Implementation Aspects of Quantum Computing
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Lecture – 36
Understanding Quantum Measurement, Entanglement etc in Quantum Computing using Density Matrix

We have been looking at density matrices and its implications on the implementation of quantum computing that we have been looking at until the last several weeks. In this week we had based our studies from the beginning of density matrices this was prompted as I mentioned in the very beginning of the week by some questions raised by the students and as a result of that we have gone ahead to discuss the details of density matrix in relation to the measurements and the importance of it; now that we have come to the point where we have hopefully understood density matrices to a point that we can use it in a much more effective manner to these implementations and understandings that we have been carrying forward.

Let us now look at the general quantum operations, which include Decoherence, partial traces and measurements based on density matrices.

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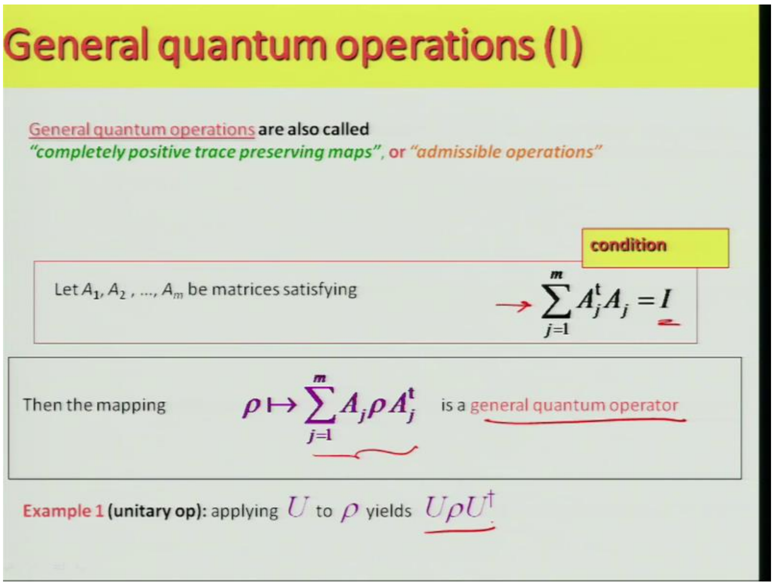

So, the general quantum operations are also completely positive trace preserving maps or admissible operations, as a result we basically choose several matrices that satisfy the condition. Let us consider matrices A 1 through A 1 M, which follow the condition as provided here that I is the identity matrix A j t is the transpose of A j then the mapping of the density matrix to this set of matrices can be such that it becomes a general quantum operator. So, the mapping in terms of the density matrix written in terms of these matrices that we just defined can be written in terms of this general quantum operator. So, for example a unitary operator applying U to rho, we already know gives rise to this result rho U dagger.

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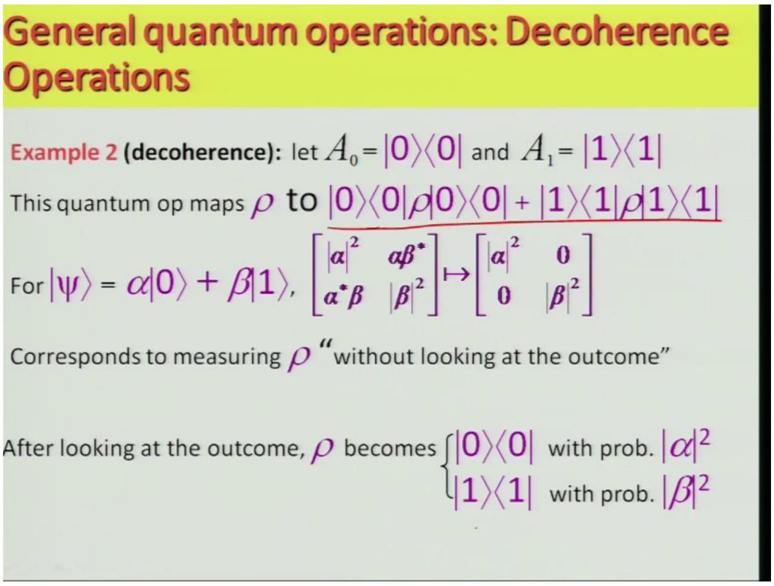

Let us take the example of Decoherence; let us consider A 0 matrix which is an outer product of 0 states and A 1 matrix which is a outer product of 1 states; this quantum operation maps rho into this form and now if we take a state vector psi which is having super position of 0 and 1 with alpha and beta coefficients, then this mapping would essentially result in alpha squared and beta squared formation in this map, this would correspond to measuring rho without looking at the outcome. After looking at the outcome rho would become 0 0 outer product with probability alpha squared and 1 1 with probability outer product with probability beta square.

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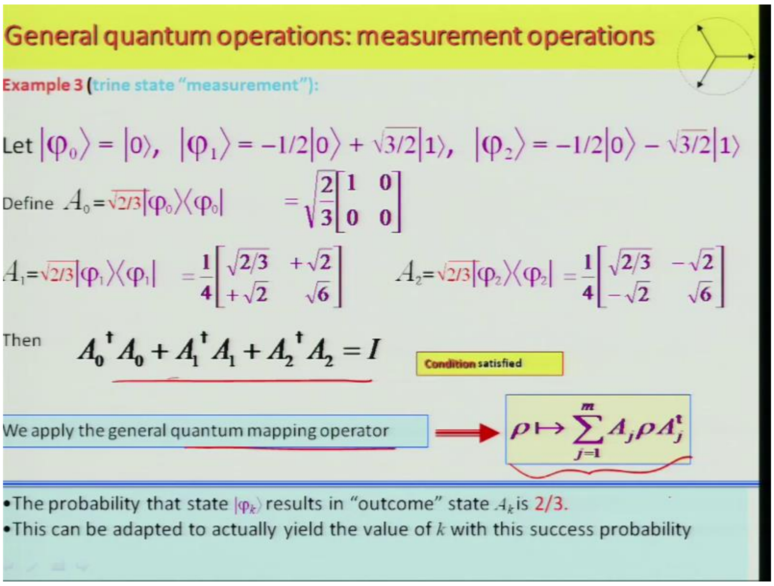

Another example is the trine state we had done this state in one of our example problems maybe home assignments, where we have these states which are given in terms of 0 and linear combinations of 0 and 1 in certain order; if we define a naught as two-thirds square root outer product of phi naughts, which will give rise to this matrix whereas, A 1 has two-third of outer product of phi 1 and A 2 has two-third outer product of phi 2 then we will be getting each of them in such a way so that their transfer products and their sums are equal to identity which means that our condition is satisfied. So, we can apply the general quantum mapping operator, which is that sum of this is going to give rise to this operation that we mention.

The probability of the state psi k results in an outcome state A k which is two-third; this can be adapted to actually yield the value of k with this success probability. So, that is the reason why this particular principle that we can have the general quantum mapping operation work is very important because once this condition is satisfied, we apply the quantum of mapping operator we can make a measurement with probabilities that can be the outcome of the state with the exact values that we are interested in terms of the state. So, that is the reason why this is a very important measurement principle.

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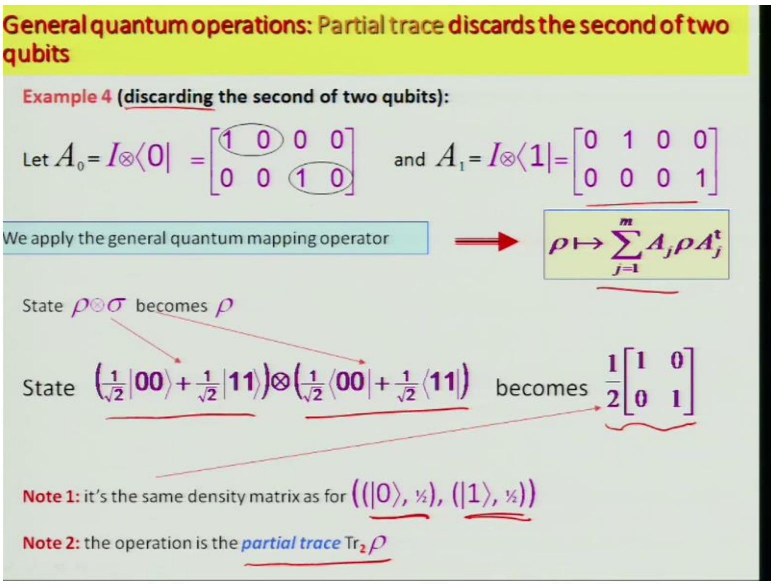

Another generalized quantum operation advantage is the partial trace which discards the second of the 2 qubits and this could be the example which is of the kind where we discuss. Here if we take a matrix A naught, which is an identity times the tensor product to the bra of state 0 then we construct matrix of the form this and the other 1 of the form this and we apply the general quantum mapping operator as we have discussed before and then the state becomes rho tensor product of sigma gives rise to the state which will become this particular form half 1 along the diagonals is the same density matrix for 0 with half probability and state 1 with half probability.

It is the operation is the partial trace of rho. So, what we have essentially done is basically we have looked at a way of looking at the partial trace of an operation, and that is an important operation that is often used. So, this is almost like we are discarding the second of the 2 qubits and that is why it is trace 2 rho considering that as unity and we get the other part which is a solution.

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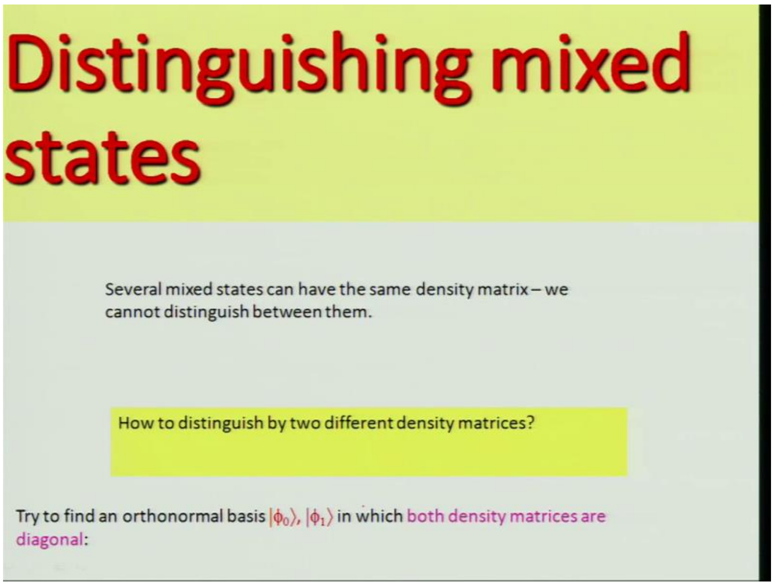

So, we are interested in distinguishing mixed states, we have mentioned before that several mixed states can have the same density matrix which we cannot distinguish. So, how to distinguish by 2 different density matrices? So, one of the options is to try to find an orthonormal basis phi naught phi 1 in which both density matrices are diagonal.

