Advanced Mathematical Methods for Chemistry

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Module – 05
Lecture – 02
Nonhomogeneous 2nd Order ODE, 2nd Order ODE with Constant Coefficients

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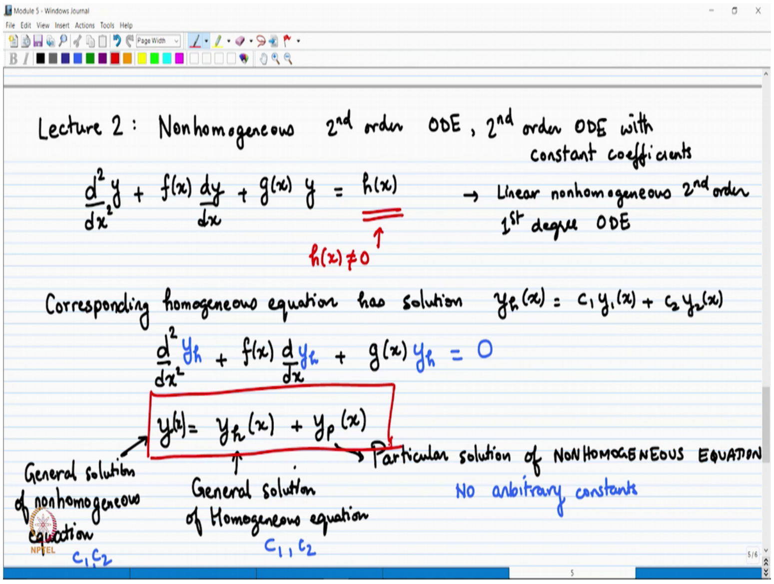

In this lecture, I am going to talk about non homogeneous 2nd order ODEs and I am going to talk about a particular type of second order ODE in which you have constant coefficients.

Now let us first start with non homogeneous 2nd order ODEs. So, what is the non homogeneous 2nd order ODEs? So, that is a differential equation that has this form f of x d y by d x plus g of x y equal to I will just say h of x. So, what you have is, so h of x is not equal to 0. So, if h of x is not equal to 0 this is a non homogeneous 2nd order ODE and I should emphasize that these are all linear non homogeneous 2nd order ODEs as second order and first degree ODE. So, we have been talking only about linear ODEs. So, now, what is important is that this term is what makes it non homogeneous. So, h of x not equal to 0 then the equation is non homogeneous.

Now, what you can do with this equation. So, suppose I, now let us write the corresponding homogeneous equation has solution y h of x as a general solution, has a has a general solution y h of x equal to c 1 y 1 of x plus c 2 y 2 of x. So, what that means, is that suppose I had d square by d x square and instead of if I had y h and then I had plus f of x d by d x of y h and I had plus g of x and y h then I would get that equal to 0. So, what I did was I had the same f of x same g of x, but instead of y I had y h, so then this gives 0. Now this is the homogeneous equation and y h is the solution of this homogenous equation. So, now given that you know y h of x can you determine the solution of the non homogeneous equation.

So, now we can determine the solution of the non homogeneous equation in the following way. So, we can write the solution of the non homogeneous equation as y is equal to y h of x plus y p of x. So, y p of x is a, so y h of x is a general solution and it has 2 arbitrary constants this is a, but this is a general solution of homogeneous equation and this is a particular, any particular solution of non homogeneous equation. So, basically if you can find out any particular solution you might find it just by trial and error just by inspection sometimes you can identify particular solutions and if you know the general solution, then you can write the solution I will write y of x y of x is the solution of this is the, then you get the general solution of non homogeneous equation.

Now, let me emphasize the difference between general and particular solutions general solution to a second order differential equation will have 2 arbitrary constants a particular solution will have no arbitrary constants. So, now, the particular solution is one that satisfies whatever boundary conditions are chosen. Now when you set up your differential equation now sometimes you just want to find a general solution of the differential equation, now if you want to find a general solution of the differential equation you can take the general solution of the homogenous equation and add to it any particular solution of the non homogeneous differential equation and you will get a general solution of the non homogeneous differential equation. So, the general solution of the non homogeneous differential equation will have the 2 arbitrary constants which actually come from the homogenous equation. So, this will have c 1 c 2, this will have no arbitrary constants and this will have the same c 1 and c 2, c 1 and c 2 that came from this general solution.

