Advanced Mathematical Methods for Chemistry
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Module - 04
Lecture - 02
1st Order ODEs, Exact Differentials, Integrating Factors

In the last lecture we learnt about the different kinds of differential equations, like linear, non-linear, homogeneous, non homogeneous etcetera and we learnt about what kinds of solutions we can have. Today's lecture I will talk about first order ordinary differential equations and in particular first order and first degree differential equations. So, just to remind yourself let us consider an independent variable x and the dependent variable y.

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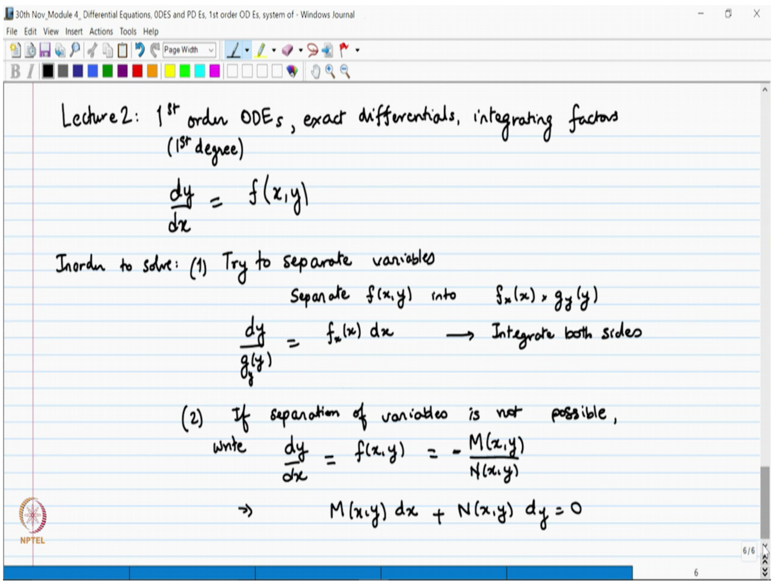

So, then the first order differential equation, and first order and first degree equation can be written in this form dy by dx is some function of x y.

Now, if you want to solve this equation then what you have to do is you have to in order to solve this. So, the first thing you try is try to separate variables. So, what I mean by separating variables is you break the right hand side, you break f of x into terms that contain only x and terms that contain only y, you take the terms that contain only y to the left hand side and take dx to the right hand side and then you integrate both sides. So, separate f of x y into let f x of x into g y of y. If you have this kind of separation of variables then you can write dy by g of y is equal to you can just call it g y of y is equal to fx of x dx and then you can integrate both sides. So, this is the first thing that you try to do.

Next thing that you try to do now sometimes it is not possible to separate the variables. So, if it is not possible to separate the variables, if separation of variables is not possible then what you do is you write dy by dx is equal to. So, what you have is f of x y. So, you write that as M of x y divided by N of x y and let me just put a minus sign, I will just put a minus sign. So, I just write it in this form, you can always write f of x y in this form. And then what you do is multiply out and you write you multiply it out and you write it in this form. So, M of x y dx plus N of x y dy equal to 0 the first order first degree equation we wrote in this form we wrote it as M of x y dx plus N of x y dy equal to 0.

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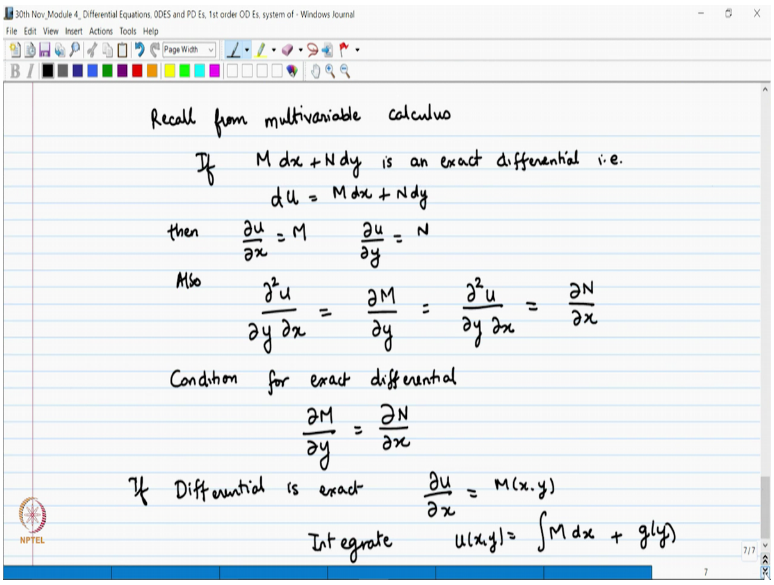

