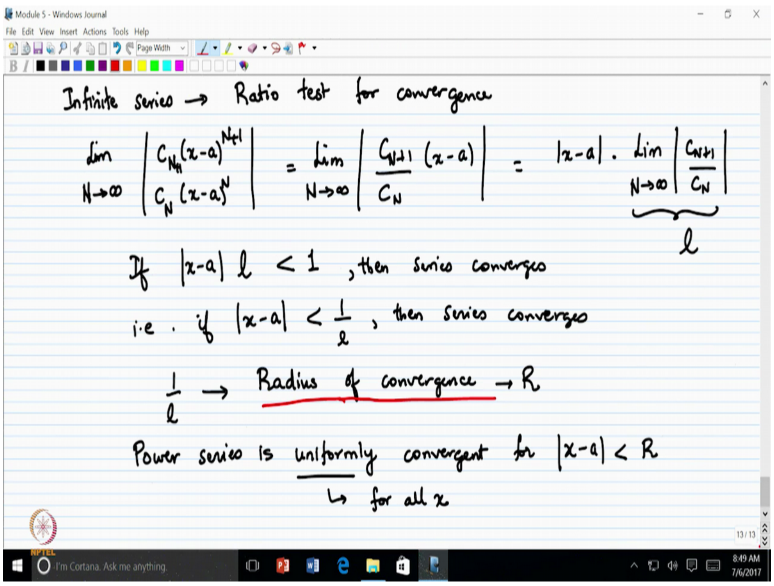

So, now there is a simple test for infinite series you can use something called the ratio test for convergence and the ratio test for convergence basically says that limit applied to this particular series. So, limits as N tends to infinity. So, if you take the N plus if you take the N plus 1th term N plus 1 term is c N x minus a raise to N plus 1 and you divide it by the Nth term c c N plus 1 c N x minus a raise to N. So, this limit absolute value of this if this limit is equal to 0 then the series will be convergent. So, let us take this limit and see what we get. So, if you take this limit you will get this will be limit as N tends to infinity c N plus 1 divided by c N times x minus a. Now, x minus a is should be a little careful with the notation this is capital N. So, this should be capital N plus 1 plus 1 capital N capital N capital N plus. What you have is I can write this as absolute value of x minus a times this limit as N tends to infinity c N plus 1 divided by c N.

So, now, what will happen is that if this ratio is less than 1, if x minus a times let me call this factor l, I will just call this l this absolute value if x minus a times l is less than 1 then series converges that is if x minus a less than 1 by l then series converges. So, now, this value of 1 by 1 is referred to as the radius of convergence. So, this radius of convergence is usually denoted by r and what is important is that the power series is uniformly convergent for x minus a less than r. So, it is uniformly convergent; that means, it converges for all x, the word uniformly is used to say for all x.

So, now what happens is that you have this radius of convergence of the series and this is a very important concept in series and this will tell you when your infinite series will actually be well defined. See if you have an infinite series and each term get keeps getting bigger and bigger and it does not tend to any particular value then the series becomes almost it is not very useful, but on the other hand if you have an infinite series where the successive terms keep getting smaller and smaller then such a series is extremely useful. So, this radius of convergence is then, is an idea that tells about the utility of this power series.

So, now, let us look at, let us look at homogeneous ODE.

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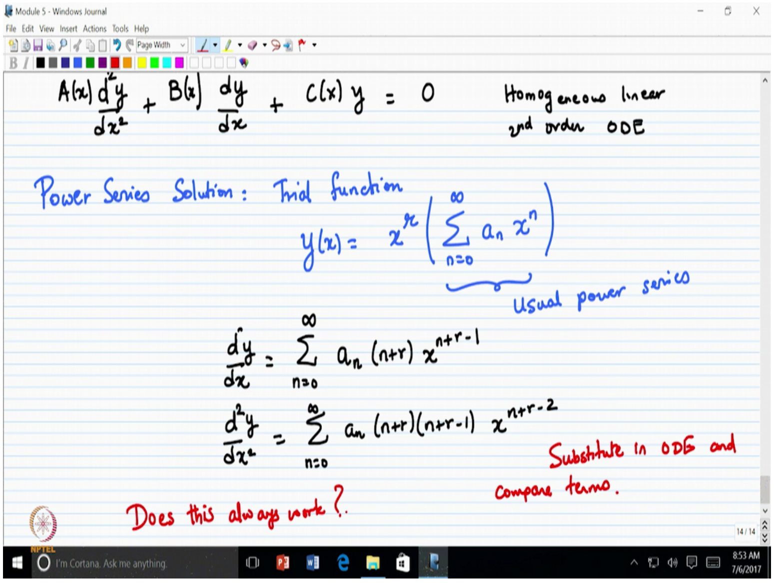

So, the homogenous ODE that we have been looking at for some time I will write it in a slightly different way. So, I will write A of x d square y by d x square plus B of x d y by d x plus C of x y equal to 0. So, this is a typical homogeneous linear second order ODE. Now what we want to do is, so the cracks of the power series method solution. So, the most important idea is that you write a trial function that has a particular form. So, you write y is equal to, so your trial function y of x is equal to x raise to r times sum over n equal to 0 to infinity a and x raise to n. So, r is some value, r is some value and a n are coefficients.

