Implementation Aspects of Quantum Computing
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Lecture – 38
Measurements: Single vs Ensemble Averaged

We have been discussing in this week an overall summary of whatever we have read throughout this course as this is our final week. We have just discussed the basics from the classical concept of computing to how we get to the point of quantum computing. Let us now continue in that direction and see what we mean when we have entered the concept of qubits.

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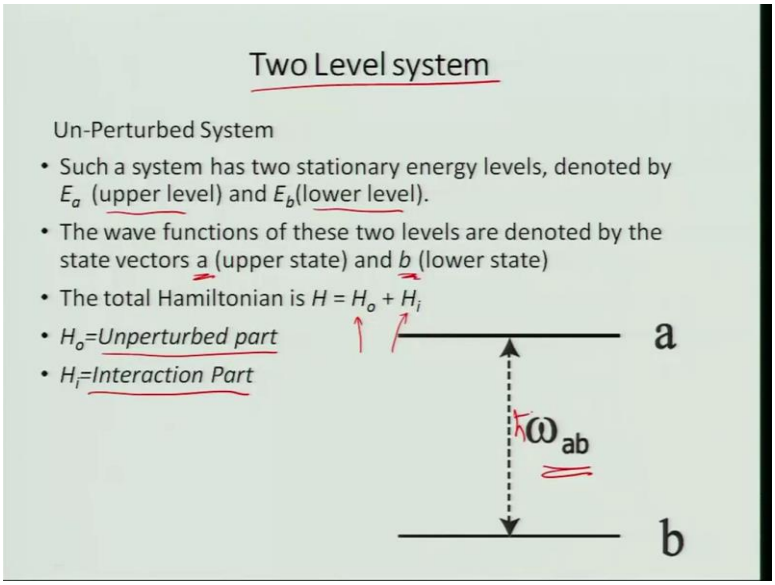

So, typically qubits as we have been discussing are two level quantum systems. So, for an unperturbed quantum system, there are 2 stationary energy levels denoted by say the x upper level E a and the lower level E b. The wave functions for these 2 levels are denoted by state vectors a upper level and b lower level. The total Hamiltonian is H which is the 0 th order one and H i which is the interaction one. So, H 0 is the unperturbed part and H i is interaction part. So, as far as a steady state goes this is how the system looks like there is an energy gap, which corresponds to omega a b in terms of energy, it would be h cross omega ab.

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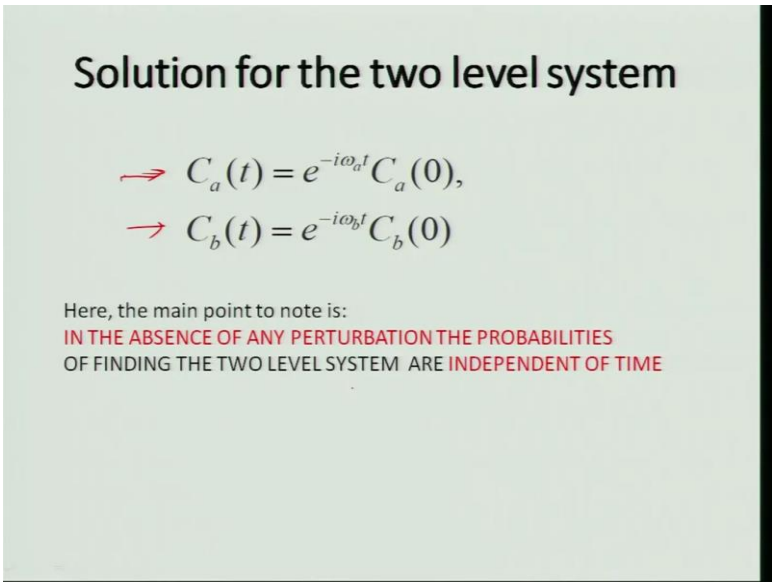

The solution for the two level system would essentially mean that we time the coefficient would evolve as has been shown here for these two cases. The main point to note here is that in the absence of any perturbation the probabilities of finding the two level system are independent of time.

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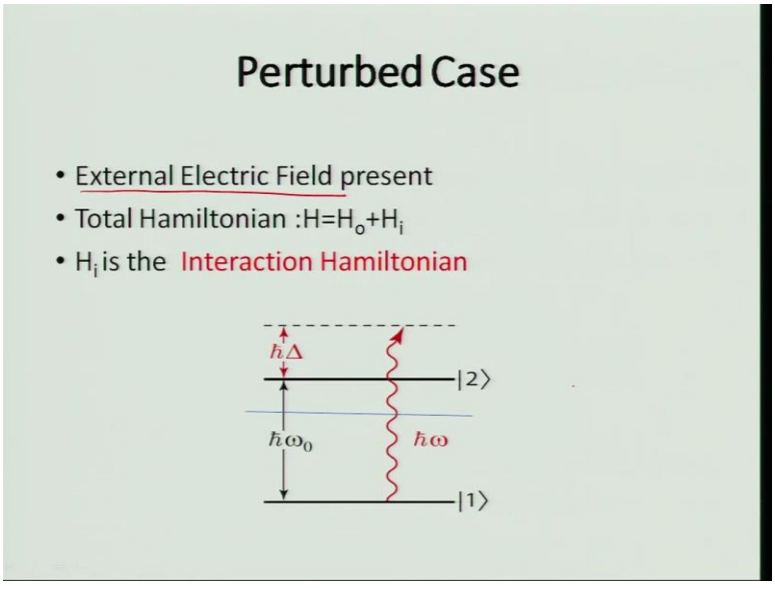

However, when we perturbed system say for example, by providing an external electric field, the total Hamiltonian in this case the interaction Hamiltonian is the one which is going to influence how this evolves. And now we represent this a b is by say 1 and 2,

then this is how the interaction picture looks like the system could either H i is the interaction Hamiltonian and the system would evolve as a result of the external applied electric field.

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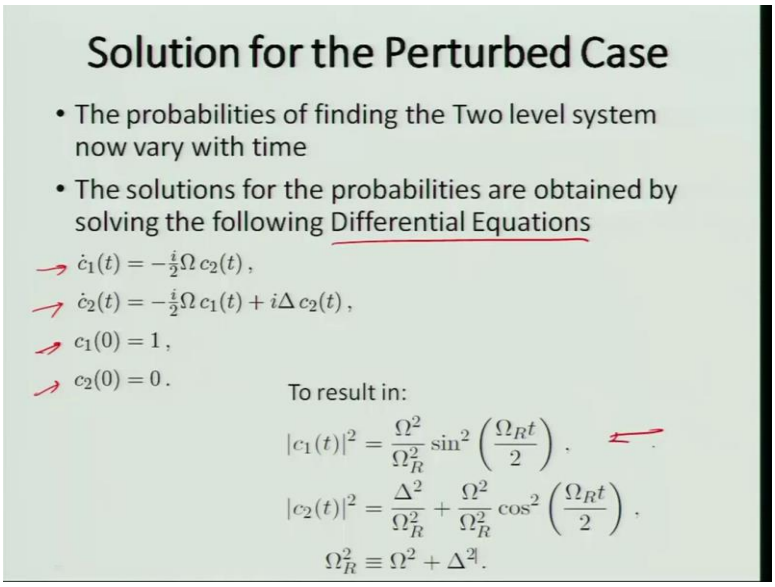

The probabilities of finding the two level systems now vary with time. The solutions for the probabilities are obtained by solving the differential equations which are based on how the amplitudes or the probabilities of each of these components of the states change and we get the time dependent result which is shown here.