Now, what is the advantage of writing it in this form? So, the advantage is very clear because what this looks like is this looks like recall from multivariable calculus if M dx plus N dy, I am not I am not writing the expressive dependence on x and y. If M dx plus N dy is an exact differential that is d of u is equal to M dx plus N dy, then dou u by dou x and u is a function of x and y. So, I have a partial derivative of u with respect to x is equal to M and dou u by dou y equal to N and also now if you take dou square u by dou x dou y. So, you have dou square u by dou x dou y is equal to dou M by dou y, no should

I write it in the other order for first you take with respect to x then you take with respect y. Now, this should be equal to dou square u by dou y dou x is equal to dou N by dou x.

So, therefore, we have the condition exact differential dou M by dou y is equal to dou N by dou x. So, this is the condition for this differential to be exact. So, then what you do now you go back. So, suppose you are not able to separate the variables then you try to look you see, if you can write this as an exact differential you see if this condition is satisfied. So, you check for exact differential. If differential is exact, so exact then you can use each of these conditions. So, then you can use dou u by dou x equal to M of x y then you integrate and what you will get is u of x y is equal to integral M and dx plus some function of y plus I will just call it g of y, this is.

So, now when you integrate a partial differential equation then instead of getting a constant of integration you can have any function of y because if I take a partial derivative with respect to x, the partial derivative with respect to x of a function that is only a function of y will go to 0. So, you will get something like this and then the next step is to is to now you use the second condition dou u by dou y equal to N is equal to integral dou M by dou y dx plus dou g by dou y and what you can do is you can just solve for g. So, solve for g.

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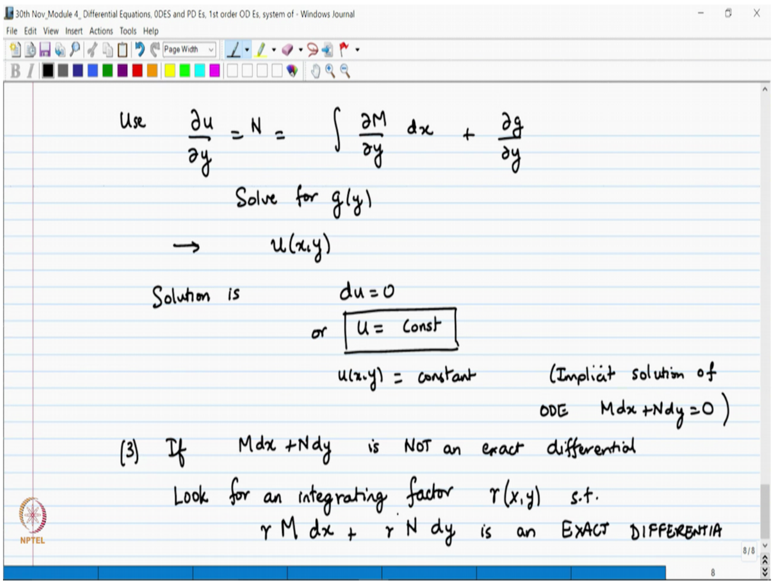

So, then that will give you, that will give you u of x y and what do you do with u of x y. So, our differential equation solution is du equal to 0 or u equal to constant. So, the solution, once you solve for u of x y your solution is u equal to constant. So, that is the, this is the procedure that you use when you can write this as an exact differential. So, what we have seen is that you can take a first order first degree equation, you first try to separate variables. If you are able to separate the variables then you can easily integrate both sides and you can get the solution, if you are not able to separate variables then you write the equation and see whether it is an exact differential. If it is an exact differential then we saw that you can easily we can easily integrate it out and we can solve for this exact differential and you can solve the differential equation.

So, I should emphasize that the solution, solution is an implicit solution u of x y equal to constant this is the implicit solution of differential equation, this is the implicit solution of ODE M dx plus N dy equal to 0. So, our ODE our original ODE was M dx plus N dy equal to 0 the implicit solution is u of x y equal to constant. Now what happens if the third case if M dx plus N dy is not an exact differential then you can still, you can still go ahead and solve it. So, there are, if this is not an exact differential then the look for an integrating factor I will just call the integrating factor r, r and in general it is a function of x and y such that r times M dx plus r times N dy is an exact differential that is what we will do.