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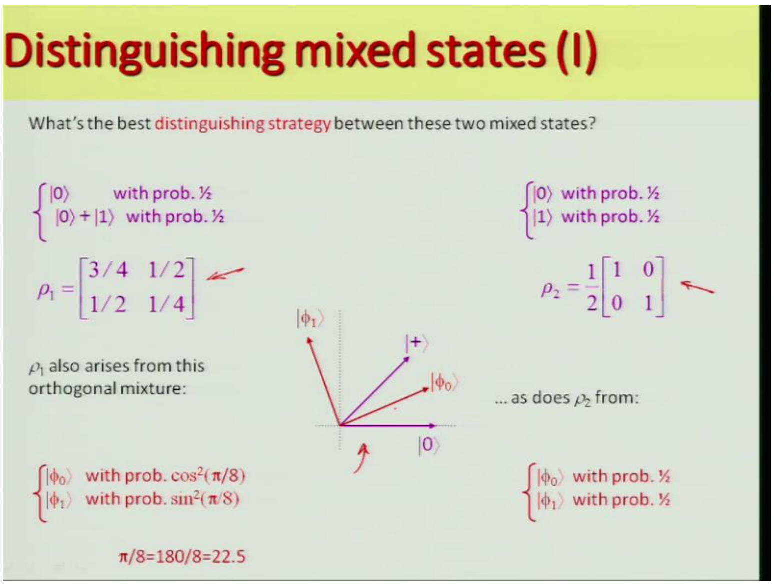

The best distinguishing strategy between these 2 mixed states is to go with a rotation about the state. So let us consider a state 0 with probability half and super position of 0 and 1 with probability half giving rise to rho 1 which is given here and another wave function, another is mixed state with state 0 with probability half and state 1 with probability half giving rise to a density matrix rho 2 of this kind; rho 1 can also arise from other orthogonal mixture where rho phi 0 and phi 1 have probabilities of cosine theta, rho 1 can also arise from this orthogonal mixture where the states are in phi 1 and phi 0 with probabilities cosine squared pi over 8 and sine squared pi over 8, which is essentially in the rotated condition.

Similarly, rho 2 can arise with probabilities half when they are in other states.

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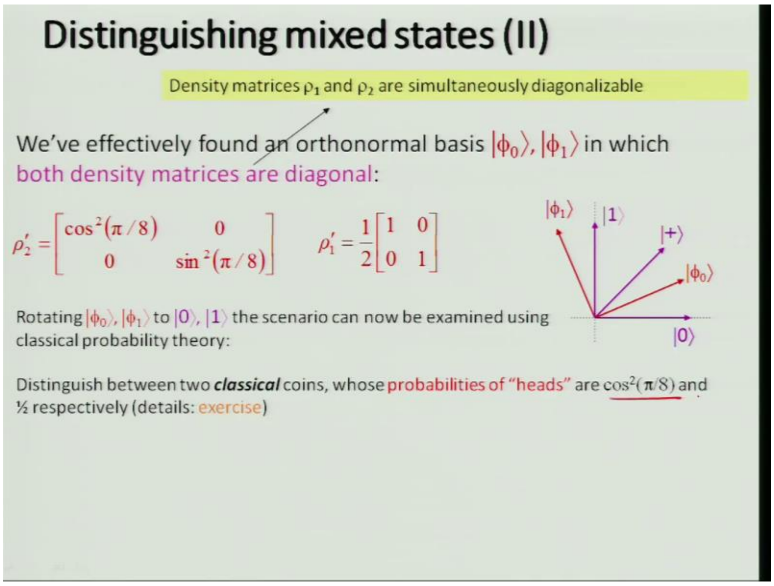

So, if we can effectively find an orthonormal basis rho 1 and rho 2 that are simultaneously diagonalizable, then we can get the solution. So, rotating phi naught phi 1 to 0 and 1 the scenario can now be examined using classical probability theory, you can distinguish between 2 classical coins whose probabilities heads are cosine square pi over 8 and half respectively. So, rotating phi naught and phi 1 to 0 and 1 the scenario can now be examined by using classical probability theory.

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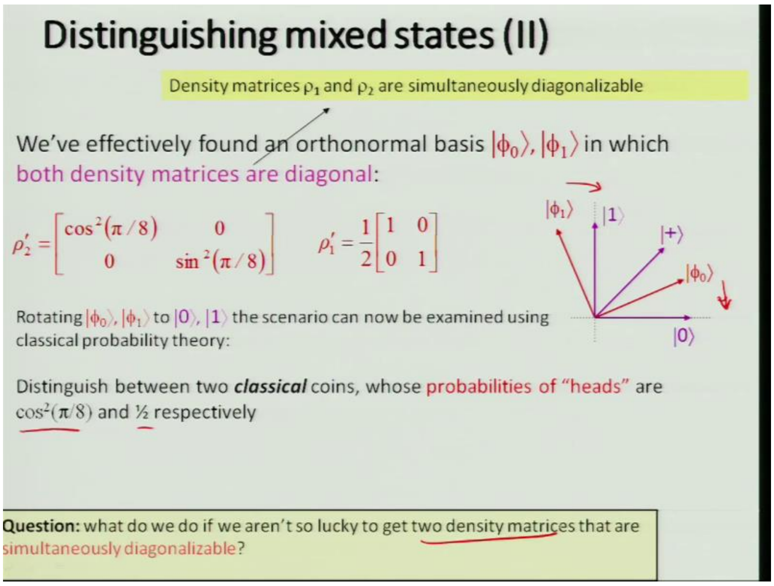

So, the rotation actually allows us to go from one state to the other. So, we can distinguish between the 2 classical coins whose probabilities whose heads are now, cosines squared pi over 8 and half respectively and possible as we have found the orthonormal basis, where both the density matrices are diagonal. Now the question is what do we do if we are not so, lucky to get 2 density matrices that are simultaneously diagnosable?

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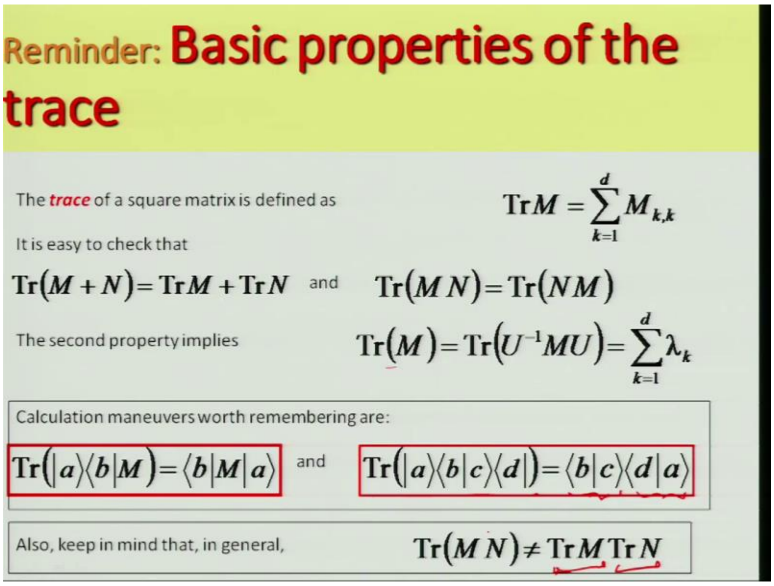

We should recheck the basic properties of the trace as we have done once before the trace of a square matrix is defined as we mentioned trace of the matrix which is the sum of the diagonal terms, it is easy to check that the trace of 2 matrices are essentially the trace of individual matrices summed together and the product of the trace of 2 matrices is cumulative, which means trace of M N is equal to trace of N M the second property implies; however, that the trace of M would be equivalent to the trace of the unitary transform of M, which means that its essentially looking at the sum of all the eigen values or the diagonals.

So, the calculation maneuvers which are worth remembering are the fact that the outer product of the trace a and b with a matrix M, would give rise to the expectation value of M and trace of outer product of a b and c d is equivalent to getting the expectation value of is equivalent to writing out outer product of b c times that of d a; trace of this a b outer product of a b and c d it give rise to inner dot product of b c and d a together. We can all we should also keep in mind that in general trace of M of N is not equal to the trace of M and trace of [noise], that is not quite generally true you cannot make the trace apply on individually, although they are commutative in terms of the product of the matrices.

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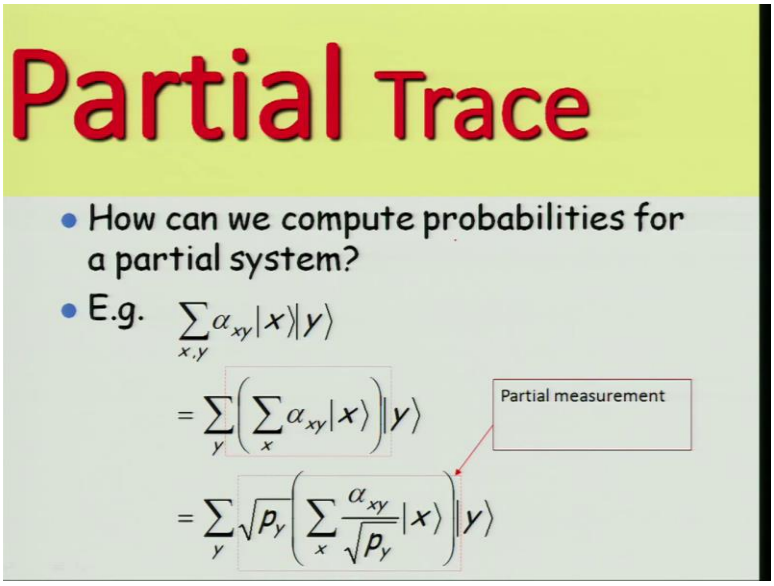

So, the idea of partial trace that I mentioned before can be looked at now, how do we compute the probabilities for a partial system for example, we have 2 states X and Y with a probability of xy, which is with all their sums then if we look at the probability of one of them with respect to the rest then that is the partial measurement of one with respect to the other. So, this is the form that we are looking at this is the partial measurement where the term which gives rise to the part, which is correlated to only the part summing over all the y is giving rise to the partial measurement for that.

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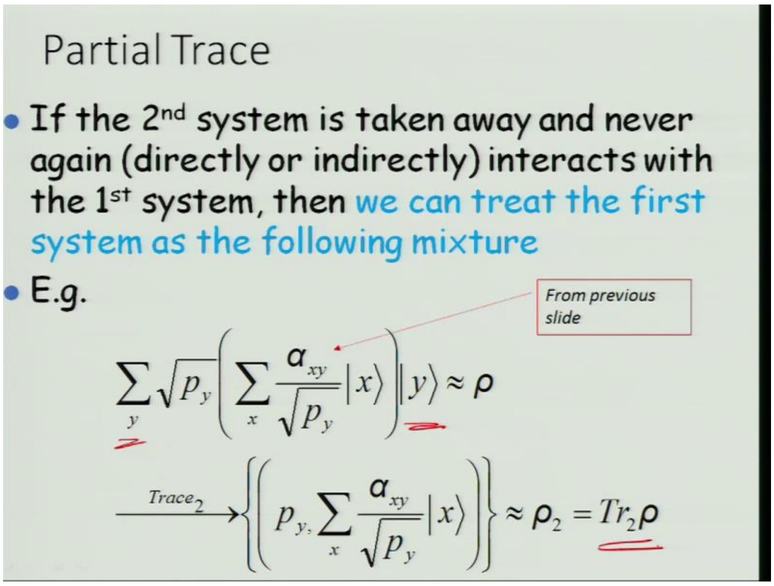

So, if the second system is taken away and never again directly or indirectly interacting with the first system, then we can treat the first system as the following mixture.

