Implementation Aspects of Quantum Computing
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Lecture – 35
Understanding the Ensemble of Qubits from Density Matrix

We have been looking at density matrices, as it is 1 of the most important aspects that helps us in implementing quantum computing and quantum information processing. And in the first lecture of this week we have looked at some of the basic fundamentals of density matrices which have been implicitly use by us, but it was important for us to start off on this area to make sure that the understanding is complete in this field.

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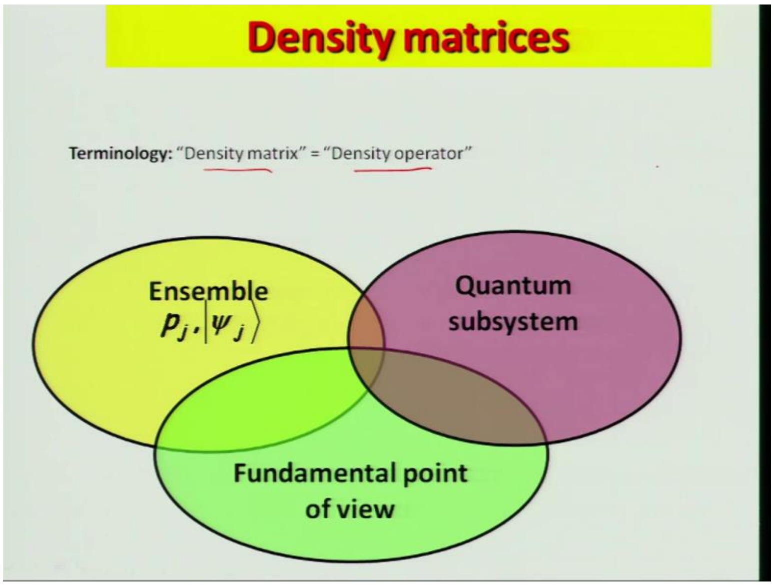

So, on going forward we want to just ensure that we are in this topic of density matrix where we are using reversibly the terminology of density operator as the same as density matrix. And we are looking at statistical ensemble where the quantum sub system these being treated with some probability of the way functions that essentially create the subsystem quantum sub system. And this is a very fundamental point of view as it enables us to understand the system even though we have we have difficulty in gaining information about it.

So, in some sense for pure states, we know that we can find out the expectation value of the system by applying; it helps us to know about the system even though we have very little knowledge about the system.

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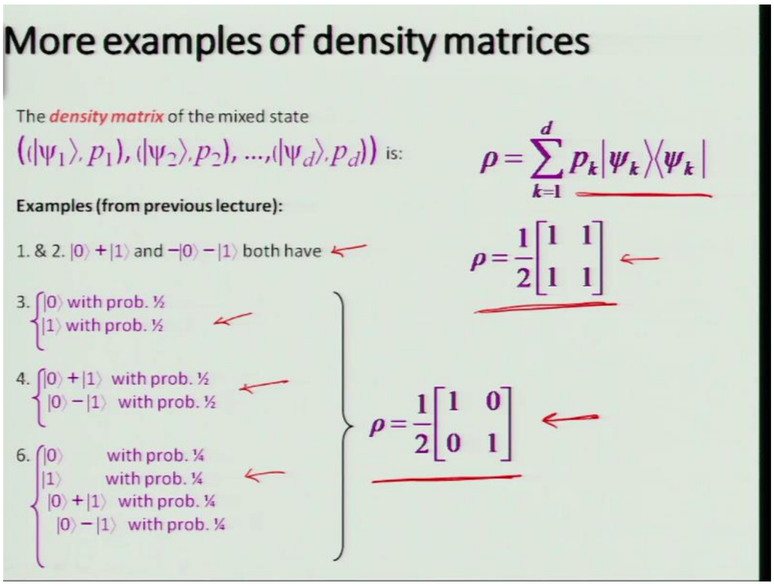

So, going ahead with this principle of the density matrices; here are some of the examples that we have been starting to look at. The density matrix of mixed state can be written as a sum total of the in a product and the probability of observing that particular state.

From previous lecture we have few examples that we had looked at; a super position of 2 states whether they have a face factor would give rise to the same way density matrix. So, whether it is ket 0 plus ket 1 or ket minus 0 minus ket 1 both of them will have the same density matrix. Similarly with ket 0 and ket 1 with probability half, ket 0 and ket 1 with probability half super position of ket 0 and 1 with probability half, super position when the in mutual sin is minus with probability half versus a super position of ket 0 ket 1 0 plus 1 and 0 minus 1 each of probability quarter will all have the same density matrix half 1 0.