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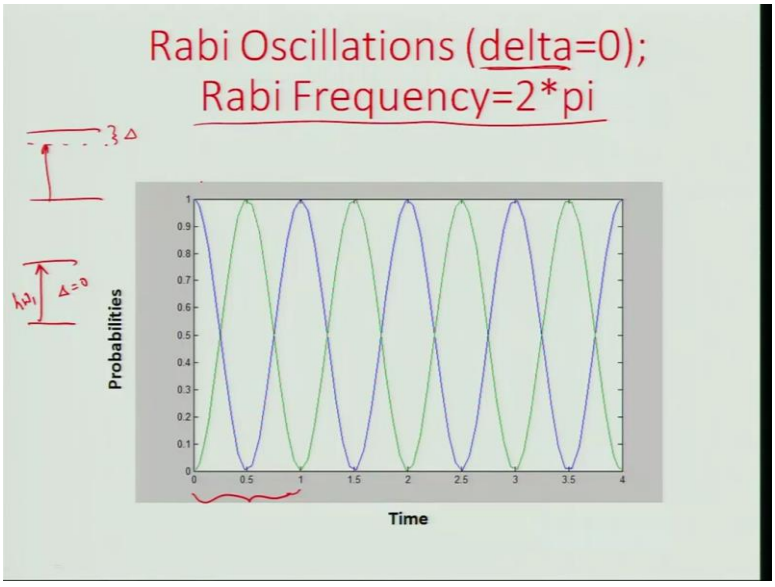

So, this result in the Rabi oscillations, this delta is the detuning or the difference in the energy from the excitation to the actual state. If it is resonant, which is basically the case where delta is equal to 0 then this is the case where it is exactly going to have an excitation which will correspond to the gap.

So, we get the population oscillate from the ground to the excited and back and forth and this is known as the Rabi flopping with a frequency which is known as Rabi frequency which is equivalent to 2 pi. So, at every 2 pi the population is back to its ground state and it goes on cycling, and that is true for both the ground and the exiled state. The total population always remains conserved which is one.

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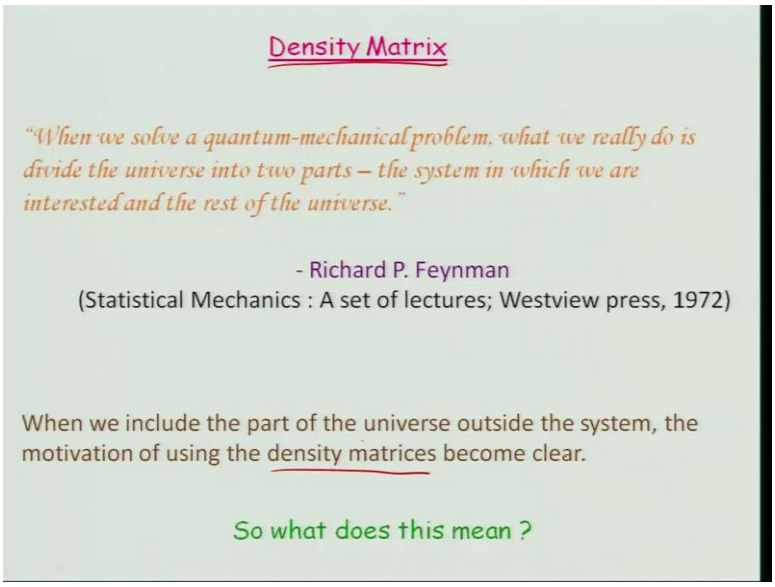

So, this particular picture that we just discussed is true when we are talking about A single quantum state. In principle we have an ensemble of states and that is when we need to invoke something known as a density matrix. So, a density matrix includes the statistics of the problem and as pointed out by Richard Feynman in statistical mechanics book; when we solve a quantum mechanical problem what we really do is divide the universe into two parts, the system in which we are interested and the rest of the universe. When we include the part of the universe outside the system, the motivation of using the density matrices become clear; so what does this mean?

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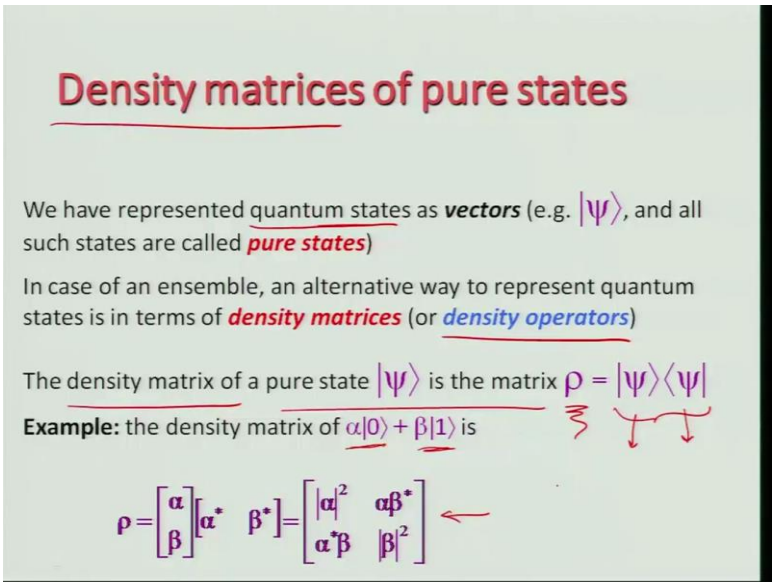

It means that when we are talking about a pure state as for example, the quantum state that we just talked about the vectors or the states that we discussed in terms of the states that we looked at. In case of an ensemble, a collection of these states would be representing the density matrix, often also known as density operators. So, as per definition we have discussed this earlier, the density matrix of a pure state is the matrix where rho which is the outer product of the 2 vectors: psi ket and psi bra. So, this is the representation we have used and we know that they basically represent the matrices which are row and column and therefore, they can form the matrix that we are looking for.

So, for example, the density matrix of alpha 0 and beta 1 is given by this form where we get the 2 by 2 matrix for this qubit which could be in any of the basis states. So, that is the importance of the density matrix, which includes all the possible conditions of the ensemble.

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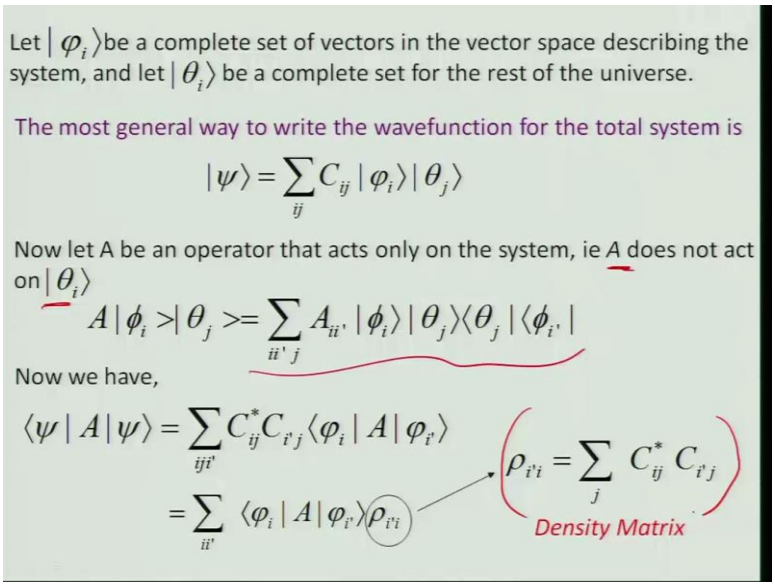

If we represent psi i ket to be the complete set of vectors in the vector space describing the system and phi i ket to be the complete set of rest of the universe. The most general way to write the wave function for the total system is to have the wave function written in this form. Now if A is an operator that acts only on the system, that is A does not act on the rest of the universe theta i, then we can write this out in this form and these we have studied earlier and this can be simplified to get to the part which is essentially give raise to the density matrix. So, this was the motive for defining the density matrix as we had done.

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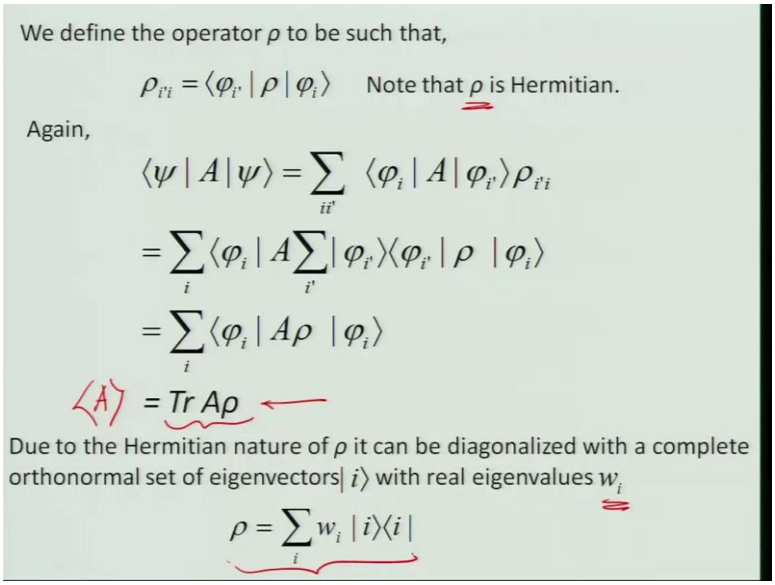

So, this operator rho is such that this follows this principle and that rho is going to be Hermitian. And one of the most important property of the density matrix that we have learned is the fact that the expectation value of the operator is essentially given by the trace of the product of the operator with the density matrix.