For example we wrote the earlier slide; in the earlier slide we came up with this term, this will be equivalent to the rho trace of 2, when we are considering that sum over all the ys are taking care of all the y component or the second component of this mixture. So, that is why it is the trace 2 (Refer Time: 14:38).

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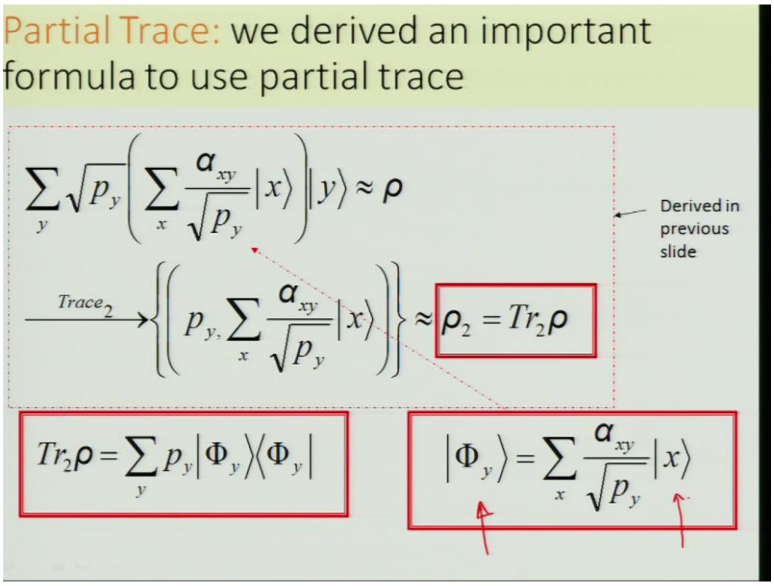

So, in terms of partial trace we derive an important formula to use; which means that we have rho 2 which is the trace 2 of rho is basically sum over all the ys for the particular probability of P of y with the outer product of the psi's, where the psi y is written in terms of all the sum of the x vector.

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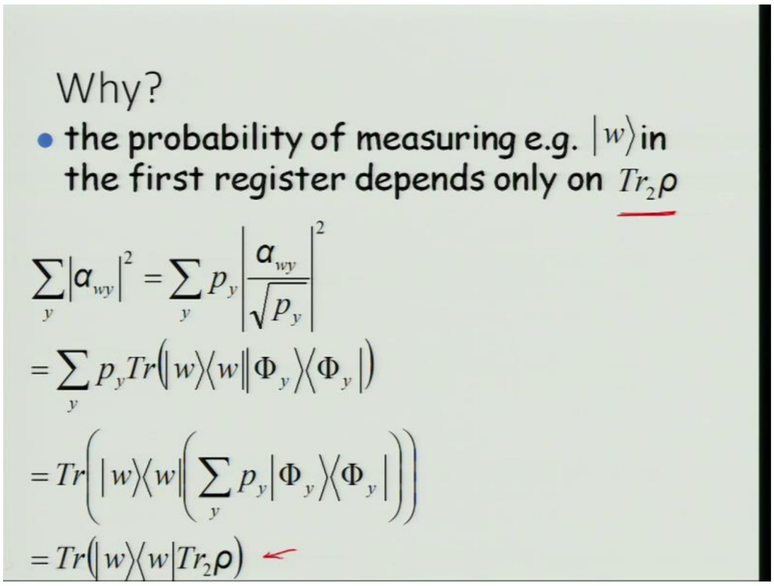

The probability of measuring for example, w in the first register depends only on trace rho now. So, the sum over all the probabilities of w y would be given by probability of y summed over with the weightage factor w y with respect to root P y square give rise to this particular set where we get the trace of the measuring the outer product w times the trace of rho 2.

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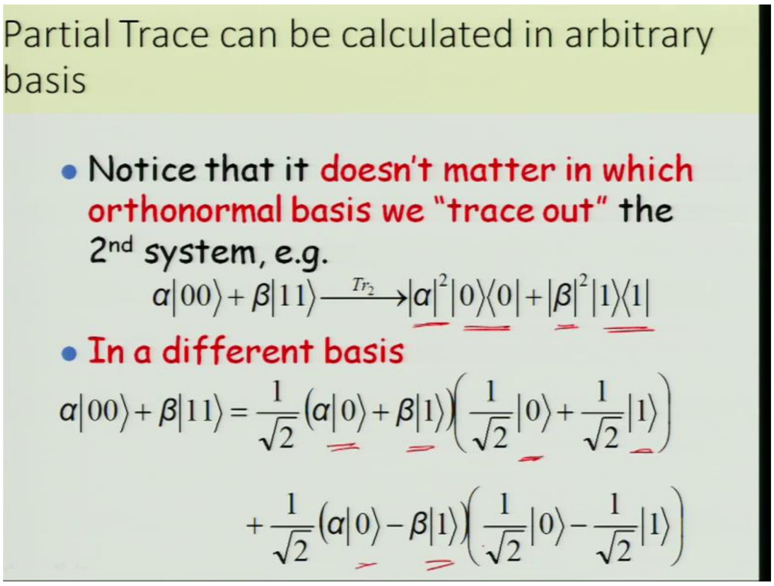

So, the partial trace can therefore, be calculated in an arbitrary basis it does not matter in which orthonormal basis we traced out the second system for example, if we have alpha 0 0 and beta 1 1, and we do the trace out of the second system, then we are left with alpha squared outer product of 0; beta square outer product of 1 in the same; however, in a different basis you can write this in terms of alpha 0 0 beta 1 1 as in a different basis alpha 0 beta 1 and root 1 over root 2 0 and 1 for both and with the with another one which is in this different basis.

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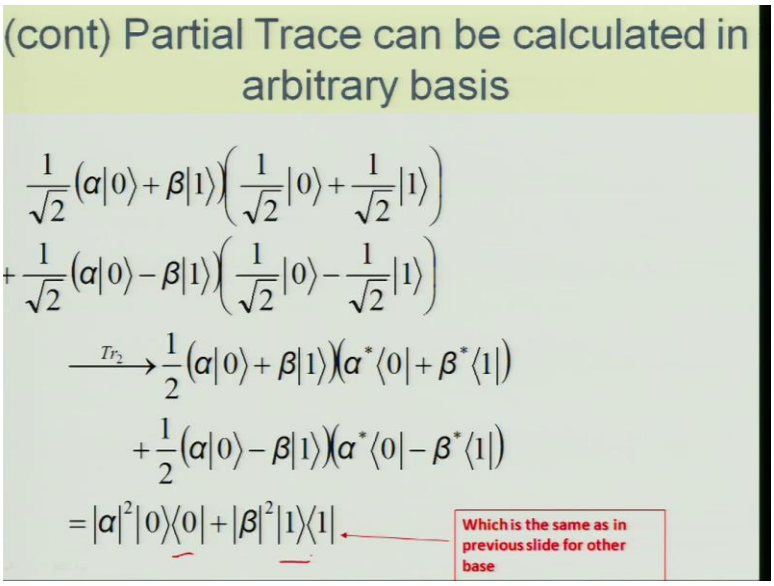

Now, if we do a partial trace can be calculated on a arbitrary basis then the trace 2 will finally, again give rise to alpha squared outer product of zeros and beta square outer product of one and that is exactly the same as the earlier one. So, what it means is the partial trace is invariant to the orthonormal basis that we choose and that is actually a very important statement is that is the way how the properties of the system can be determined without problem.

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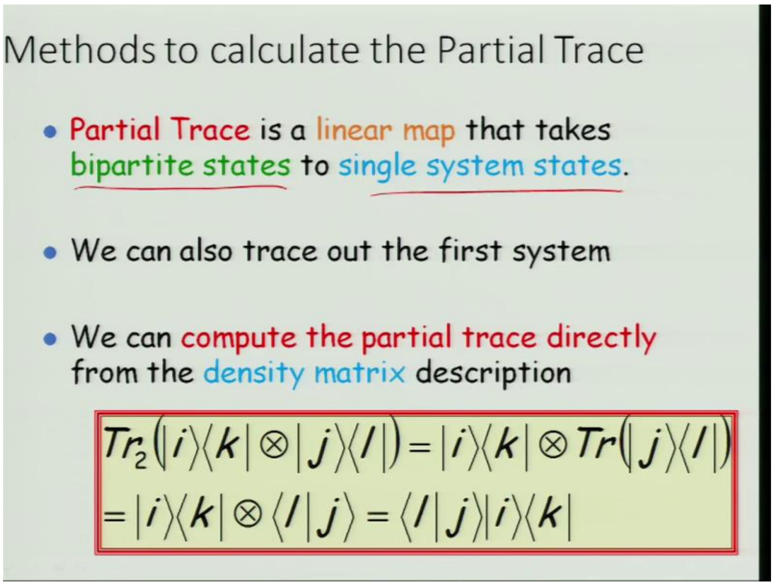

Now, methods to calculate partial trace; partial trace is a liner map that takes bipartite states to a single system states. So, what we have essentially done is we have taken a mixture of 2 states into a single system state by tracing out one of them and that is through a linear mapping as we just did. We can also trace out the first system if necessary is the choice how we like to do this, we can compute the partial trace directly from the density matrix description it is another important part of this exercise.

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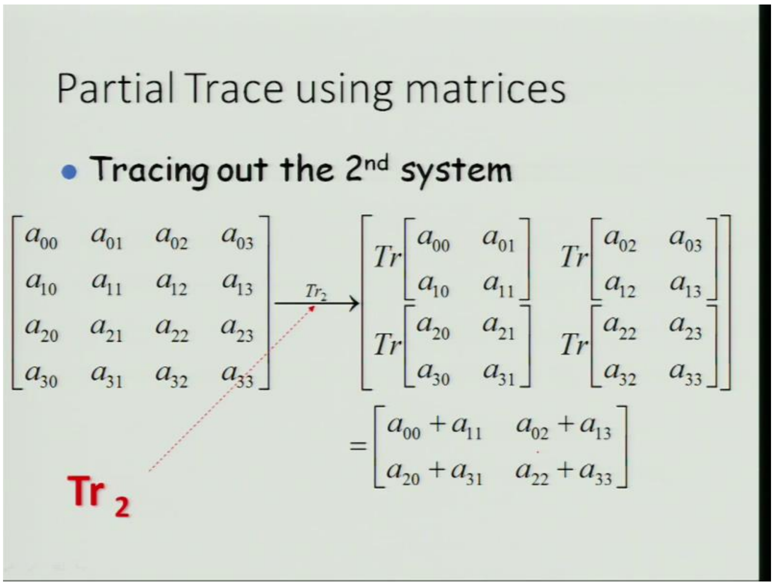

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{Tr_2} \begin{bmatrix} Tr\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} & Tr\begin{bmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{bmatrix} \\ Tr\begin{bmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} & Tr\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{00}+a_{11} & a_{02}+a_{13} \\ a_{20}+a_{31} & a_{22}+a_{33} \end{bmatrix}$$

So, partial trace using matrices tracing out the second system would be of this kind where you can set up the traces by individual parts and then we can write it out in these kinds of forms.