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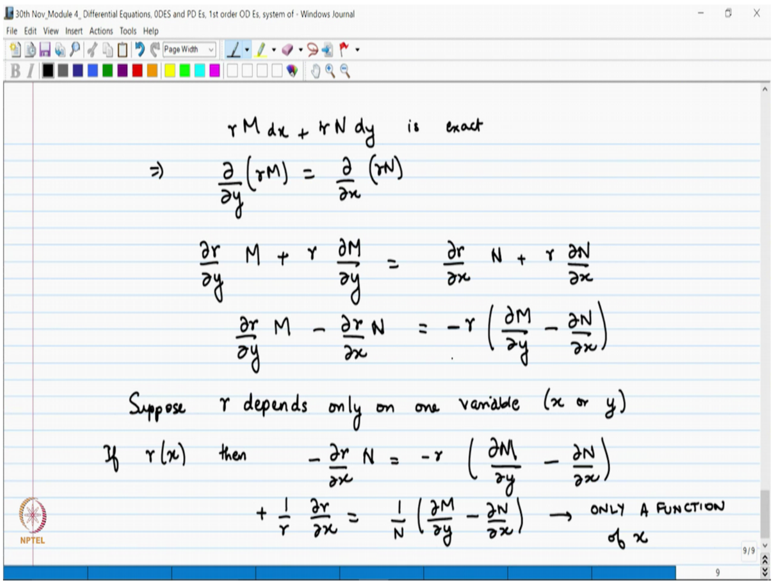

So, what we are going to say is that r times dx plus r N dy is exact. So, this implies that dou by dou y of r M is equal to dou by dou x of r N. In other words, if I expand this out what I will get is dou r by dou y into M plus r times dou M by dou y should be equal to dou r by dou x into N plus r times dou N by dou x. We will just rewrite this in a slightly different form and then we will look for the solution. So, suppose I write this as let me take terms with derivative of r s to the left. So, if I write dou r by dou y into M minus dou r by dou x into N is equal to r times or let me write a minus r times dou M by dou y minus dou N by dou x. So, I have written it in this form and now suppose, so this is a way to look for this integrating factor r, now suppose r depends only on 1 variable x or y. So, suppose r depends only on 1 variable x or y.

So, for example, if r depends only on x then dou r by dou y will be 0. So, the first, this term will go to 0. So, example if r of r, r is equal to r of x if r of x. So, r depends only on x then we have minus dou r by dou x into N is equal to minus r dou M by dou y minus dou N by dou x and what you can do is you can just r is a function only of x. So, let us take it to the left. So, what I can write is, I can write this as plus 1 by r dou r by dou x is equal to 1 by N dou M by dou y minus dou N by dou x. So, this is a very convenient form to write.

So, remember we said that this quantity in the parenthesis dou M by dou y minus dou N by dou x, this is equal to 0 if M dx plus N dy is an exact differential, but since M dx plus N dy is not an exact differential so this is not equal to 0, now this will be some function of x and y. So, now, if I divide it by N which is also some function of x and y then that should be equal to this quantity. So, if that is a function only of x, this is only a function of x because the left hand side is only a function of x; that means, what happened is the following that. So, the procedure is you calculate you calculate this quantity and see if it is a function only of x, if it is a function of x then you know that you have an integrating factor that depends only on x.

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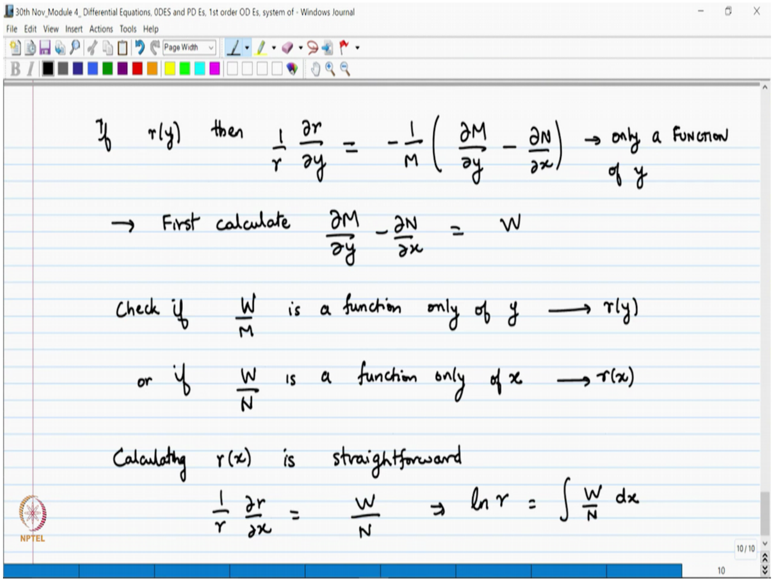