Due to the Hermitian nature of the density matrix, it can be diagnosed with a complete orthonormal set of eigen functions i with real eigen values W i and so we can have a final form which looks like this.

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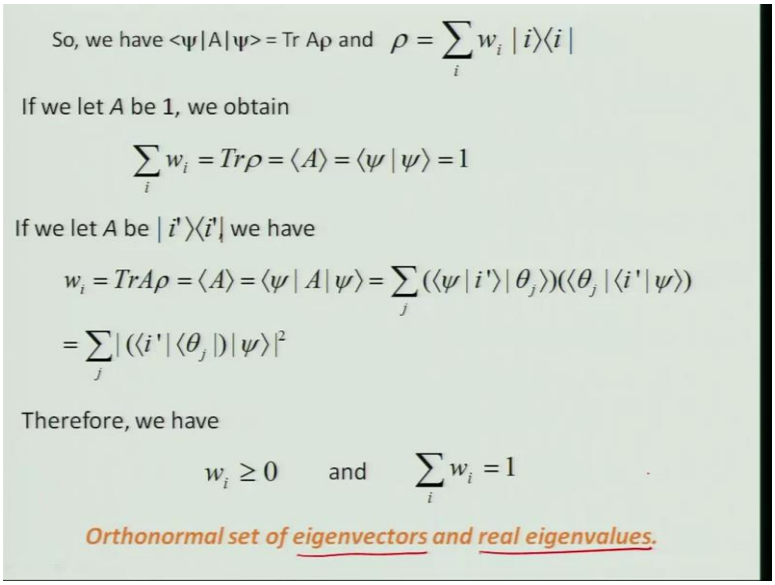

And we have gone ahead to show that the orthonormal set of eigenvectors and eigenvalues are the ones which give raise to density matrix that we are interested in. So, that is the power of the density matrix, which incorporates the elements along the diagonal which gives the probability of presence of the individual state and this can be explored further here.

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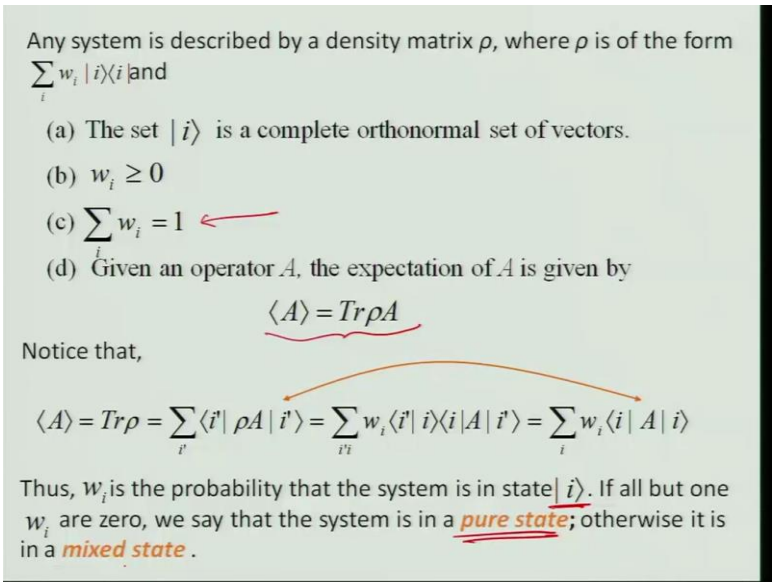

Any system described by density matrix rho, which has this form would have these properties. One that the set i is complete orthonormal set of vectors, the diagonal elements would be greater than equal to 0, essentially non negative elements, the sum of the diagonals equal to 1, given an operator A the expectation value is going to be this; trace rho A. This essentially ensures that the total probability of the system is going to be 1.

So, W i is the probability that the system is in state i as I was trying to say; if all, but one of W i is 0, we say that the system is in a pure state; so W i is like the weightage or the contribution of a particular state to the density matrix. So, if it is only one state then it is a pure state, otherwise it is a mixed state; so as the most important part about the density matrix.

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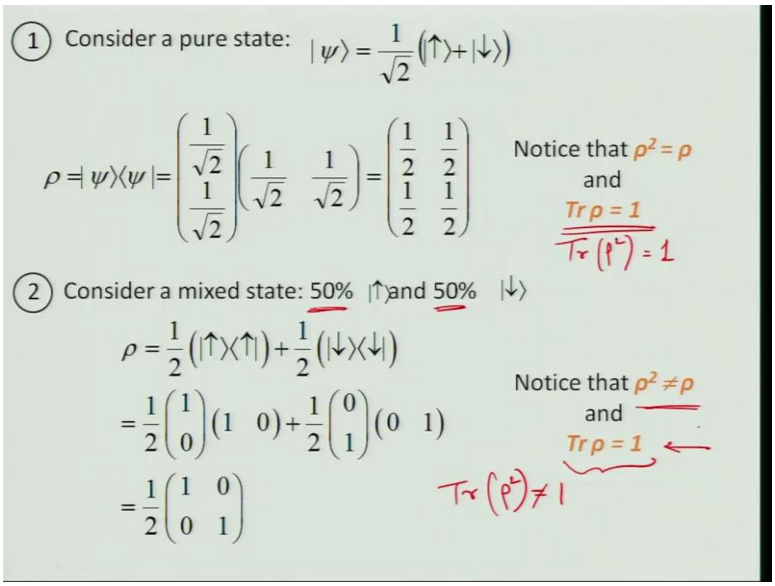

So, when we have a pure state we also have found out that the trace has discussed before is anyway going to be equal to 1, but it is also true that their square of the matrix is also equal to the matrix itself. So, the square of the trace of the matrix is also equal to 1; however, when it is a mixed state for instance 50 percent of up and 50 percent of down, then though the trace is going to be one trace of the matrix because that has to happen because that ensures the total probability of the system is 1 which is the requirement, the trace of rho square is not equal to 1 and rho square is also not equal to rho. And so that is the most important part of mixed state.

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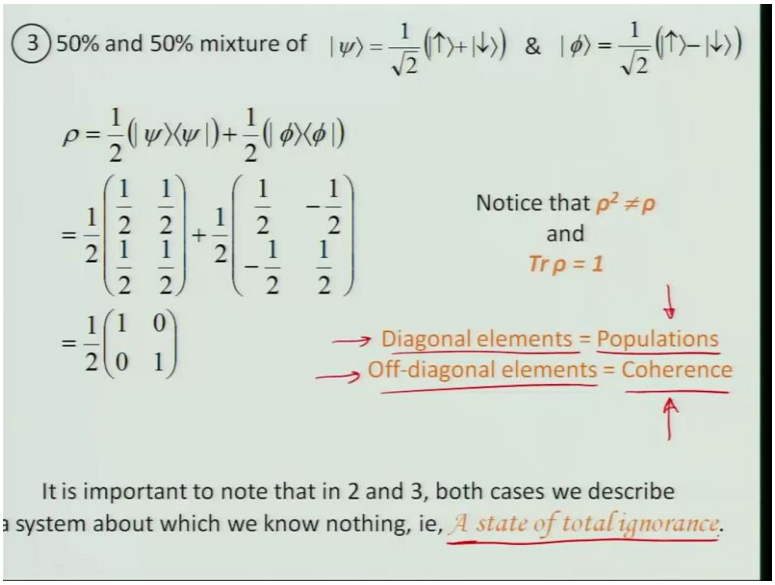

So, one of the most important part of density matrix that we have discussed is the fact that the diagonal elements represent the populations whereas, the off diagonal elements represent coherence and therefore so which is the very important part of understanding quantum mechanics in an ensemble, because most of the issues that we have dealt with in terms of quantum information relies a lot on coherence, so density matrices have the critical information of coherence also present in its elements.