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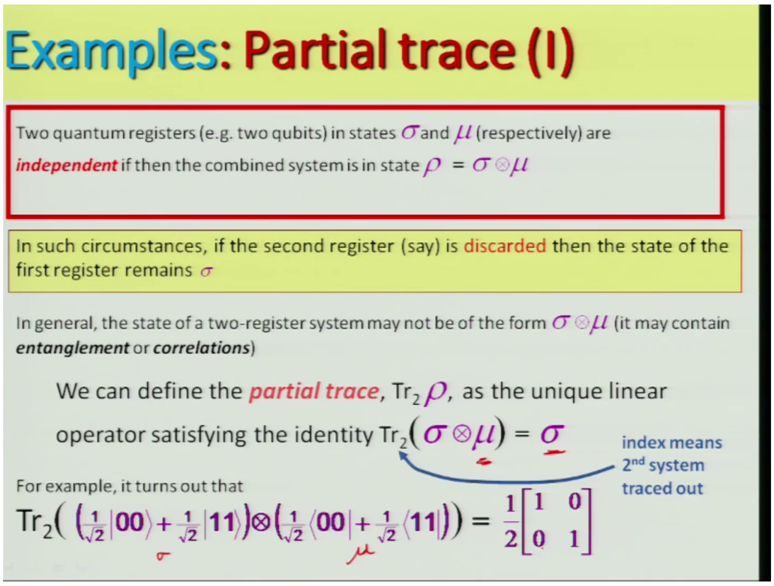

So, here are some examples on partial traces: 2 quantum resisters example 2 qubits are in states sigma and mu respectively are independent if they can be come if the combined system is in the state, rho which is a tensor product of the two. In such circumstances if the second register say is discarded then the state of the first register remains in sigma. In general the state of a 2 register system may not be of the form sigma tensor product with mu, it may contain entanglement or correlations; we can define the partial trace 2 rho as the unique linear operator satisfying the identity at trace 2 sigma tensor product mu is equal to mu the second the index 2 here essentially means that the second system is being traced out second system is mu.

So, for example, it turns out that trace 2 of this particular form that we have written here is half 1 1 diagonal element, it turns out that if we take this particular set of sigma and mu then we get a result which is of this kind is the half.

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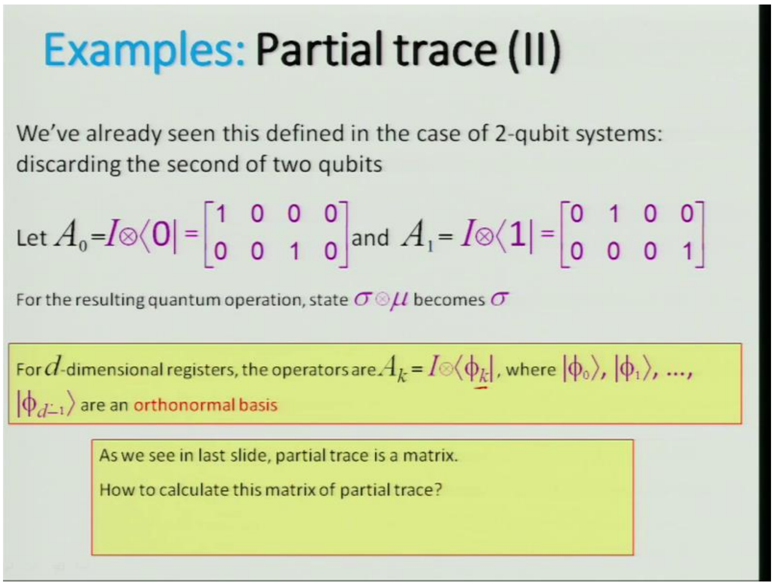

We have already seen this defined in the case of 2 qubit systems discarding the second of the 2 qubits, let a equal to I times the tensor product of bra 0 state then we have this and A 1 is the tensor product of I times 1, for the resulting quantum operation state which is the tensor product of these 2 becomes sigma. For d dimensional registers the operators are A k times I tensor product of bra phi k, where phi naught phi 1 all the way up to phi d minus 1 are an orthonormal basis. As we saw in the last slide partial trace is a matrix how to calculate this matrix of partial trace?

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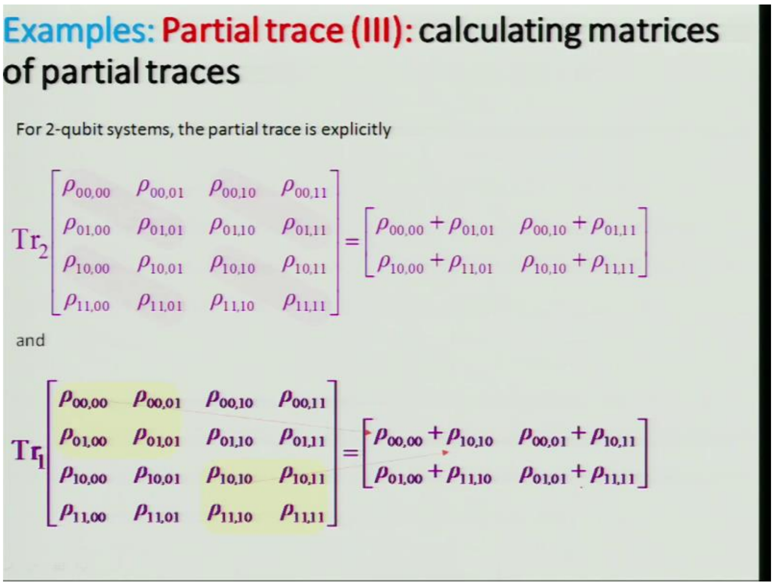

So, calculating matrices of partial trace is for 2 qubit system the partial trace is explicitly given in terms of the traces of each individual parts which add up as we had shown before and we trace them out and we get this and as a result we can sum them up as we show here into these individual traces and we can write them out in this form.

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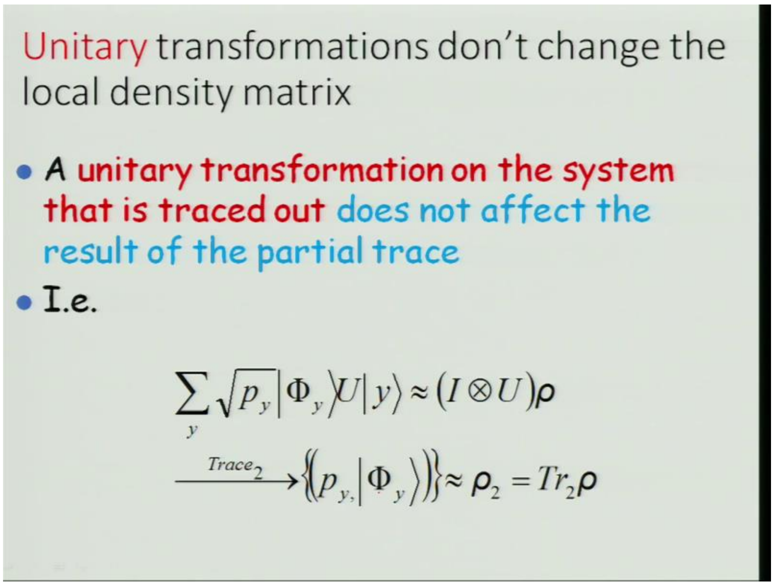

The unitary transformation do not change the local density matrix that is a very important property, a unitary transformation on the system that is traced out does not have an effect on the result of the partial trace, which means that if we take a system which is sort of defined in this form and we take the partial trace over 2 the second system, then the unitary transform in the process does not affect the state.

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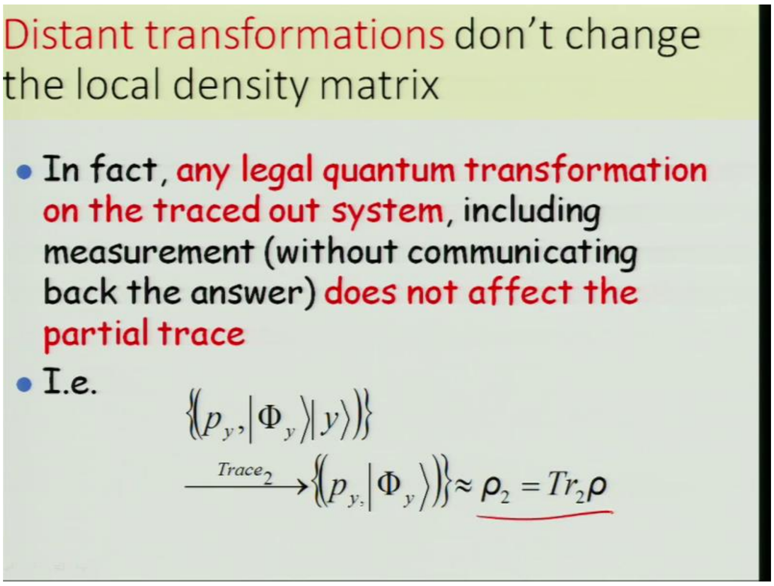

The state also distant transformations do not change the local density matrix also. In fact, any legal quantum transformation on the traced out system including measurement without communicating back to the answer does not affect the partial trace also. So, here is the meaning of that that if we mention it in this format then the partial trace will remain the way it is.

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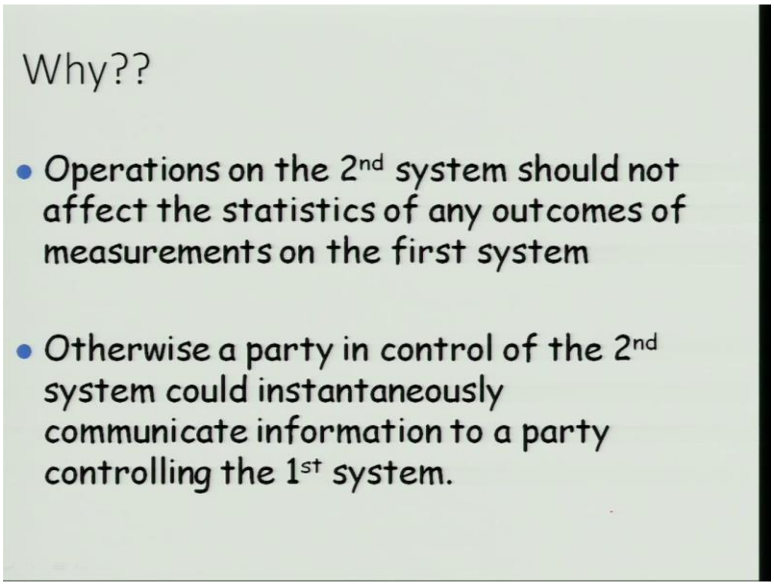

Now, these are because of the fact that operations on the second system should not affect the statistics of any outcomes of the measurement of the first system, otherwise the parity of the control of the second system would instantaneously communicate information to controlling the first system, which would be violation of information control and that is why individual operations that we showed until now essentially has no effect on the other operations as we were discussing.