So, now, if on the other hand r depends only on y then we have dou r by dou y and I will just go ahead with the same algebra. So, 1 over r dou y by dou y is equal to minus 1 over M dou M by dou y minus dou N by dou x and this is only a function of y. Rather I should say a function only of y. So, this is a function only of y this is a function only of x.

Now, so the procedure is the following. So, what you do is first calculate dou M by dou y minus dou N by dou x. So, that is the first thing that you will do, then check if 1 over M and let me call this, I will call this W, just as a short notation just check if W by M is a function only of y or if W by N is a function only of x.

Now, if the first case is true then you find integrating factor r of y, in this case you find r of x. So, this is the procedure and let me just show you that you know doing this integral is also very straightforward. So, let us take examples. So, calculating r of x is straightforward let me manual, let me emphasize that because our equation is 1 over r dou r by dou x is equal to W by n. So, this implies, so if I integrate this I will just get natural log of r is equal to integral W by N dx. So, natural log of r is equal to integral W of N dx and you can put a constant factor, but that is not really necessary.

So, by this method you can calculate the integrating factor. Similarly you can calculate the integrating factor even if r is a function only of y.

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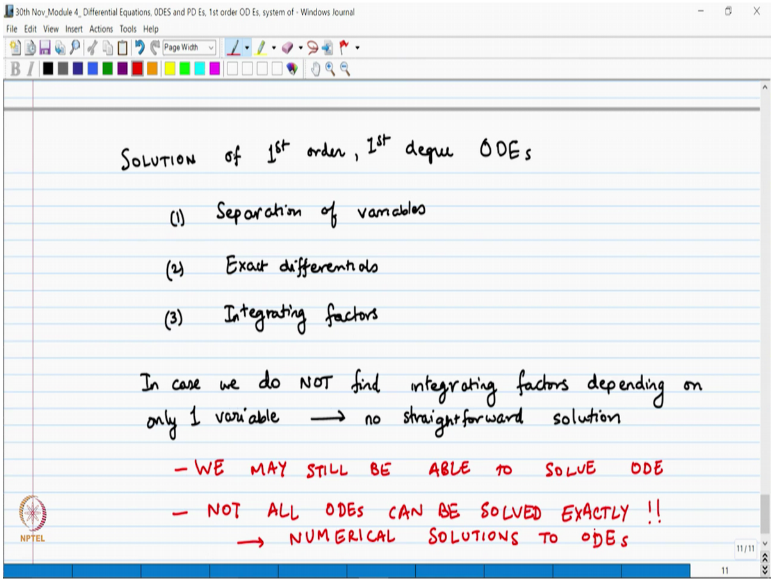

Just to conclude so the solution of first order first degree ODEs. So, first step is a separation of variables. So, that is the first thing you should try then the second step is you look for exact differentials and if that does not work out look for integrating factors.

So, this is the general method, now in case you do not get an integrating factor, in case both these are both these conditions are not satisfied then there is no way to solve it, then there is no simple way to solve it. So, in case we do not find integrating factors depending on only 1 variable no straightforward solution. So, I am saying that there is no straightforward solution, but there are 2 things to emphasize here. So, the first thing is, is that we may still be able to solve it, solve ODE by some tricks by some tricks which are not, which might be which might work for that particular equation.

So, just because you do not get an integrating factor that does not depend on your x or y does not mean you cannot solve it, you may still be able to solve it using some tricks, but there is no straightforward way to, there is no standard way to solve it.

The next point that I want to emphasize is that not all ODEs can be solved exactly. So, exactly means you cannot write an analytical solution of to that ODE so. In fact, this idea that you know you cannot solve all ODEs exactly is what gives rise to the field of numerical solutions of ODEs and I will not be doing this in this course, but that is a very very important part of modern applications of mathematics that you know you know we know that not all equations can be solved exactly. So, we look for numerical solutions.

So, in today's lecture I have tried to explain to you the different ways in which you can solve first order first degree ODEs. In the next lecture we will go to system of first order odes and then we will go to second order ODEs.

Thank you.