So, while the populations are extremely important, the coherence which is one of the critical components of this information processing quantum information processing, is also embedded in the density matrix formalities and when we have mixed states it gets difficult to get the information on the individual states, which could lead to a condition when they are completely mixed to a case when there is a state of total ignorance unless and until it is measured.

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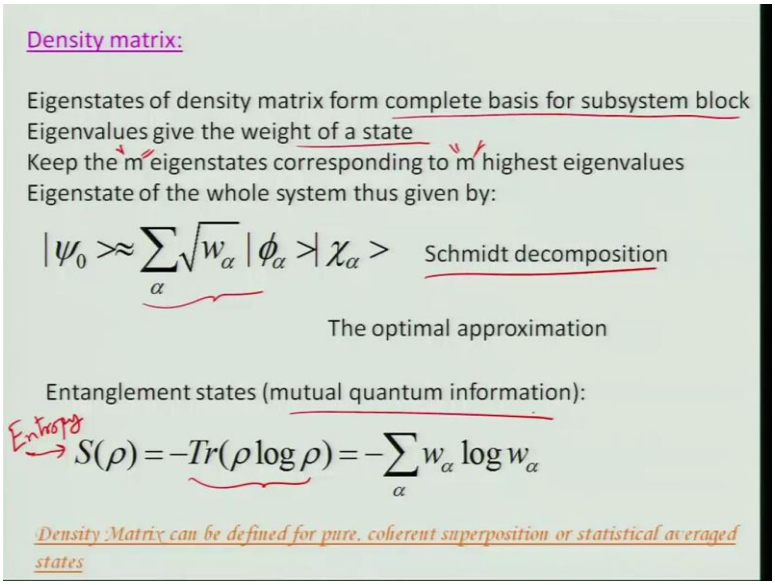

The other most important part of the density matrix which we point out here is the fact that density matrices can be defined for pure, coherent superposition or statistically averaged states. The eigenstates of density matrix form a complete basis for subsystem block; the eigenvalues if the weight of the state.

We can keep the m eigen states corresponding to the m highest eigenvalues, the eigen values of the whole system can thus be given by the sum of the root of the weight of the individual elements which is often known as Schmidt decomposition, this is the optimal

approximation for this and for entanglement state the mutual quantum information is possible to be found from the entropy of the system, this was a part of many of the exercise which was given in relation to the density matrices which is essentially minus trace rho times log rho and that is a very important part of understanding the system.

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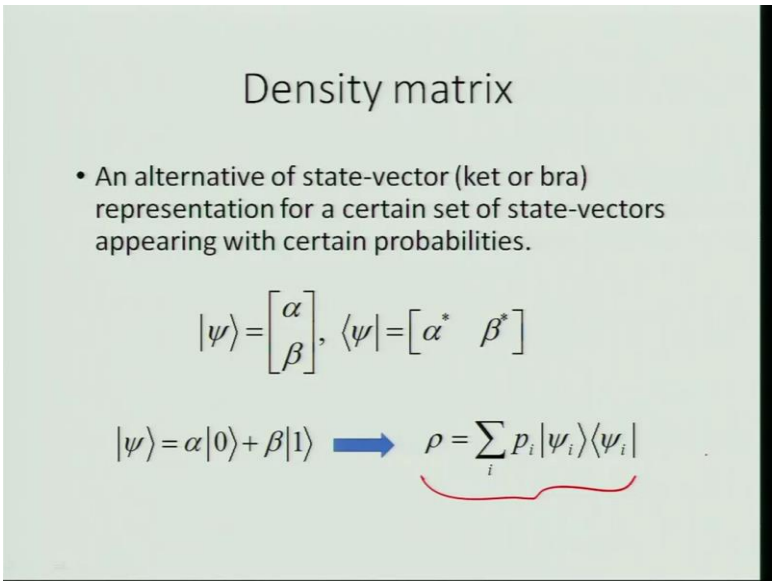

So, the density matrix is the alternative state vector the ket or the bra representation for a certain set of state vectors appearing with certain probabilities as we have mentioned giving raise to the form that we have shown before.

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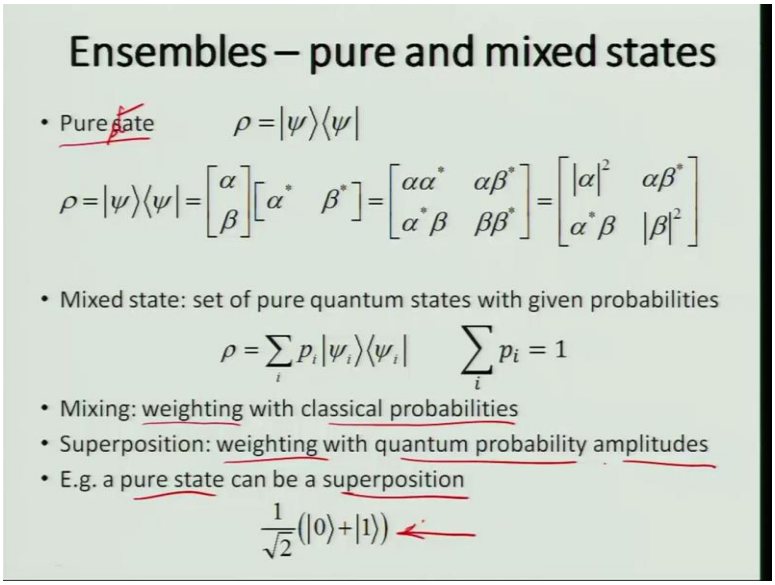

And we have discussed in terms of the pure state as well as the mixed state; mixing give raise to weighting with classical probabilities, superposition is weighting with quantum probability amplitudes, which means that the weighting can be written in terms of square roots. So, for example, a pure state can be a superposition, so that the weightage factor can be written as a square root that is this Schmidt decomposition.

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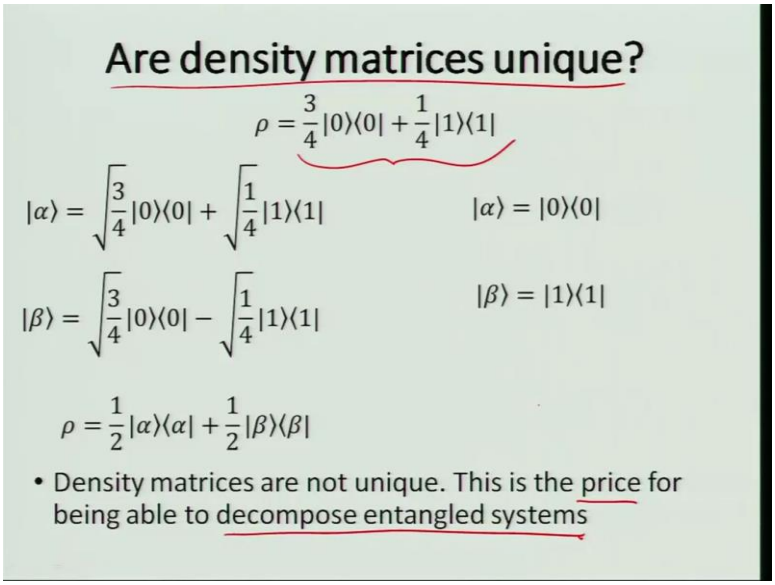

One important point to realize is that density matrices are not unique. So, here is an example which kind of shows the taken a density matrix of this kind, how they can be formed from different conditions and that is the price for being able to decompose entangled state because in no other form of understanding can be look at density matrices.

Density matrices are not unique, this is because by using density matrices we are able to handle or treat entangles states to some level and that is the price for being able to a certain concepts related to entangled states; however, it is still extremely important to be able to have this mathematical formulation and this non uniqueness has its advantage also.