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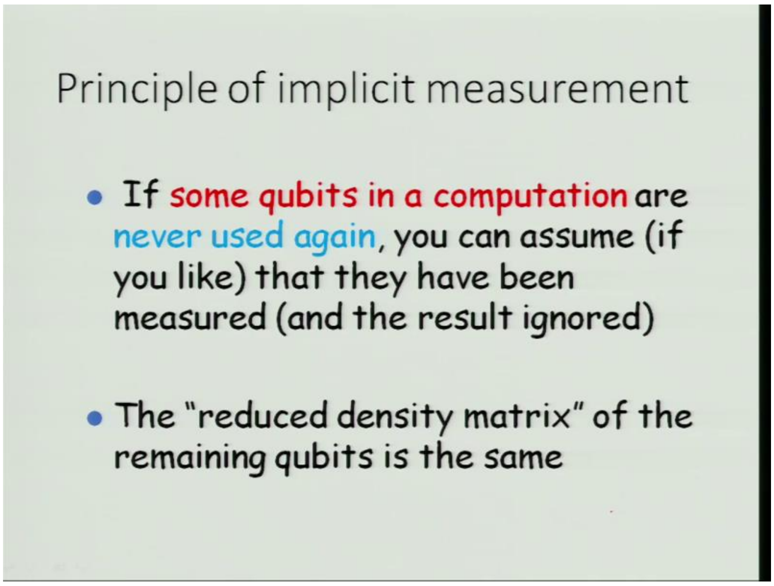

Principle of these implicit measurements lie on the fact that, if some qubits are in a computation are never used again, you can assume if you would like that they have been measured and the result ignored, the reduced density matrix of the remaining qubits is the same.

So, these are the very important aspects of implementation that we have already used while we were doing the different operations that we looked at in the earlier weeks and so these measurements are ratified in terms of the density matrices and their developments and their behaviors as we are discussing here.

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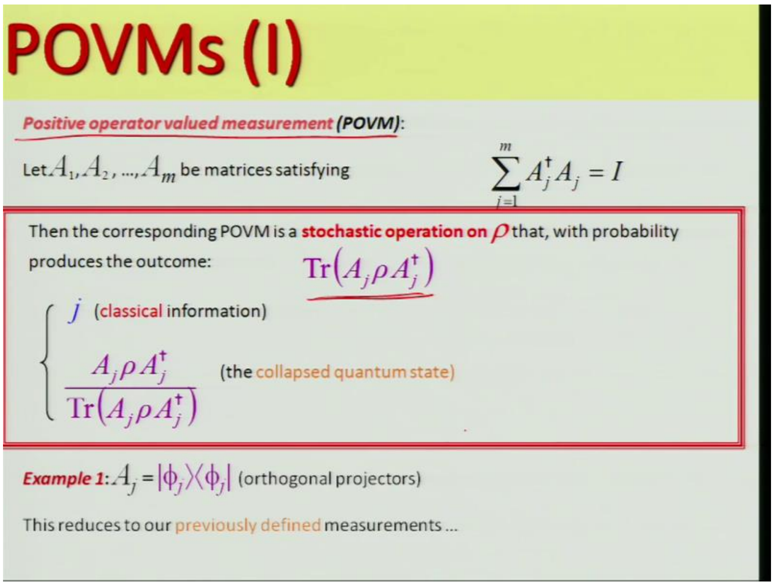

One of the most important measurement aspects which we have utilized all the time is the positive operator valued measurements POVM. If we had matrices which satisfy the form that the adjoined of that products of the adjoineds are identity, then the corresponding positive operator valued measurement POVM is a stochastic operation on density matrix that with probability produces the outcome which is the trace of the matrix and its products with the density matrix and its. So, basically the trace of the observable of the particular matrix and so it leads to the classical information and the collapsed state is the one where it is the product of the jth matrix with the density matrix and its dagger normalize with respect to its states.

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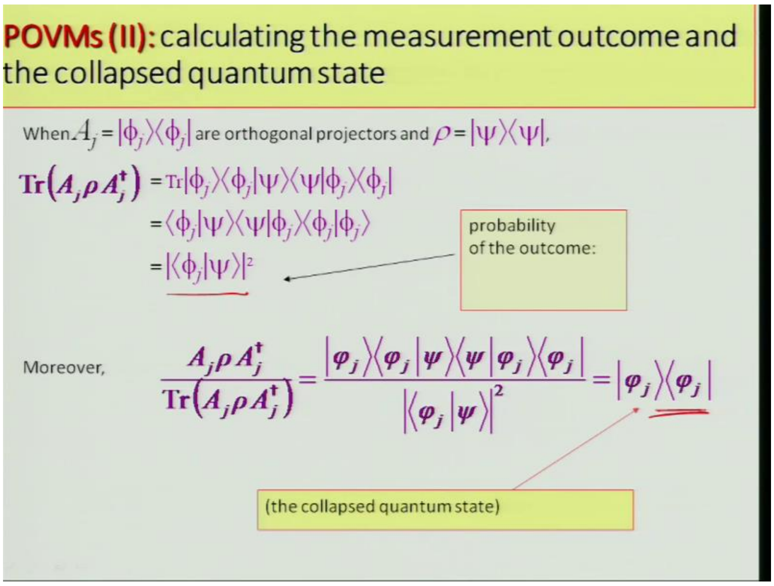

So, for example, if we have A of j which is an outer matrix, outer dot product of phi j which orthonormal projectors this reduces to our previously defined measurements; when A of j is phi of j is outer product are orthonormal projectors and rho is the outer product of psi's then the trace would be the probability of the outcome which has been show here.

And similarly the collapsed state would essentially be the state phi j outer product and that is the collapsed quantum state as expected; because we are basically finding the projection of the collapsed quantum state and its probability. So, this is the reason why this particular line of development is very essential because this is essentially connected to measurements.

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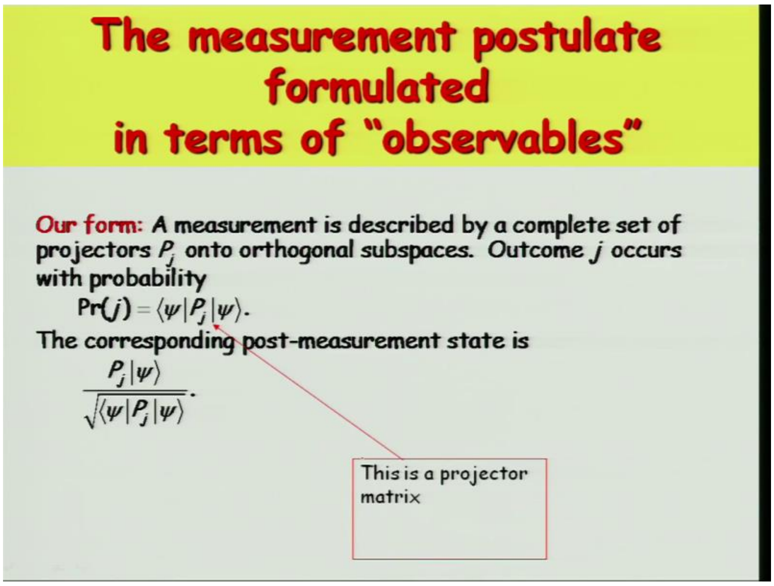

The measurement postulate formulated in terms of observables are exactly in the form that we have been discussing, now our form a measurement is described by a complete set of projectors P of j onto the orthogonal subspace, the outcome j occurs with probability, probability of j which is the inner product of psi with respect to P of j, the corresponding post measurement state is projector of the state with respect to its.

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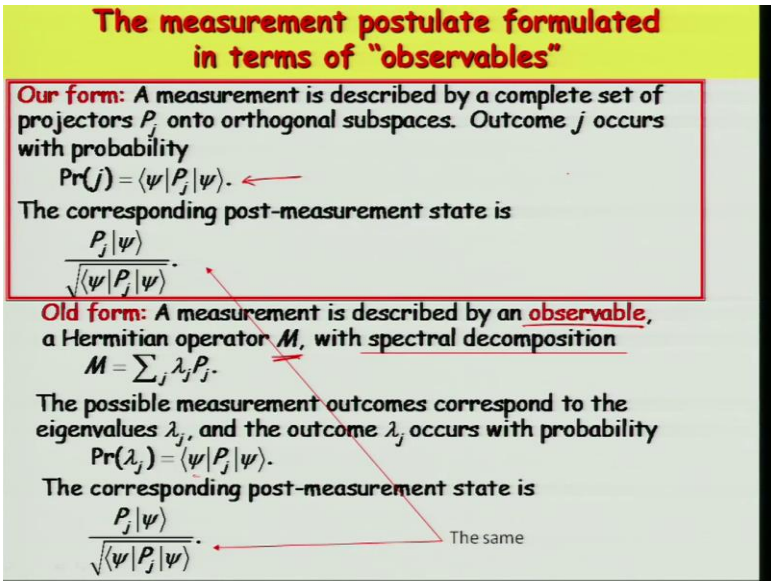

So, this is the projector matrix that we have been discussing earlier, the measurement is described by a complete set of projectors P of j onto the orthogonal subspace, the outcome j occurs with probability j which is given as this form, the corresponding post measurement stat is of this form. The old form of measurement which was being discussed is the measurement is described by an observable, a Haitian operator M with spectral decomposition M which is equivalent to the sum of the diagonal element times the projector.

The possible measurement outcomes correspond to the Eigen value lambda j and the outcome lambda j occurs with probability of lambda j psi p of j psi. The corresponding post measurement state is essentially the same. So, either we look at it in the hermitian operator principle state by using Eigen state properties and Eigen values or in the projector way of looking at it, they both essentially give rise to the same result as is expected either from the density matrix formalism or from the Schrödinger's formalism of solving a Schrodinger's equation.

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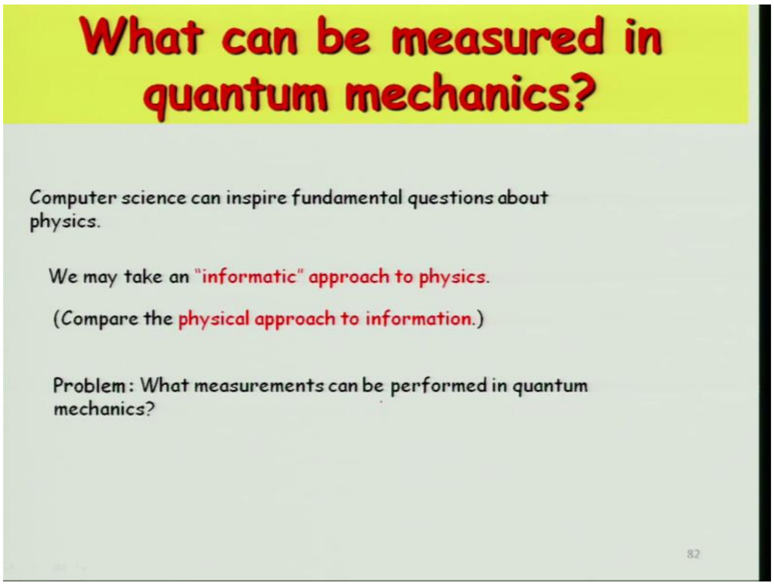

So, in terms of quantum mechanics that we have been utilizing for making the quantum computers and information quantum information, computer science can inspire fundamental questions about some of these we can take a informatics approach to physics, compare the physical approaches to information, what measurements can be performed in quantum mechanics that are of interest.