Now, this is a very powerful result because what it says is that if you have a non homogeneous equation you can solve it, you can write the solutions based on the homogenous equation, but still it is incomplete because how do we find a particular solution. Now sometimes particular solutions can be obtained just by inspection other times you need you need some standard method.

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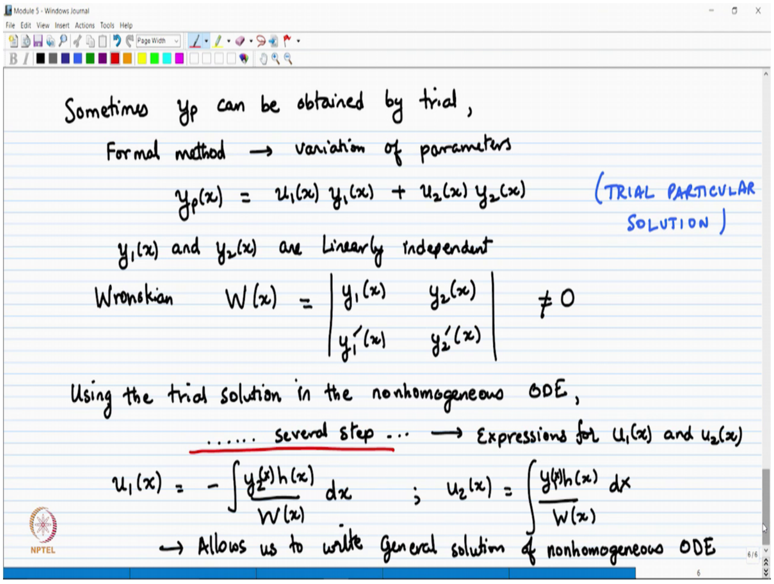

So, I will emphasize that sometimes y p can be obtained by trial, but a more formal method is variation of parameters and in this you try y p of x is equal to I will say u 1 of x, y 1 of x plus u 2 of x y 2 of x. So, you write a trial solution in this form now note that that y 1 of x and y 2 of x are linearly independent, now there is a mathematical statement that if these 2 if y 1 of x and y 2 of x are linearly independent then the there is something called the Wronskian W of x this is a function of x and this is this is defined as y 1 of x y 2 of x, y 1 prime of x, y 2 prime of x. So, if 2 functions are linearly dependents Wronskian is not equal to 0. So, the Wronskian is constructed from the function from the 2 functions and the derivative. So, it is the determinant and this determinant is not equal to 0. So, you have something called W of x. So, using this trial solution let me emphasize that this is a trial solution, trial particular solution. So, we use the try using the trial solution in the non homogeneous ODE and I will say several steps I am skipping several steps and finally, you get an expressions for you can get expressions for u 1 of x and u 2 of x.

So, again let me emphasize what we are trying to do. So, we had a general non homogeneous 2nd order ODE and we have a first order, we have the homogeneous second order ODE whose solution we knows y 1 of x and y 2 of x. So, we know y 1 of x and y 2 of x the 2 linearly independent solutions. Now the method of variation of parameters says that u multiply y 1 of x by a function u 1 of x multiply y 2 of x by another function u 2 of x and you use this as a trial solution and there are several steps we you can find these in the textbooks I am not going to discuss them in detail, but at the end of the day you want to know what should be u 1 of x and u 2 of x, if you know u 1 of x and u 2 of x then you know the particular solution. So, when you follow these steps you get expressions I am not deriving these I am just writing the expression u 1 of x is equal to. So, I will write minus integral y 2 times now remember the right hand side had h of x, h of x divided by W, W is again a function of x, d x this is u 1 of x and u 2 of x is written as y 1 times h of x divided by W of x integral d x.

So, what you can do is you can you can go through this procedure and you can actually show that you get this results, but basically this allows you to, allows us to write general solution of non homogeneous. So, what we did was we wrote a general solution of the non homogeneous ODE and this is actually not very difficult to do I mean I should emphasize y 1 of is also a function of x. So, all these are functions of x. So, what we showed is that you can write, you can write a general solution of the non homogenous ODE if you know the general solution of the homogenous ODE using this method of variation of parameters. That there are say I should emphasize that there are several steps in this derivation and, but finally, you get these expressions and once you get these expressions you can write the, you can use your, you can use this expression to get the general solution of the non homogeneous ODE.