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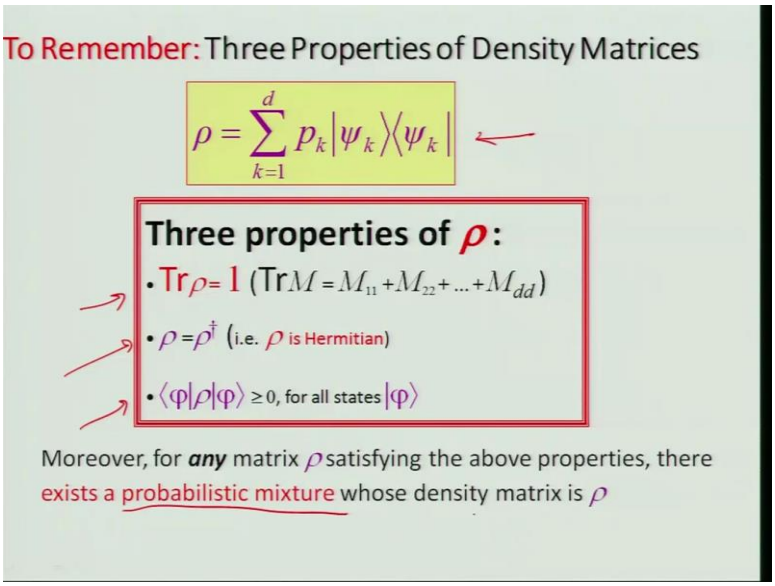

We need to remember three properties of density matrices which were mentioned during the regular classes also A how it is written; B that the trace is always going to be 1 of the density matrix, that it is going to be Hermitian and that the expectation value of the is always going to be positive definite for any state. Moreover for any matrix satisfying the above properties there exists a probabilistic mixture, whose density matrix is rho.

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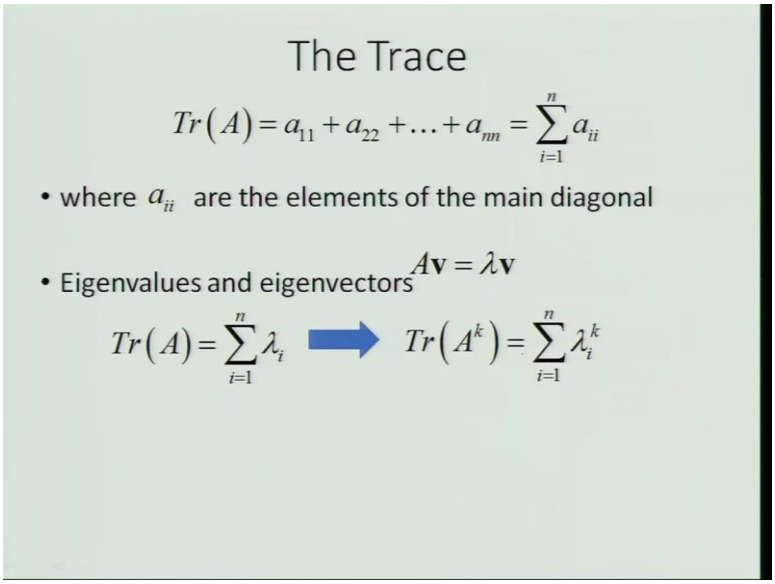

The trace of the matrics has been defined several time before is essentially the some of the elements of the main diagonal, and they can be decomposed into the eigenvalues and the eigenvectors.

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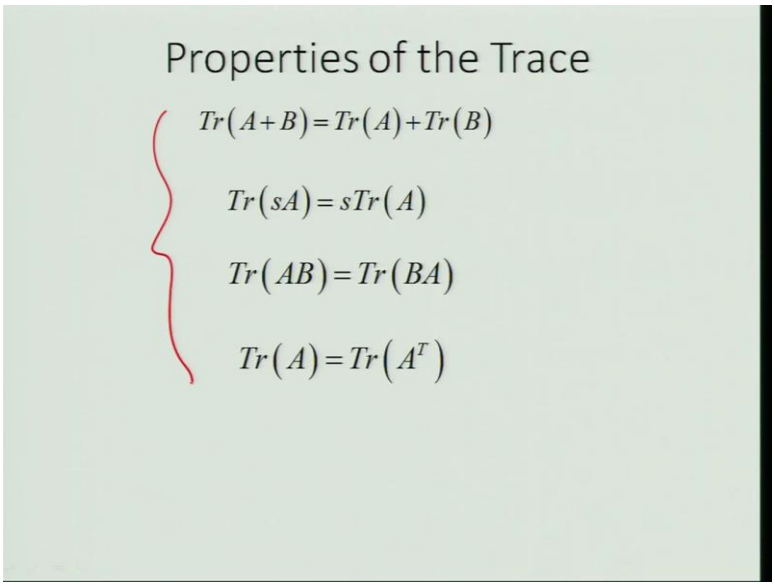

And the properties of the trace are summarized here, these are thing which we are done earlier again, but just been put up for given their importance.

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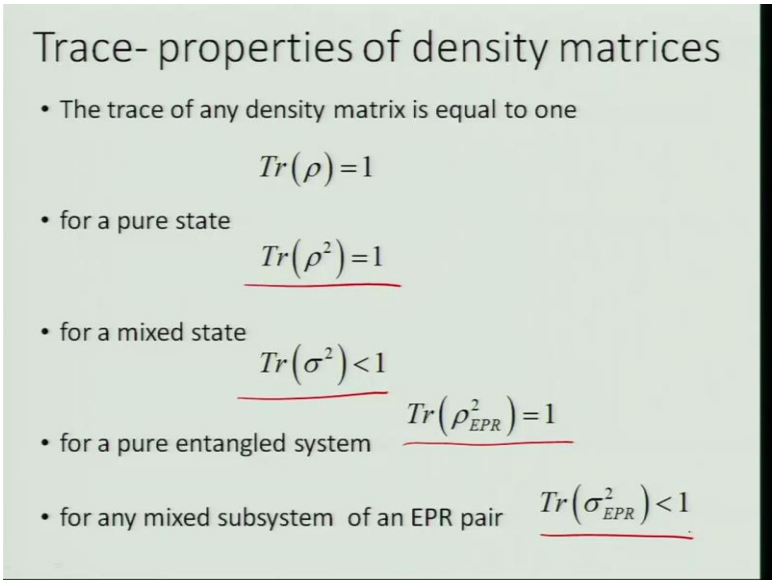

The trace of any density matrix is equal to 1, for a pure state square of the trace is also 1, for a mixed state the square of the trace is less than 1. For a pure entangled system the

trace of this square of the density matrix is 1, for any mixed subsidy of an EPR pair therefore, is going to be less than 1. So, these are consequences of how we treat density matrices.

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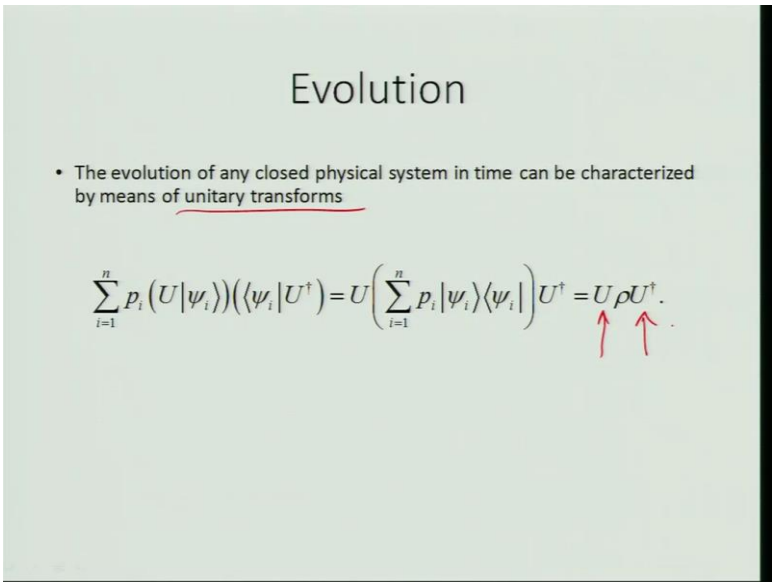

The evolution of any closed physical system in time can be characterized by means of unitary transforms and that is true for any quantum systems and that is the how density matrices are also treated.