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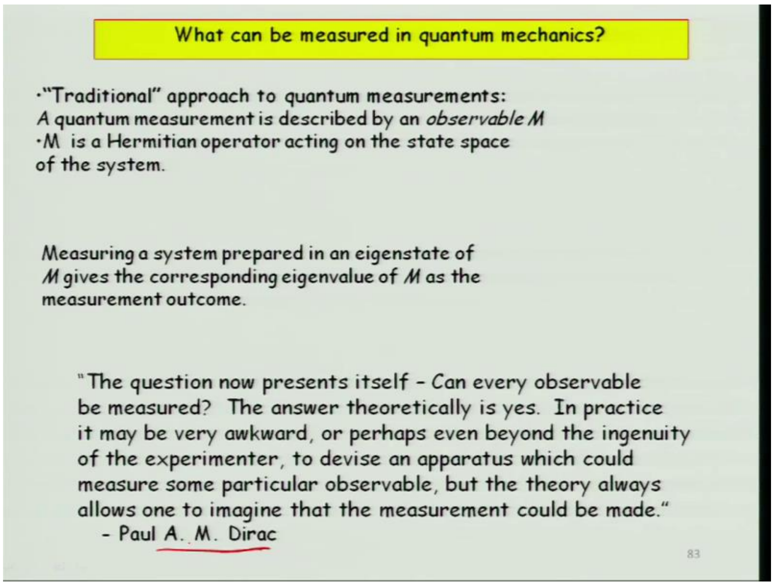

Traditional approaches to quantum measurements is a quantum measurement is described by an observable M. M is a hermitian operator acting on the state space of the system, measuring a system prepared in an eigen state of M gives the corresponding eigenvalue of M as the measurement outcome.

The question now present itself can every observable be measured; the answer theoretically is yes, in practice it may be very awkward or perhaps even beyond the ingenuity of the experimenter to devise an apparatus which could measure some particular observable, but the theory allows one to imagine that the measurement could be made. Now this is a statement made long back by Dirac essentially understanding, essentially pointing out the importance of the concept of measurement and to the real measurement aspect.

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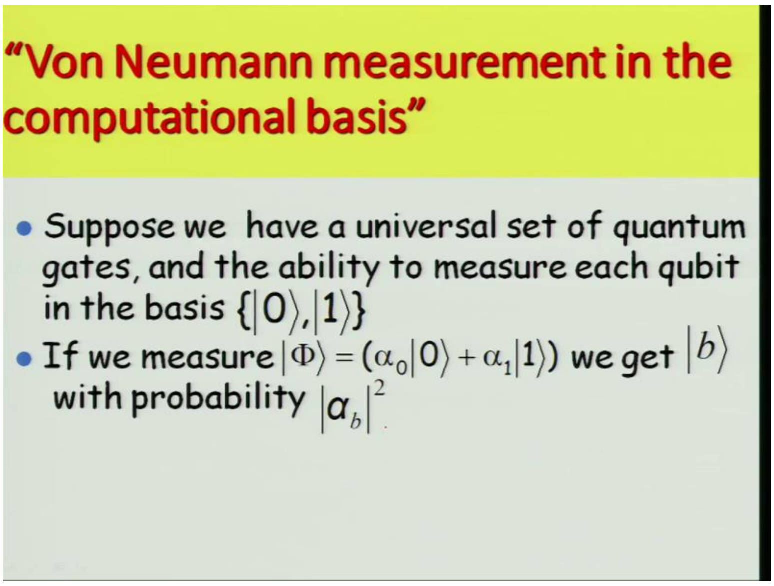

So, one of the most important work in this area has been the Von Neumann measurement aspect that is because that has been related to the idea of entropy and all the states put together, so in that respect if we have a universal set of quantum gates and the ability to measure each qubit on the basis 0 1, if you measure say phi state, we get b with probability alpha b squared as it is expected.

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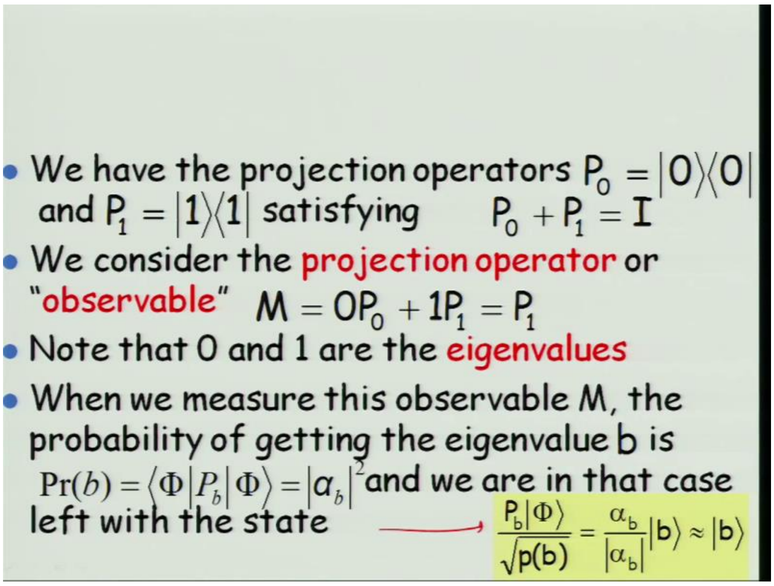

We have the projection operators p naught, which is the outer product of state 0 and p 1 which is the outer product of state one satisfying the sum total is one, we consider that the projector operator or observable to be M which could be written in these terms and we note that 0 and 1 are the eigenvalues. When we measure these observable M, the probability of getting the eigenvalue b is probability b is equivalent to psi bra times, the probability of b is essentially given by this form in is called quantum mechanics and we get it equivalent to the alpha b mod square and we are in that case left with the state which is the P of b times psi over the probability of the process. So, essentially we are left in the state which is given by b.

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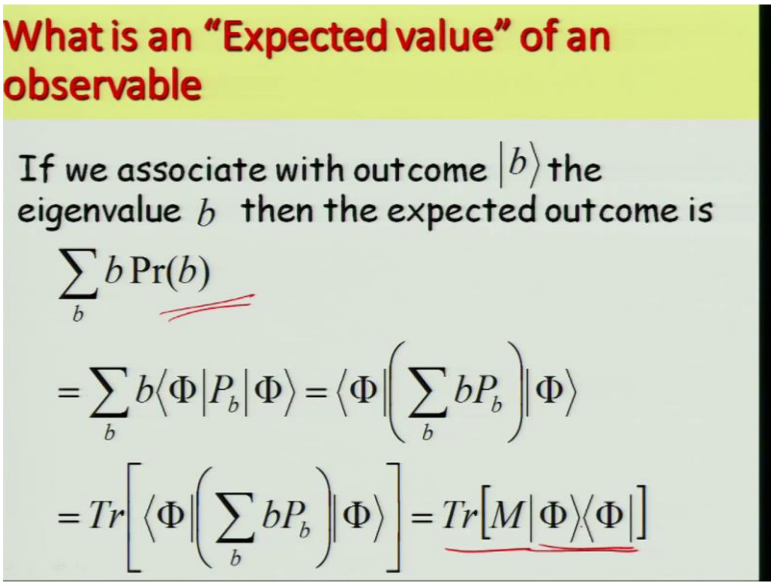

The expected value of an observable therefore, can be associated; if we associate with the outcome b the eigenvalue b, then the expected outcome is given by this summation which can be then related to the trace of the matrix and the outer product of the phi as been done before and the Von Neumann measurements can therefore, give us a universal set of.

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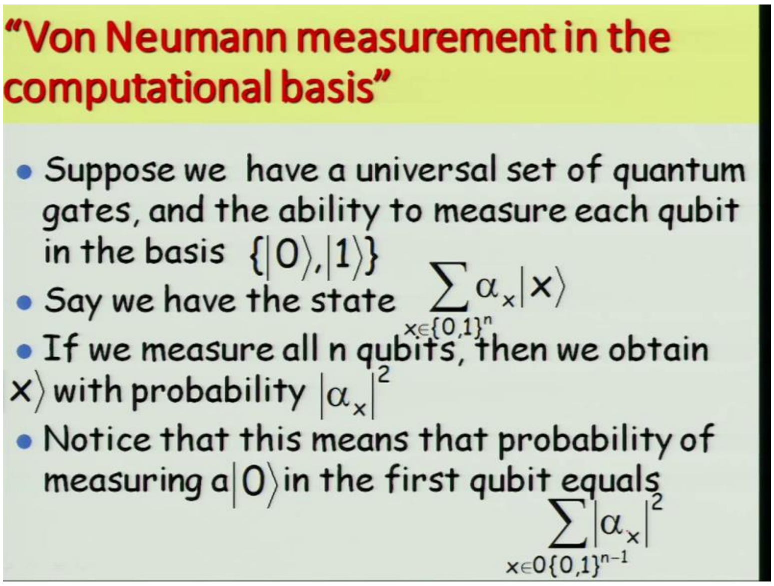

Suppose we have a universal set of quantum gates and the ability to measure each qubit in the basis 0 1, say we have the state alpha x X where x goes anywhere x is an element of 0 and 1 if you measure all n qubits, then we obtain the state x with probability alpha x mod square; we have to notice that this means that our probability of measuring a 0 state in the first qubit equals sum of alpha x mod square for all the values of x or n bits.

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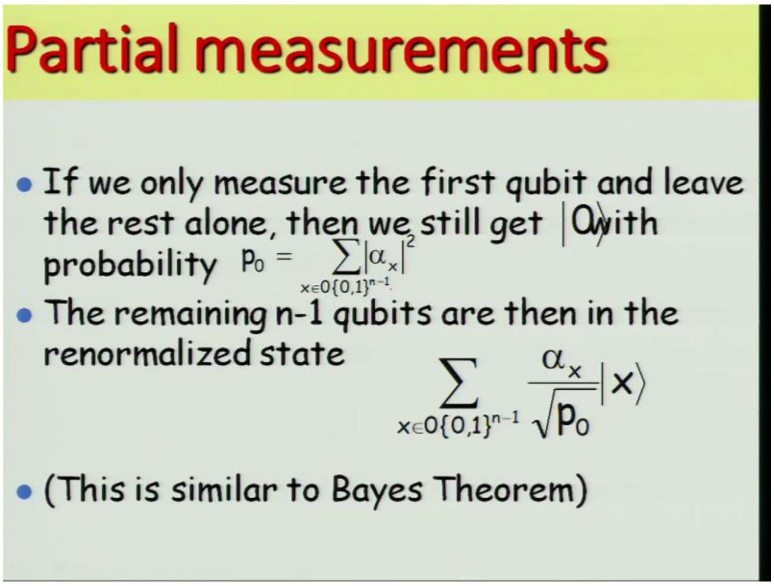

If we measure only the first qubit and leave this rest alone, then we still get 0 with probability of the state 0 with probability P 0, which is a sum of all this states the remaining n minus 1 bits, the remaining n minus qubits are then in the renormalized state.