So, with that, so there is still one problem that remains that is how do you solve the how do we get the general solution of a homogenous ODE and that we will discuss a little later. So, we will show how the power series method can be used. So, that will be discussed in the following lectures, but before that I just want to take a particular case which is something that you see very often. This is a second order homogeneous linear ODE with constant coefficients.

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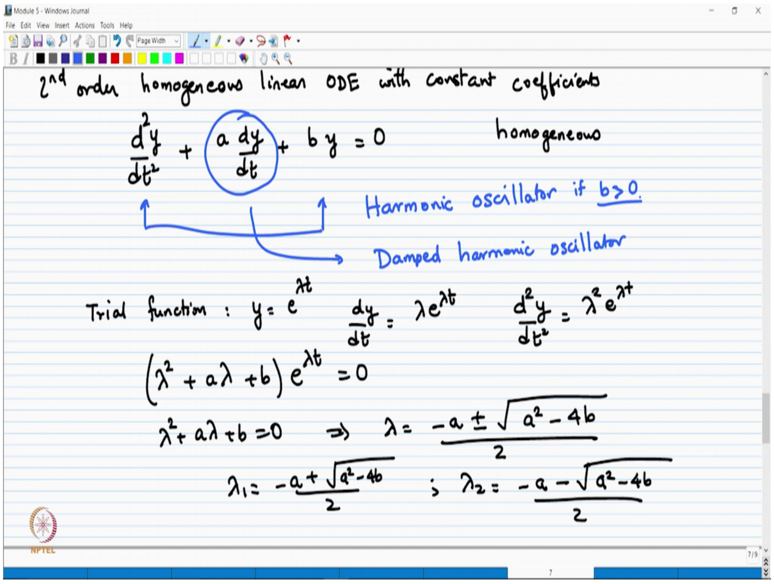

So, this is something like d 2 y by d t square plus a d y by d t plus b y. So, so first we will take the homogenous case. So, equal to 0 this is homogeneous and, so if you have something like this I should also emphasize where just appears. So, if you did not have the, if you did not have the intermediate term, if you did not have this term, then this would be a this would look like a, if you just add these 2 terms this would look like a harmonic oscillator later or it could just be yeah. So, we will just say it is a harmonic oscillator if this is there then it will be it becomes a damped harmonic oscillator and I should emphasize that all these are conditional to the values of, to the values of a and b. So, if b is greater than 0 then you have a harmonic oscillator and under certain conditions. So, we will get a damped harmonic oscillator.

So, let us solve this equation. Now since it is linear equation we will use a slightly different method to solve it. So, we will use a trial method. So, you say your try trial function y is equal to e raise to lambda t. So, then d y by d t is equal to lambda e raise to lambda t d square y by d t square is equal to lambda square e raise to lambda t. So, then when you substitute this you will get lambda square plus a lambda plus b this whole thing times e to the lambda t equal to 0 and since you want the solution to be valid for all times you get a quadratic equation in lambda. So, lambda square plus a lambda plus b equal to 0 and this implies lambda is equal to negative a plus or minus square root of a square minus 4 b divided by 2. So, you have you have, so you have lambda 1 is equal to minus a plus square root of a square minus 4 b by 2 and lambda 2 equal to minus a, minus square root of a square minus 4 b by 2.

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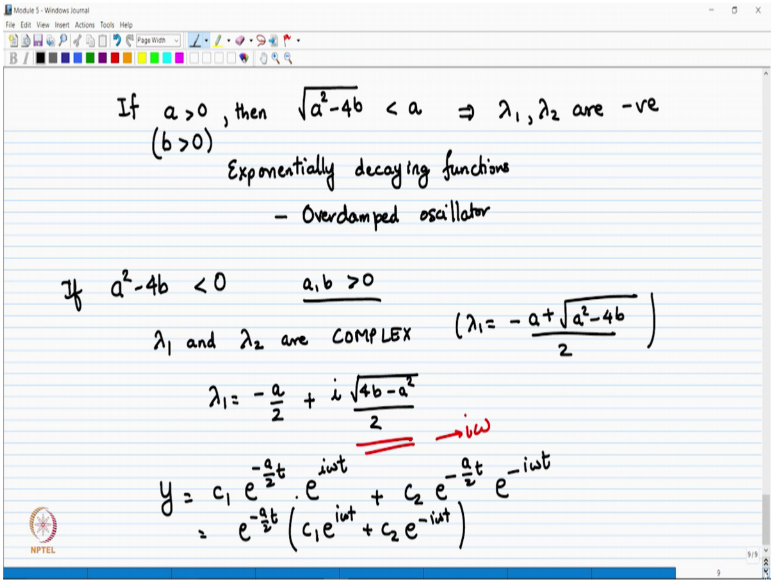