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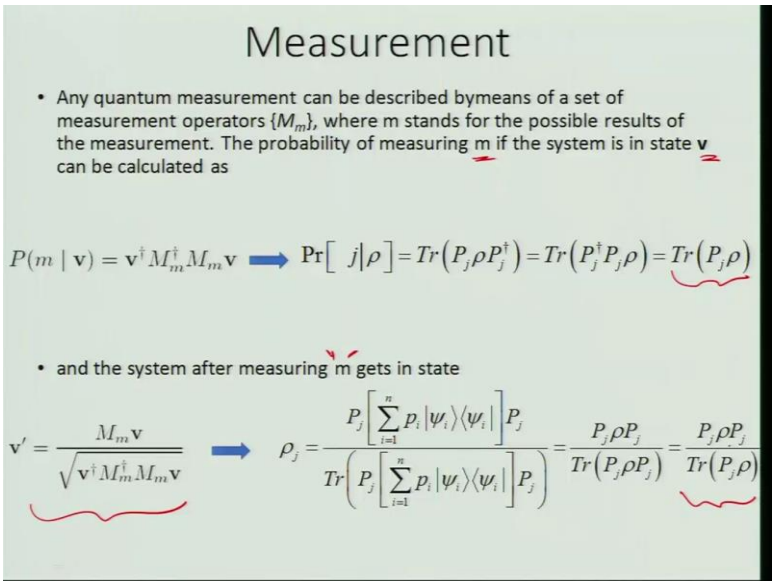

So, we can devise the unitary operator, which can operate on the density matrix and any quantum measurement can be described by means of a of measurement operators M s of m; where m stands for the possible results of the measurement small m. The probability of measuring small m if the system is in state v can be calculated as below, which we show here; essentially it is the projection of the state into the frame of reference that is being measured.

So, that is how it is the trace of the projection operator with the density matrix and the system after measurement the state m is left in the state which is given by this form, which would essentially be transforming into a form which has the projection operator taking the density matrix into its form of this kind.

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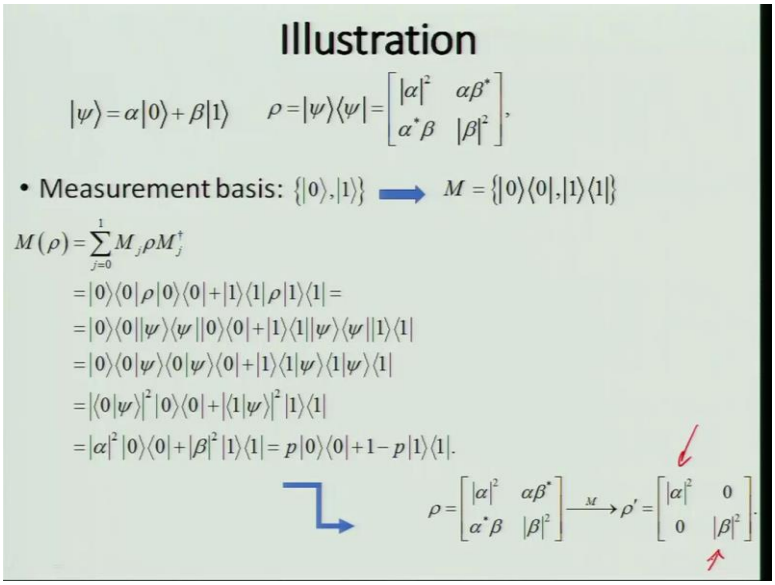

So, this can be illustrated based on the measurement basis that we use and there is an example which has been shown here for clarity. So, if alpha and beta are the constituent compositions of 0 and 1, then upon measurement would be basically getting the probabilities of alpha mod square and beta mod square as the measured values of each of the components.

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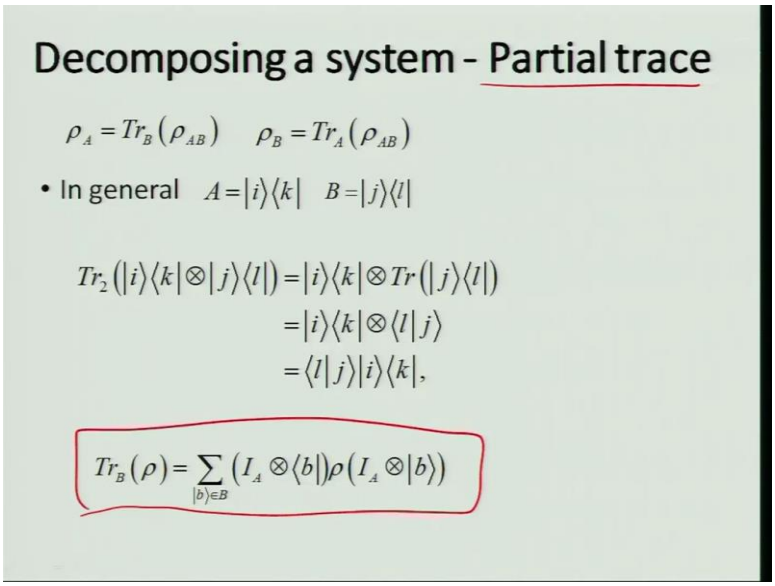

Now if you are decomposing a system into a smaller set, it can still be done by using a very powerful concept of the partial trace which is unique to density matrices and that is the part which is extremely important as we have discussed earlier in terms of understanding mixed states and others and this logic goes by the principle that we can use trace properties and the operator properties to write the trace of the overall density matrix in a way which can then be decomposed into a condition which looks like this.

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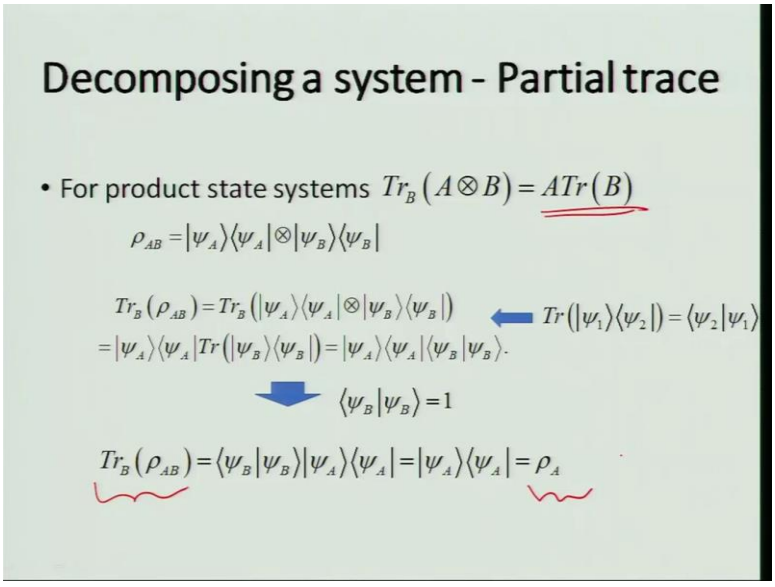

And since for product state the trace matrix can be written in this form, it is possible to trace out one of the components of the density matrix say for example, trace B of rho AB would essentially give raise to tracing out the component B to give raise to rho A.

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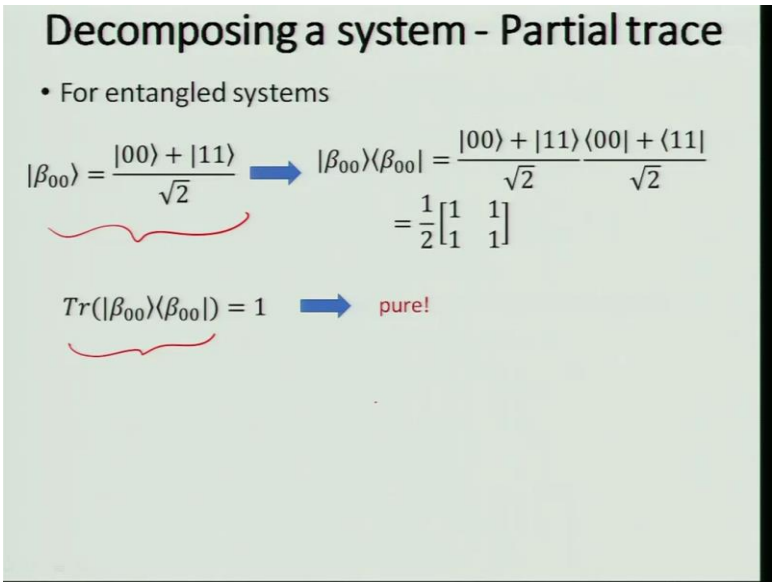

So, for entangled systems therefore, it is important that we can trace out one of the components to get the results from the other. So, for example, for the particular case that has been shown here, we can find out that you can get the trace to be equivalent with the pure state.