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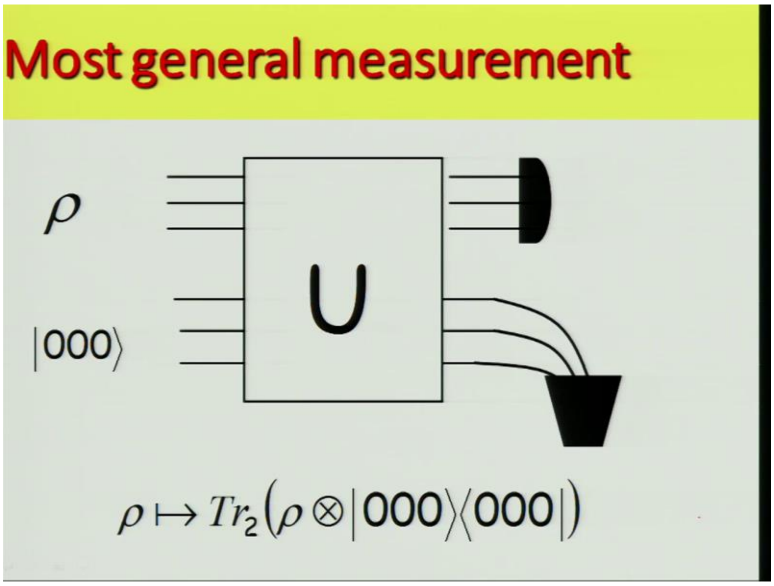

The most general measurements of these kinds can be looked at using a simple circuit, where there is a unitary operator which has the inputs coming from the states and then one of them is making the measurement.

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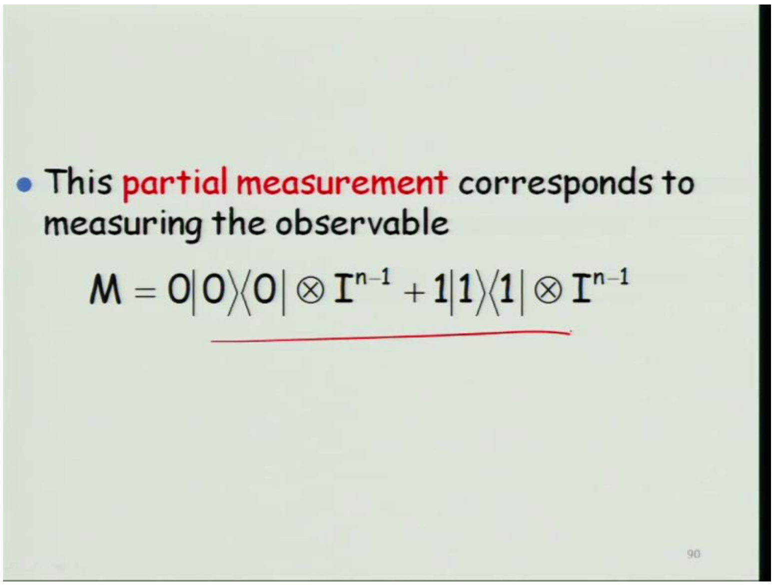

This partial measurement corresponds to measuring the observable M which is of this form.

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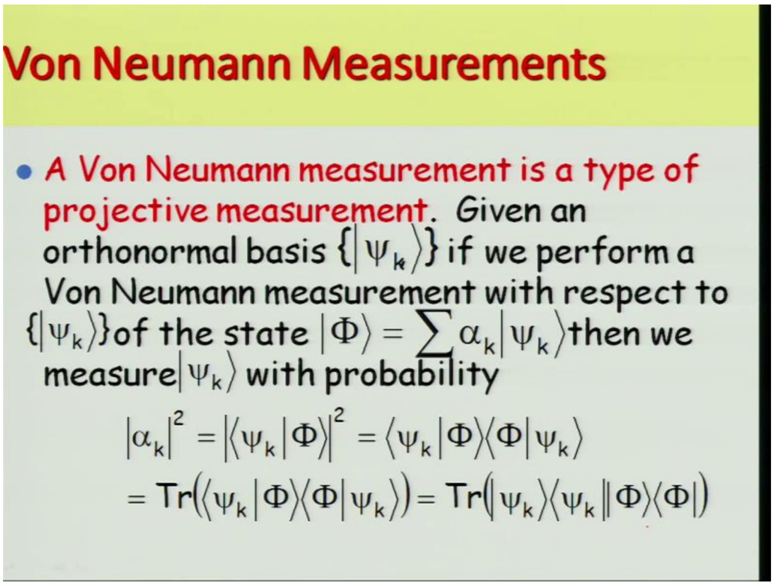

And a Von Neumann measurement is of a type of projective measurements given an orthonormal basis if we can perform a Von Neumann measurement with respect to phi k of the state with respect to psi of the state phi, which is given as alpha k psi then we measure psi with probability alpha k squared mod squared over this and that can be written in terms of the trace of the entire process.

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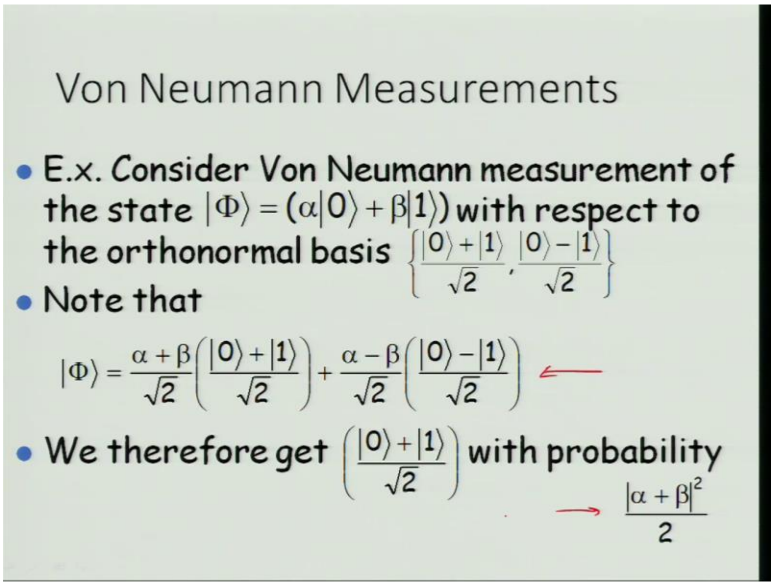

For example; if you consider Von Neumann measurement of the state phi with 0 1 with alpha beta probabilities with respect to the orthonormal basis, which is super position of

0 and 1 plus and minus we note that we get phi in the basis as which can be transformed to writ write in this form, we therefore get 0 1 1 root 2 in that basis, the probability of alpha plus beta whole squared over 2 and this is an exercise which we had looked at earlier in one of the examples that we had done in earlier lectures.

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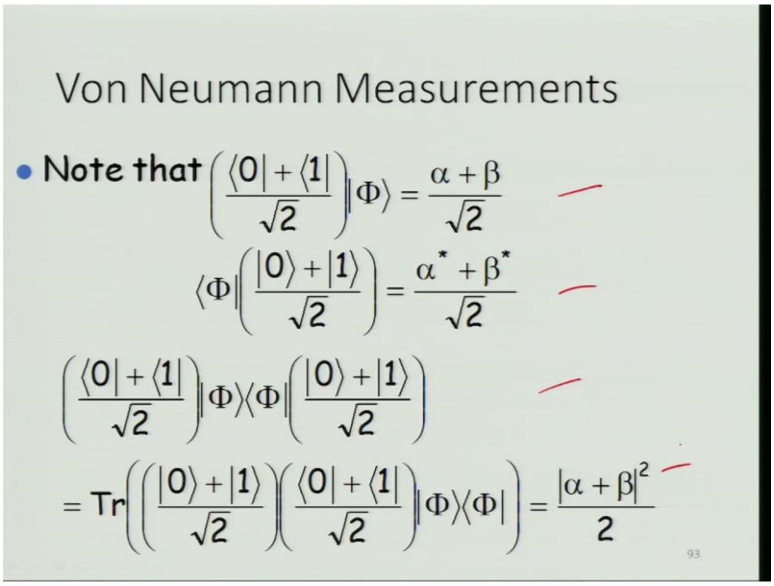

So, we note that this projective cases can be measured in this form and here is just the way of showing how this can be measured here is the little math, which you might have done as a result of the problem that I had given you in your one of the exercises.

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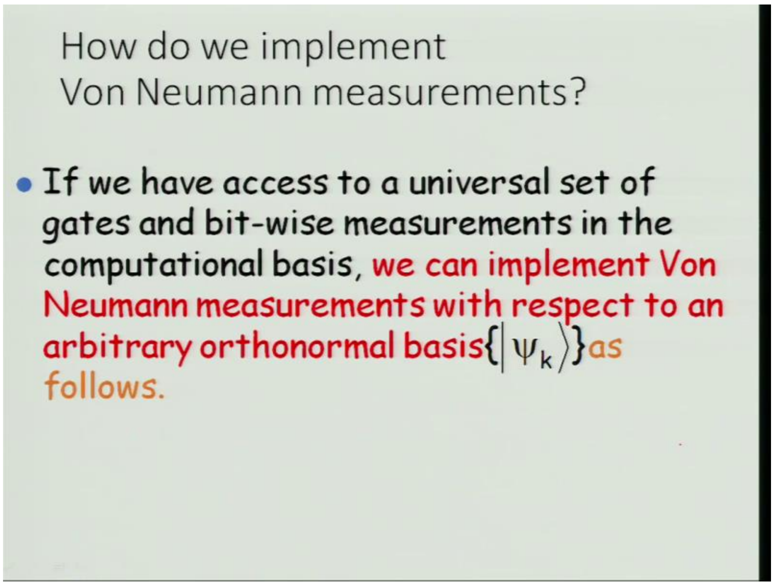

So, I think the last point we would like to point out is how do we implement Von Neumann measurements, which is what we have essentially done when we were doing all our quantum computing implementations. If we have access to a universal setup of gates and bit-wise measurements in the computational basis, we can implement Von Neumann measurements with respect to an arbitrary orthonormal basis psi k as follows.

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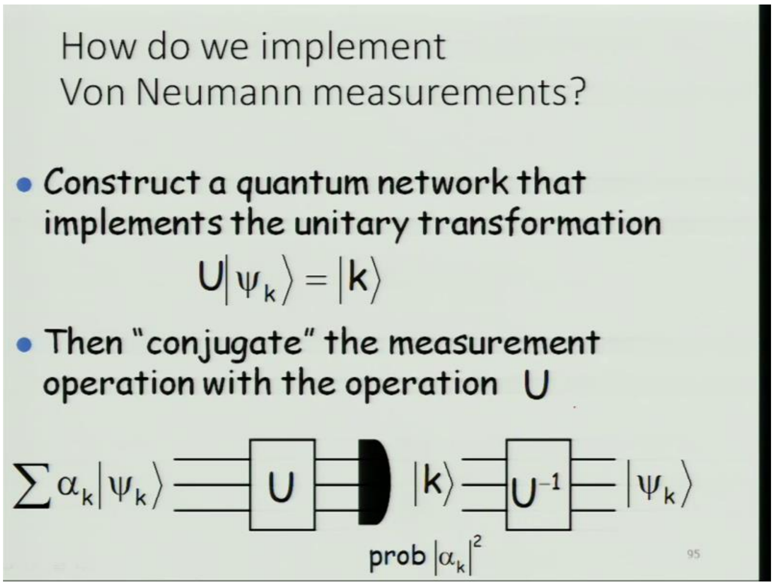

We can construct a quantum network that implements a unitary operation psi operating on psi k to give rise to a k ket vector, and then conjugate the measurement operation with the operation U which is a unitary operation. So, I will get a probability of alpha k mod square.