So, now there are if lambda 1 and lambda 2 are distinct then e to the lambda 1 t and e to the lambda 2 t are linearly independent and you can write the solution y is equal to c 1 e to the lambda 1 t plus c 2 e to the lambda 2 t. If lambda 1 is equal to lambda 2 that is a square equal to 4 b then we have e to the lambda t and second linearly independent solution can be obtained by variation of parameters by variation of parameters and this solution has the form t e raise to lambda t. You can show this that you will get t equal t e raise to lambda t and then you can write y is equal to c 1 e raise to lambda 1 t plus c 2 t now you have just have 1 lambda 1, lambda 1 t lambda 1 t. So, you have the same lambda 1 and these 2 other linearly independent solutions. So, this is the general solution. So, this becomes the general solution if lambda 1 and lambda 2 are distinct, this becomes a general solution if lambda 1 and lambda 2 are identical to each other. So, this is if lambda 1 not equal to lambda 2 if lambda 1 equal to lambda 2.

So, what we did as we just use this trial solution. Now in this case of course, we knew the solution. So, we knew what trial solution to take now there are certain features of this trial solution. So, let us get back to the values of the cases. So, if a square minus 4 b is greater than 0, if a square minus 4 b is greater than 0 then what you have is lambda 1 is equal to minus a plus square root of a square minus 4 b divided by 2 and you have a lambda 2 equal to minus a minus square root of a square minus 4 b by 2 and both, this implies lambda 1 and lambda 2 are real, real numbers. And in this case the solution y equal to c 1 e to the lambda 1 t plus c 2 e to the lambda 2 t is a sum of exponentials, is a sum of exponentials exponential functions these are real numbers. So, these are exponential functions they might be exponentially increasing if lambda 1 is greater than greater than 0, if your lambda is greater than 0 will be exponentially decreasing if it is less than 0 for t greater than 0 and so on, but basically you get you get them as a sum of exponentials.

Now, and let me emphasize one thing, one more point. If let us say if b is greater than 0. So, if you go to this equation harmonic oscillator is typically if b is greater than 0. So, if b is positive then you say it is a harmonic oscillator. So, we have b is greater than 0. So, if b is greater than 0 then if a square minus 4 b is also greater than 0. So, now, if a is greater than 0 then a square minus 4 b square root of a square minus 4 b is less than a is square; if a is greater than 0 and we already have b is greater than 0, b is greater than 0 then a square minus 4 b is we chose a particular case where a square minus 4 b is greater than 0. So, a square minus 4 b has to be less than a, that implies; that means, implies lambda 1 lambda 2 are negative.

So, both lambda 1 and lambda 2 are negative. So, lambda 1 has to be is minus a plus something, but this have, this positive part is this positive part, this is the positive part which is which is strictly less than this part negative part. In this case both are negative. So, it has, so lambda 2 is negative. So, both lambda 1 and lambda 2 will be negative in this case. So, that becomes an exponentially decaying functions, decaying functions this is what is referred to as an over damped oscillator. So, it is an over damped harmonic oscillator.

So, that is the case when a square minus 4 b is greater than 0. I will not particularly be discussing the case when a or b is less than 0, but now let us look at the case if a square minus 4 b is less than 0 and again we have a b greater than 0. So, in this case the solution, so lambda 1 and lambda 2 are complex because remember lambda 1 equal to minus a plus square root of a square minus 4 b divided by 2 and similarly lambda 2 is the same thing with a minus side. So, since the term under their square root is negative. So, this will contain the imaginary unit that is i. So, lambda 1, lambda 2 are complex numbers. So, what I can write? I can write lambda 1 as minus a by 2 plus i times square root of 4 b minus a square divided by 2. I can write it in this form and let me call this quant let me call this as this quantity I will call it as omega, I will just call this quantity as omega, so i omega. So, this becomes i omega.