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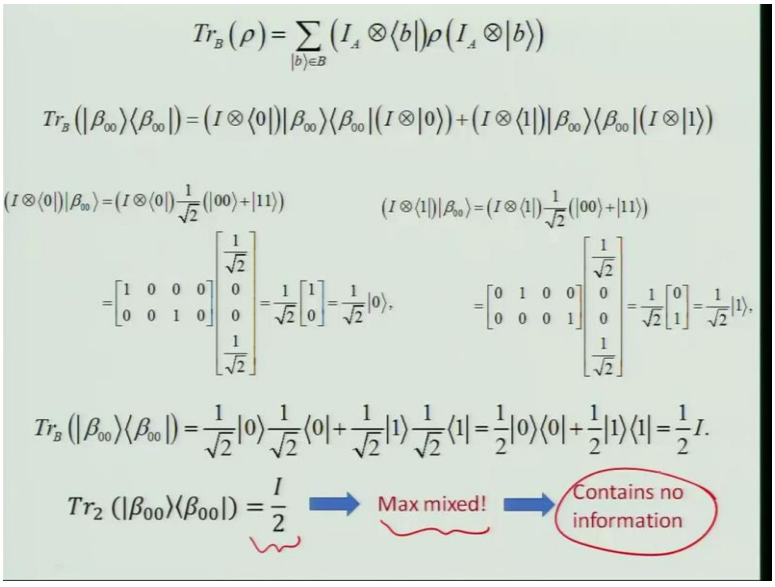

If we follow the scheme of discussion that we did until now, we can essentially find out that when we trace out the second component, we can get to a point where it is given by I over 2 which means that this essentially going to be maximally mixed and so contains no information about the system.

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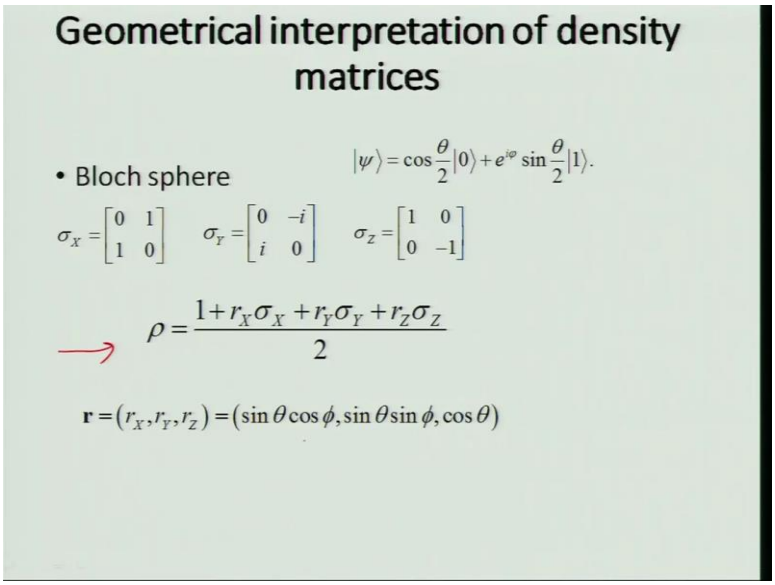

We have also done the geometrical interpretation of density matrices, where we utilize the Bloch sphere to understand how things are and so here are the 3 poly matrices and the unitary one which is utilized to get to the density matrix and its form the r is the vector which corresponds to the components in this spherical axis which is how it is shown here.

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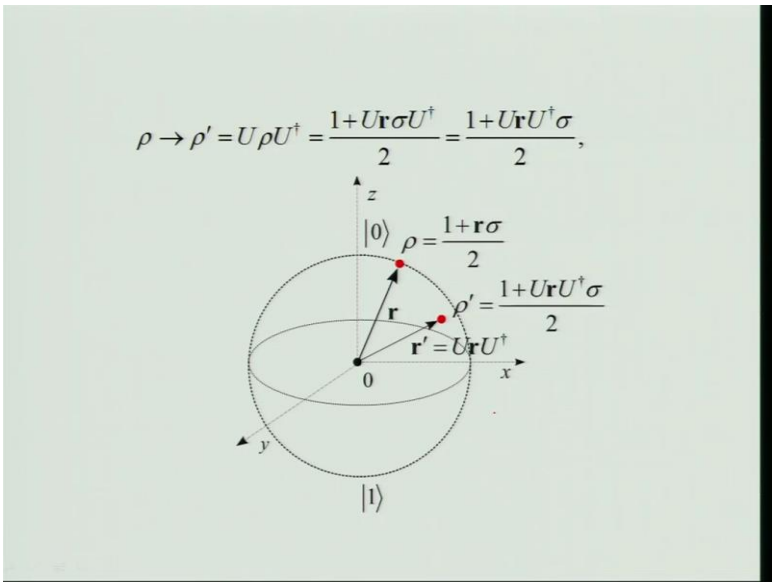

This can be looked at by utilizing a unitary transform to the entire process.

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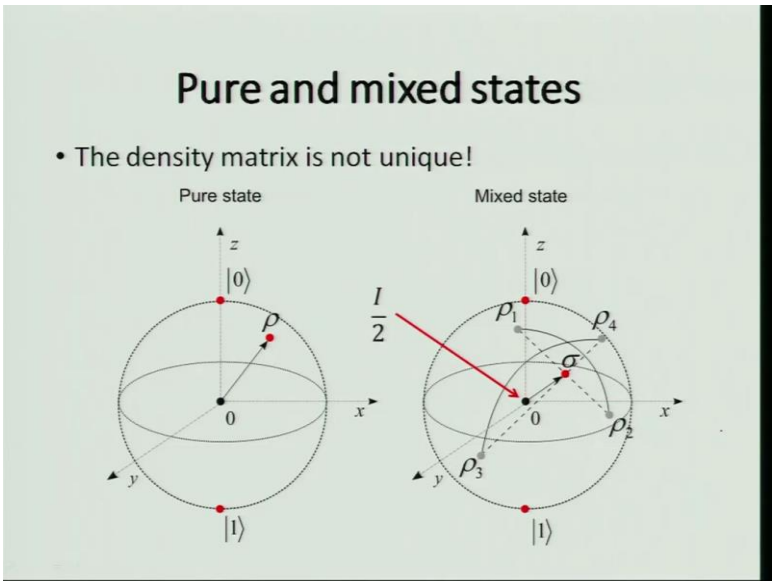

And if we do this transformations, what we essentially find out for pure and mixed state is that the density matrix is not unit as we have if do the same thing for the pure and mixed state we find that the density matrix is not unique. For the pure state is always unique however, for the mixed state we can get several results which would correspond to the same I over 2 and that is the reason why trace of the square of the matrix is always less than 1.

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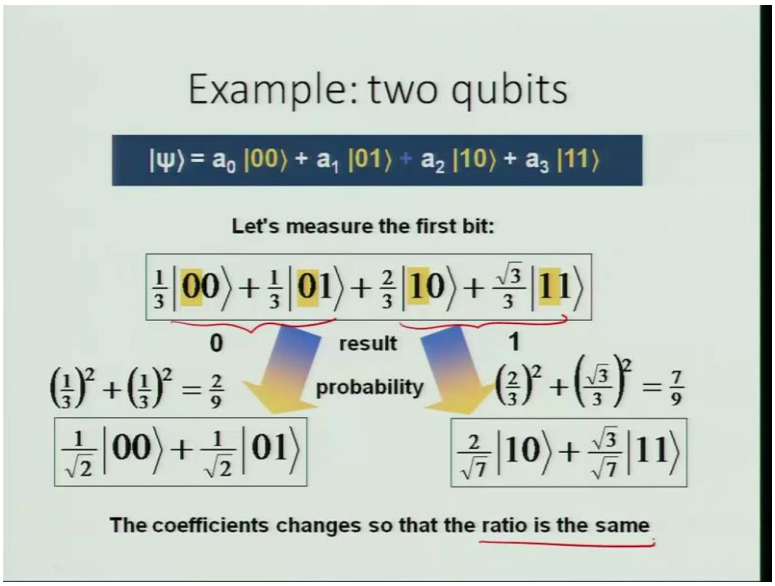

So, for 2 qubits we can take this example case of all the possible states, which are in different conditions and in this case if we measure the first bit or the second bit the results can be different. So, for example, if we measure the 0, which is the first bit and the one which is the first bit and the second set, we will end up getting 2 different probabilities just shown here, the coefficients would change so that the ratio is going to remain the same. We have done this before I was just bringing it back to make sure that you understand this important result.