So, this is like usual power series and you have an x raise to r. So, now this is, this is the basic idea of this method, so what value of r you should take will become clear as we do this method, but the basic idea is you are going to take this form of the solution and you are going to calculate the derivative. So, once you do this then you can write d y by d x is equal to now I will take the x raise to r inside because I want to take the derivative. So, now sum over n equal to 0 to infinity a n and then you have n plus r x raise to n plus r minus 1 and what you will have is d square y by d x square is equal to sum over n equal to 0 to infinity a n, n plus r n plus r minus 1 x raise to n plus r minus 2. So, once you have all these then you substitute all these back in this equation. So, substitute in ODE and compare terms. We will see this, we will see this when we actually do the example, but this is the basic idea.

Now, the question is; when does this work. So, the first question is does this always work; now answer is no it does not always work. So, there are certain conditions for applicability.

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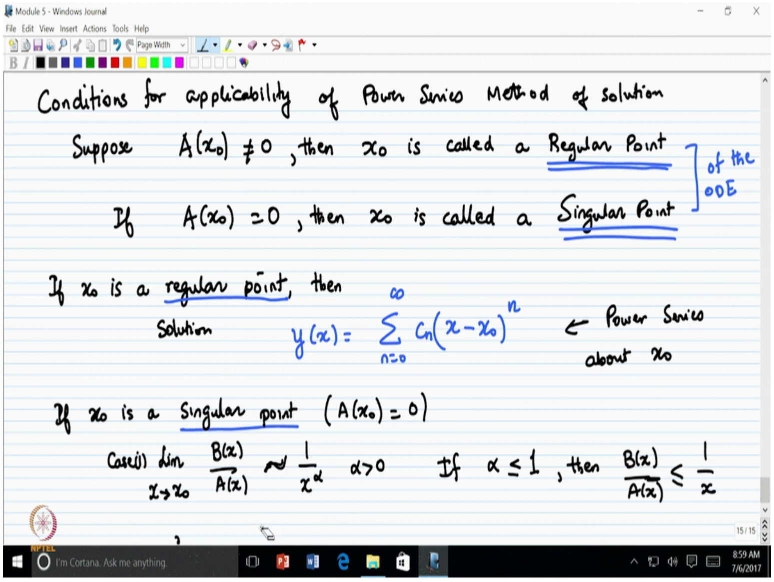

So, we will come to conditions for applicability of power series method of power series, method of solution; let us look at our series again. So, suppose A of x 0 is not equal to 0.

Suppose A of x 0 is not equal to 0 then x 0 is called a regular point, regular point. If A of x 0 equal to 0 then x 0 is called a singular point and regular and singular point is both of the ODE. So, this is a regular point of the ODE and if A of x 0 equal to 0 then x 0 is called a singular point of the ODE. So, now, if you have the regular point, if x 0 is a regular point then solution can be written in the following form. So, we can write a solution as y of x is equal to sum over n equal to 0 to infinity c n x raise to n, c n times x minus x 0 raise to n.

So, you can write it as a power series about x 0. So, this is a power series about x 0. Now if x 0 is a singular point, if it is a singular point now there are 2 cases, 2 cases. So, the first case is that limit as x tends to x 0. So, if it is a singular point then you have A of x 0 equal to 0. Now you go back to the differential equation and you take A of x and you put it and if you divide the whole equation by A of x and now you ask a question. So, B of x divided by A of x now since A of x 0 goes to 0 this limit as x tends to x 0, it is very likely that it will go to infinity. Now if it goes to infinity there are different ways in which it can go to infinity. So, typically you would expect it to go as 1 over x raised to alpha and if alpha is now if you have 1 over x raised to alpha for any positive alpha this will go to infinity, so alpha greater than 0.

Now if alpha is if this alpha that you have is less than or equal to or sorry it is ok, it is less than or equal to 1 then we say that, then you can say that B of x divided by A of x this is less than 1 over x or less than equal to 1 over x. So, in other words this ratio B of x by A of x, so limit as x tends to x 0 ok, let me write this in the next page.

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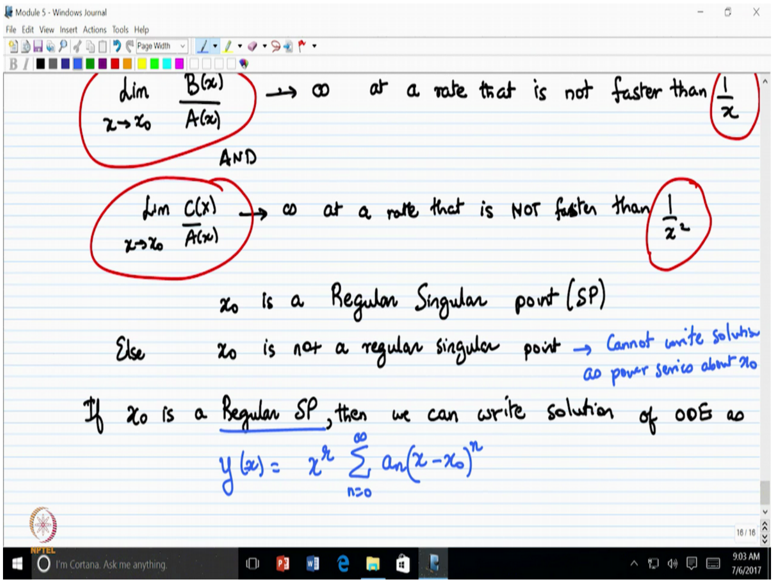