So, now your solution has this form y is equal to c 1 times e to the now you have you have minus a by 2 t times e to the i omega t plus c 2 times e to the minus a by 2 t, e to the minus I omega t. So, I can write it in this form and what I get is e to the minus a by 2 t times c 1 e to the i omega t plus c 2 e to the minus i omega t.

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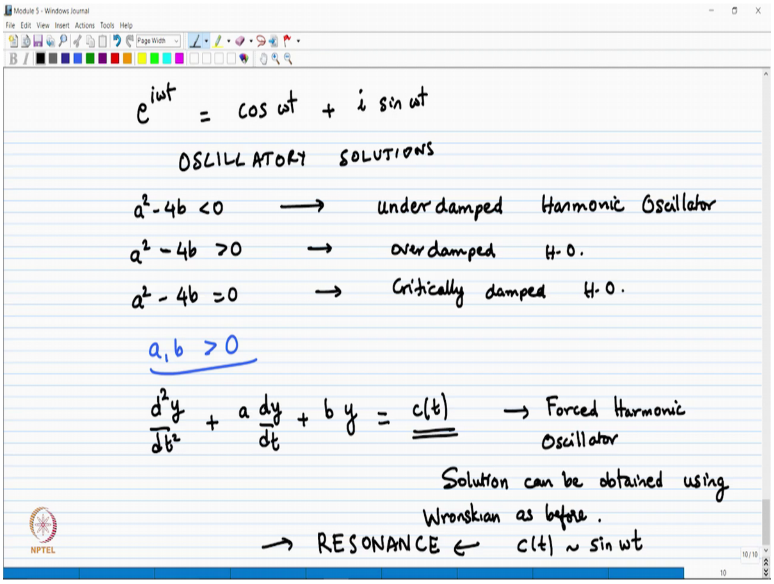

And you should recall from your basics of complex numbers that exponential of an imaginary e to the i omega t is nothing, but cos, cosine of omega t plus i times sin of omega t. So, this is oscillatory solutions and this case where a square minus 4 b less than 0 is referred to as under damped, harmonic oscillator and what we said was a square minus 4 b is greater than 0 is called an over damped harmonic oscillator, a square minus 4 b equal to 0 is called critically damped harmonic. So, these are these are the three cases. Again I emphasize that we took the case a comma b or both greater than 0 if a or b as less than 0 then denote the behavior is slightly different you do not use the term harmonic oscillator nevertheless you can still use this method to solve it, you can still use this method to solve this even if a and b are not greater than 0.

So, I will stop the discussion on second order homogeneous equations here. I just want to emphasize 1 small thing about what happens if you had a non homogeneous equation in this of this form. So, suppose you had exactly this kind of equation with constant coefficients, but you had a non homogeneous case. So, suppose you had d square y by d t square plus a d y by d t plus b y and this was equal to some constant, some c f t, some function of t. So, this is referred to as a forced harmonic oscillator and this, so based on the solution can be obtained using Wronskian as before. So, like we did we can calculate we can write the general solution in terms of homogenous homogeneous solution and the particular solution the particular solution will have these 2 u 1 and u 2 which are determined using this method of Wronskian.

So, when you get that, you can get, so the solution; obviously, depends on what your c is because if you notice your u had the right hand side h of x which was the right hand side of the homogeneous equation. So, this was there in the right hand side of the non homogeneous equations. So, this was there in the expression for u. So, basically it will depend on what your c is and basically you can look at various solutions there is one important case which is often described this is resonance. So, this happens when c of t goes as either sin or cosine of omega t where this omega is the same as the omega that is there in the solution. So, that is the case you can explore I am not explicitly solving the forced harmonic oscillator, but this is again it is a standard procedure that you can easily do once you know how to solve the homogeneous case.

So, I will conclude the discussions, I will conclude this lecture here. So, in the next lecture we will start doing the power series method which is a method for obtaining the general solution of a homogenous ODE.

Thank you.