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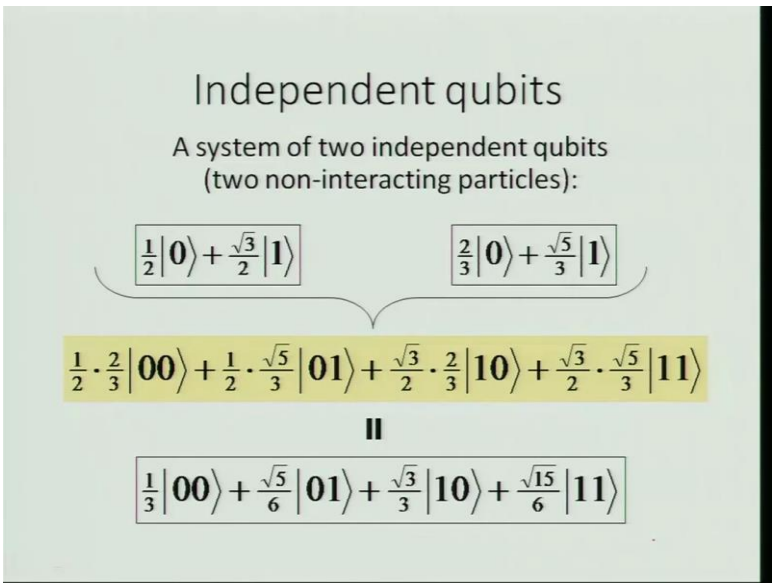

So, there is this case where the sets of qubits are independent qubits, which is a system of 2 independent qubits where the 2 are non interacting particles. So, we could write them out in such a way so that they can be put together.

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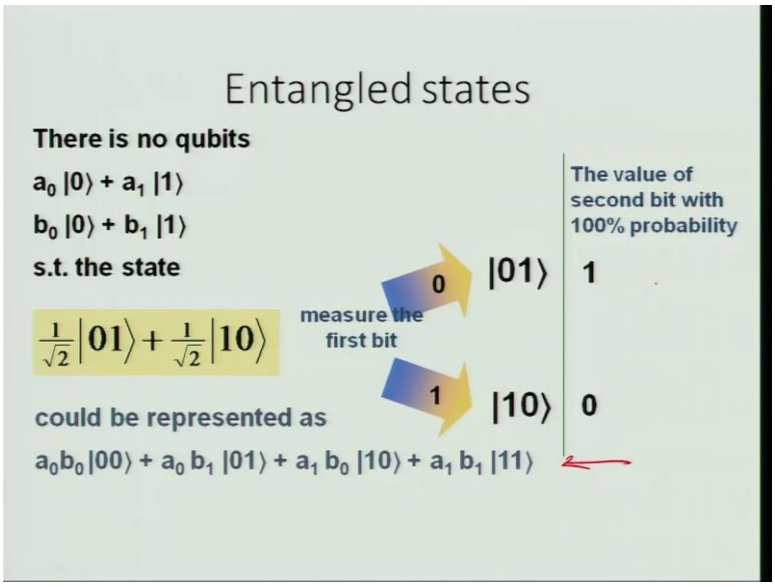

In case of entangled states; however, there are no qubits, where we in we could write down the states in such a way so that the states could be represented in the entire form as we represent here. So, in this case if we measure the first bit we get 0 in the first case and one in the second case. The value of the second bit is can be measured with 100 percent probability, when we measure one of them and that is the main principle of the entangled states where we keep on saying that if we measure the first qubit or any one of the 2 qubits we get a complete information about the other qubit, that is the most important part of entangled state, which from density point of view and partial measurements its 100 percent clear as to how things are going.

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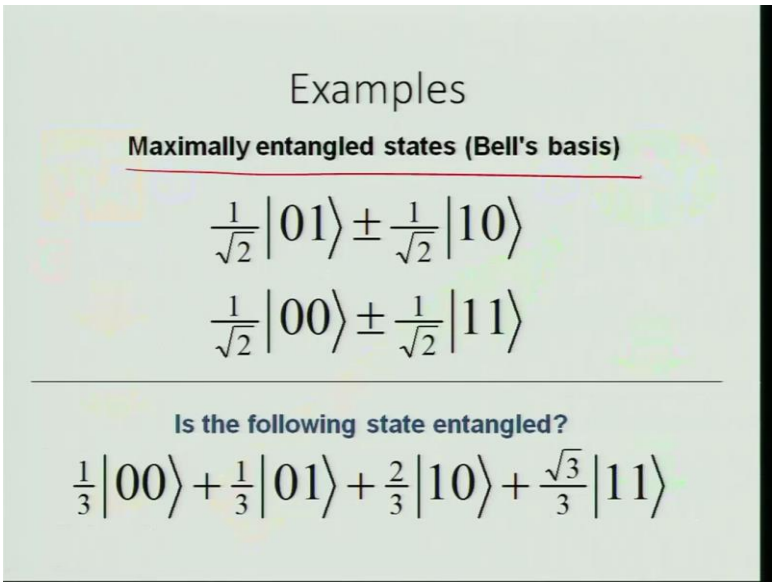

There are several examples that we are gone through for example, we have discussed the maximally entangled states, which are the bells basis and we have also looked at states whether certain state is entangled or not.

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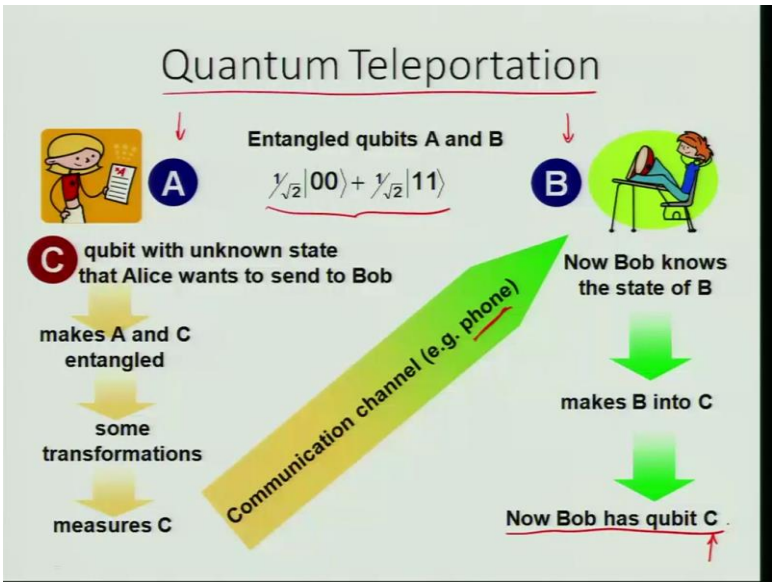

And finally, the most important thing is to realize that all of this play a critical role in the process of quantum teleportation which we have spent a lot of time about, because quantum computing and quantum information is processing one of the key relevant parts in this area is the concept of quantum teleportation; where entangled qubits A and B are

being shared; where entangled qubits are being shared between A and B, which are the kinds out the bell qubits. So initially the qubit with unknown state that Alice wants to send to bob makes the A and C entangled then there are some transformations made and then C is measured by Alice. That information is being sent say by classical channel say phone to bob.

Now bob knows the state of B. So, he makes B into C and so bob has the qubit C and that is the idea behind this entire concept. Are good enough once the communication channel is used to communicate some information which is in this particular case the qubit A, which is good enough to be able to get the value of the unknown qubit C by bob who already knows the state of B.

So, there are several issues of quantum teleportation and practicalities as well as discussions which we have earlier presented during the course work, where went through the concepts of quantum teleportation in detail and that was very important, because that is another very important area where the idea of quantum computing and information has been put to use. It is an area of strong implementation in terms of quantum computing where the principle of quantum entanglement in terms of information secure information transfer has been successful and that is one of the areas where we have shown a lot development and we have discussed this as part of this course.

So, until now what we have done in this week is we have revised the concept of bringing over the quantum principles for quantum computing and quantum information from classical rules and then we went ahead to look at one of the critical aspects of quantum computing which is ensemble principles, which are necessary to be understood, which is density matrices and based on that we were able to discuss once more the concept of quantum entanglement and information quantum information in terms of quantum teleportation; this is what we have managed to finish until now.

In the next lecture we will go about the principle of quantum computation implications, and their practicalities, their implementation issues which we have done a lot during this course. And that way we hope to have a complete summary over what we have done.

Thank you.