So, what you are saying is that limit as x tends to x 0, B of x divided by A of x tends to infinity at rate that is not faster than 1 by x; that means, this alpha is basically less than or equal to 1 so that means, it does not go to infinity see if alpha was equal to 2 then it would go to infinity faster than 1 by x. But if alpha is not. is less than equal to 1 then it then it goes to infinity, but at a rate that is not faster than 1 by x. So, this is 1 condition and, so I still have not completed case 1 and limit C of x by A of x as x tends to x 0 goes to infinity at rate that is not faster than 1 by x square. So, if both these conditions are satisfied. So, both these conditions have to be satisfied, so these limit with 1 by x and compare this limit with 1 by x square. So, if both these conditions are satisfied then you say that, if both these conditions are satisfied then you say that x 0 is a regular singular point, else x 0 is not a regular singular point.

So, basically you have 2 kinds of singular points you have the regular singular points and the not end, singular points that are not regular. So, if x 0 is a singular. So, for regular singular point, singular point I will just use the notation SP, then we can write solution of ODE as (Refer Time: 27:10) and y of x is equal to x raise to r sum over n equal to 0 to infinity a n x raise to n. So, notice let us just get back to this. So, if x 0 is a regular point you could just write the power series term, this is x minus x 0 raised to n. Now if it is a regular point then you did not have the x raise to r in front, but if it is not a regular point then you need this additional x raise to r.

Otherwise you cannot write the solution if it is not a regular singular point then you cannot do anything. So, then you cannot write solution as power series, solution as power series about x 0. There might be other points about which you can write it as a power series, but you cannot write it as a power series solution about x 0. So, this, let me summarize this.

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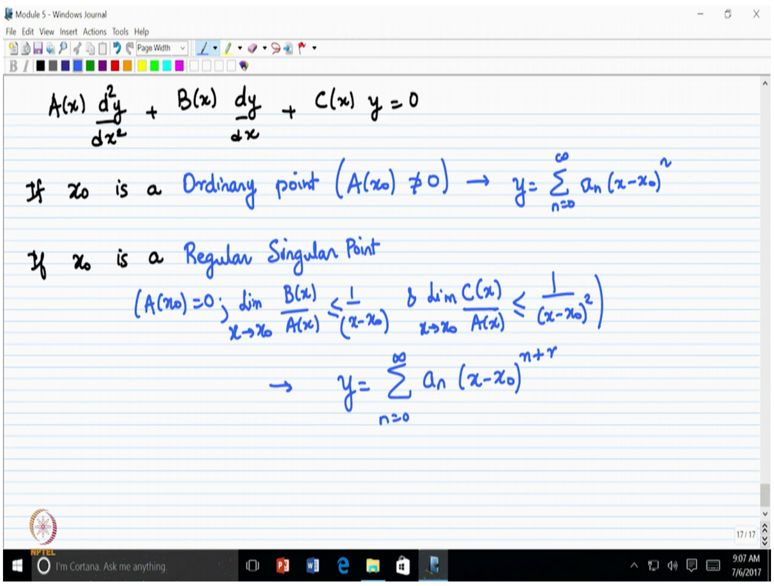

So, just to summarize if you have we had d square y by d x square times A of x plus B of x times d y by d x plus C of x times y equal to 0 and what we said is that if x 0 is an ordinary point or a regular point well I will not use the word regular. So, if it is a, if x 0 is an ordinary point that is in other words A of x 0 not equal to 0 then you write your solution y is equal to some over n equal to 0 to infinity a n x minus x 0 raise to n. And if x 0 is a among the critical points it is a regular singular point critical point can also is also referred to as a singular point and what we are referring to is here is a regular singular point that is what is the regular singular point A of x 0 equal to 0 and limit as x tends to x 0 B of x divided by A of x is basically less than 1 by x.

I am just writing in form, I am just writing it in short form 1 by x 0, less than 1 by x minus x 0 and limit as x tends to x 0 C of x by A of x is less than 1 over x minus x 0 square less than or equal to, so that is what is meant by a regular singular point. And for a regular singular point what you write is y is equal to sum over n equal to 0 to infinity you can write it in, I will write it in this form x minus x 0 raise to n plus r. So, once you have this trial solution you can substitute in the differential equation and then what you do is you compare terms and then you write the solution of the differential equation.

So, in the next 2 lectures I will be first demonstrating the operation of the method and then we look at various applications of the power series method.

Thank you.