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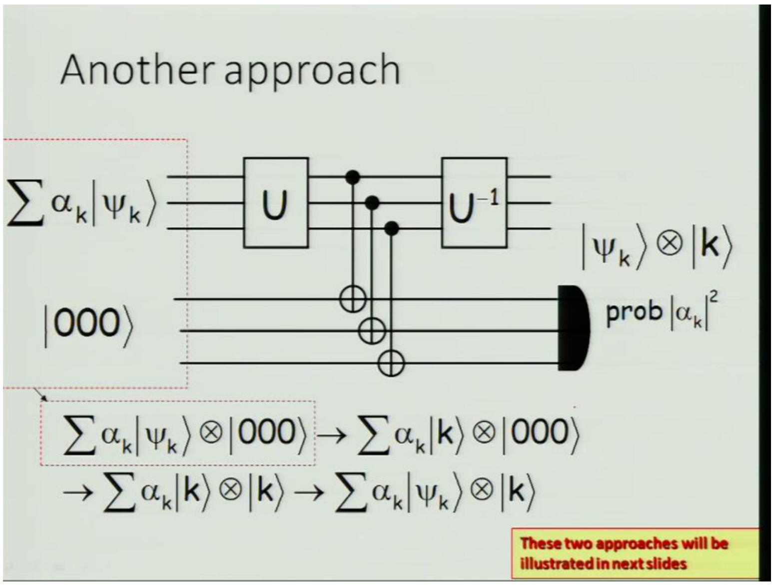

Another approach would be to have the circuit that we had shown in partial earlier to have both the inputs in 1 sen have the entangled state come in and have a final measurement of this type, both of them will have probabilities of alphas k and these 2 approaches will be discussed in terms.

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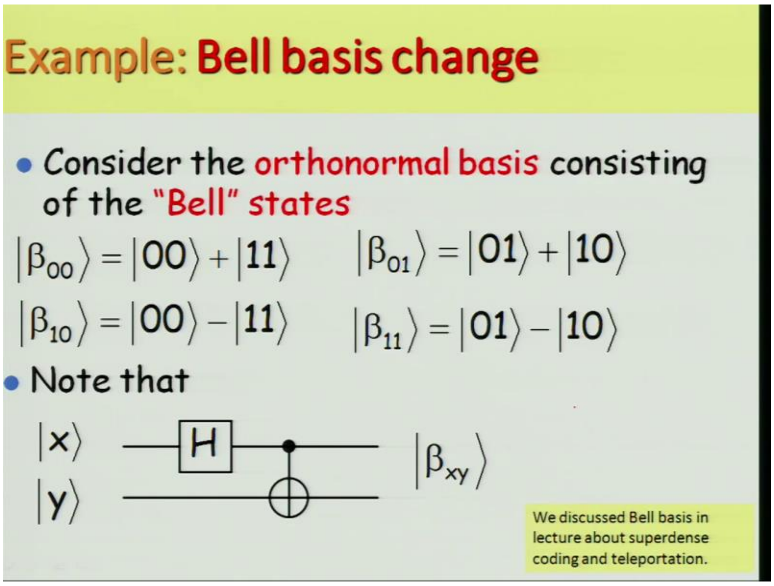

So, both of them have the same probability of alpha k squared. So, these approaches can be immediately connected to the bells inequality and the bless states that we have, it is one of the cases where we are able to connect these to the studies that we have done earlier in terms of entanglement and measurement of teleportation and states like that.

So, in the next slide if you just observe that when we take the orthonormal basis consisting of the bell states and we apply the same formalism that we have been discussing, we have discussed bell basis lectures and other things in the earlier lectures.

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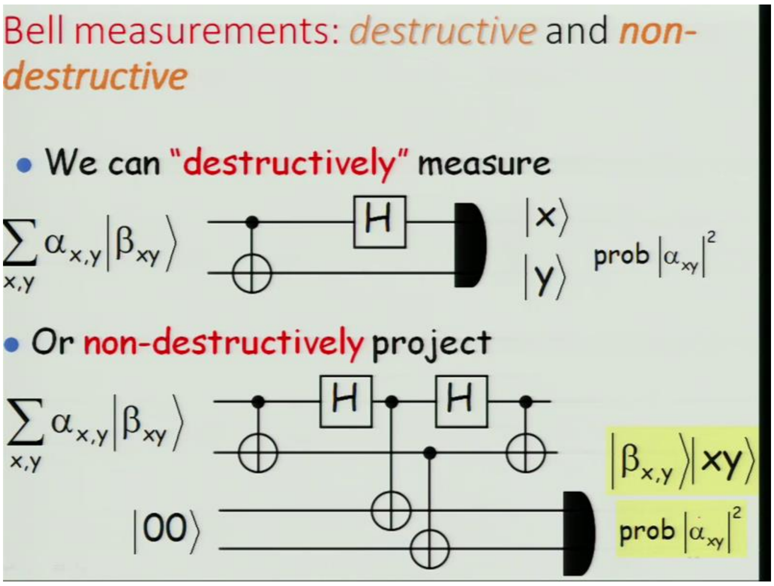

We can have destructive and non destructive measurements and we will finally, end up getting the results that will give rise to in this terminology that we have developed or discussed in this entire week about density matrices.

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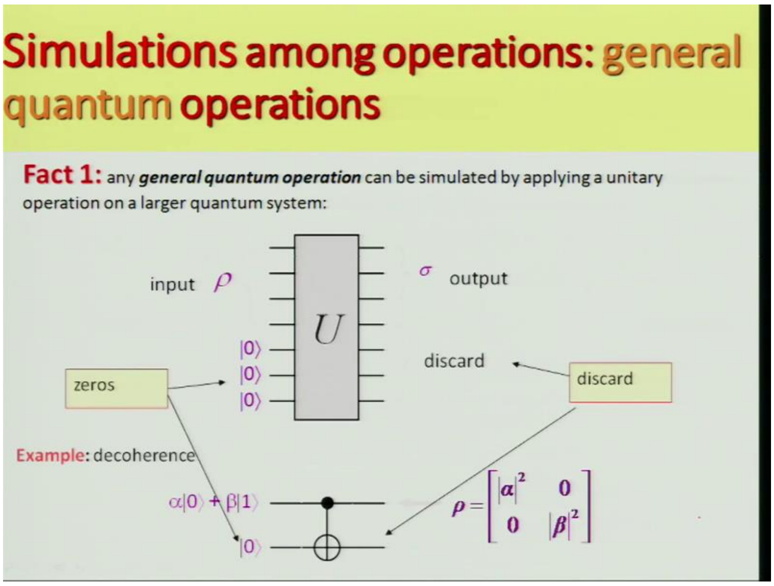

Tracing of partial nature which will give rise to the most general nature of results and this would give rise to the general cooperate quantum operation that can be simultaneously be used by applying a unitary operator of larger quantum system, which helps us in terms of discarding the parts which can give rise to difficulties in terms of the Decoherence and yet we can get to the results that we are interested in and get the right solutions and so this is a much more powerful way of generalized process, where density matrices can play a major role.

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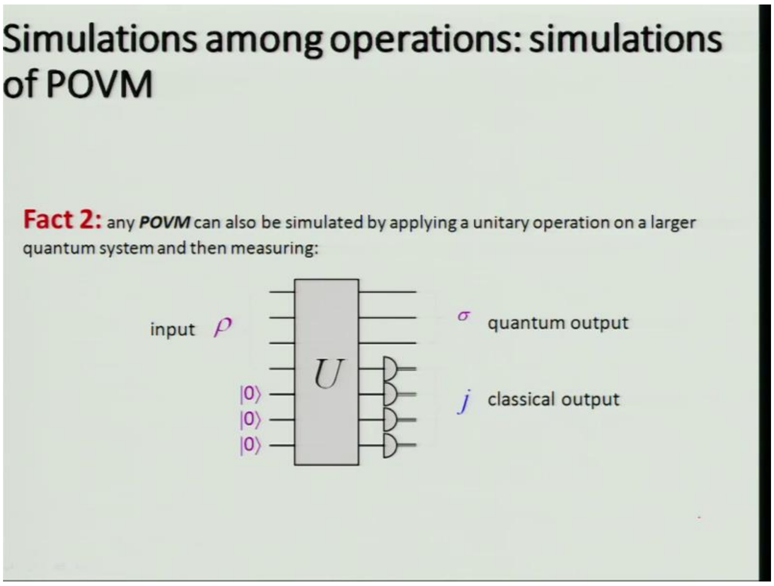

Another very important aspect is the POVM measurements which we introduced here, any POVM can also be simulated by applying unitary operation on a larger quantum system and then measuring it and whatever we get results which have their difficulties can be coupled with the classical inputs which are not going to be utilized and we can keep the preserve the quantum nature of the story by this fashion.

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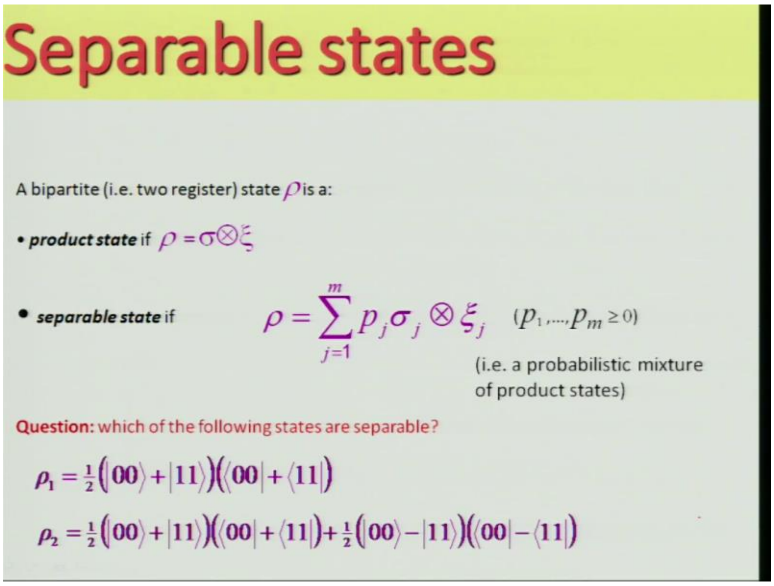

Any bipartite system for instance what we have done can be looked at in the manner that we have discussed, if they are separable then their probabilities are going to be such that we will be having a probabilistic mixture of the probabilities of the product state as we present it here.

How the nature of quantum computing goes we have looked at entanglement, where the last part that we just discussed the separability and others, in terms of density matrices have been very importantly looked at earlier and we were able to know which are the states which can be separable and which are which cannot be separable that are the entangled states or the for example, bell states and so we have looked into all these aspects and in this week we basically presented it in terms of density matrices, because that is one of the most important ways of looking at all the realistic quantum problems that we have looked at in terms of implementations, because the way the realistic measurements goes the aspects of Decoherence and other issues are inherently present and they need to incorporate density matrices to be able to address that.

So, I hope it has been a good learning experience and bridging the gap of the parts which we had earlier not looked into in detail. So, I thank the students who had prompted me to go back to the section and revise it. In the last week which will be upcoming we will be doing a summary and revision of all the aspects that we have looked into are covered in this course and I suppose it will take quite a bit of time or hours, but I have left sufficient time for the final week to cover almost all the discussions at least in summary for all the implementations of quantum computing that we have been doing until now. And we will also look into all the problem sets that we have been giving you as exercises at that last final lecture to make sure that everything is complete as far as the course goes.

Thank you, see you next week.