So, given this definition when we look at these different density matrices we realise that the properties of these 2 are indistinguishable as per as the density matrix is concerned. Similarly, the properties of all these sets combinations where all of them could occur having a total density matrix which can be given rise as a result of any of these 3 combinations 3 4 and 6will all have the same density matrix. However, when they all have the same density matrices they cannot be having distinguishable property in terms of density matrices. So, all the properties (Refer Time: 04:15) density matrices would only work on the rho and as such all the 3 cases; 3, 4 and 6 will behave similarly.

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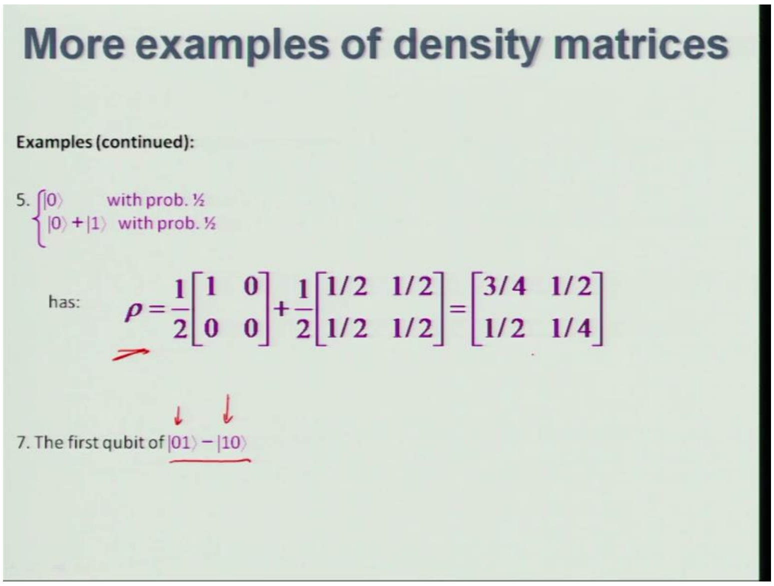

Again a case where we have ket 0 with probability half and ket 0 and 1 with probability half would have a density matrix, which would be given by rho half this plus rho half of each of them half of which give rise to a total density matrix of this kind. Now these all these individual density matrices say and this particular density matrix, they are all distinct and they will all be possibility be distinguished. And similarly taken all be looked at a independently and they will have different values.

We can also have another way of looking at density matrices. For example, if we have a combination where we have a state which is mix of 2 different super positions states, and in that case if we are measuring one of the qubits say we measure the first qubit of this combination, then the first qubit probabilities which will be 0 will be will becoming only form here and first qubit of this which will be having a probability of 1 will only come from this contribution.

So, they will have distinct contribution. We have actually dealt with these kinds of cases earlier, when we were discussing the aspects of implementation and developing the principle of quantum information and computing. So, they all very relevant and it is important to note that in most cases the information of anything which is not pure can only be determined when we are looking at the density matrices.

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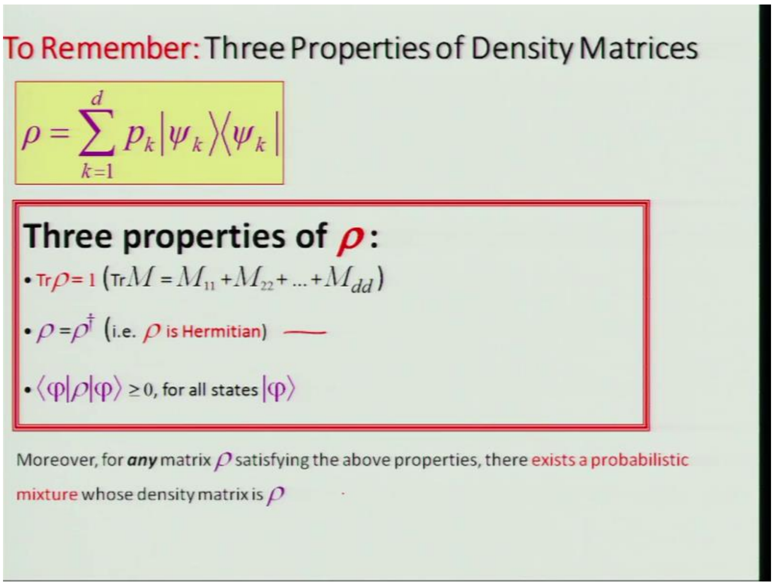

So, there are 3 properties of density matrix which are important; which is that trace of rho is going to be 1 always, which means that it is a normalize case and that is because the trace of the density matrices essentially the total probability of the system. So, it is a sum total of the diagonal elements, it is the density matrices going to be Hermitian always.

The projection of the density matrix along any of the composite states will always be either 0 or more than 0. Essentially stating that the projection of the density operator any of the states that constitute the density matrix is either going to be 0 or more than 0 as that they represent the probability of the occurrence of or the weight age factor of the contributing state. Moreover for any density matrix rho satisfying the ever properties there exist a probabilistic mixture whose density matrix is rho.

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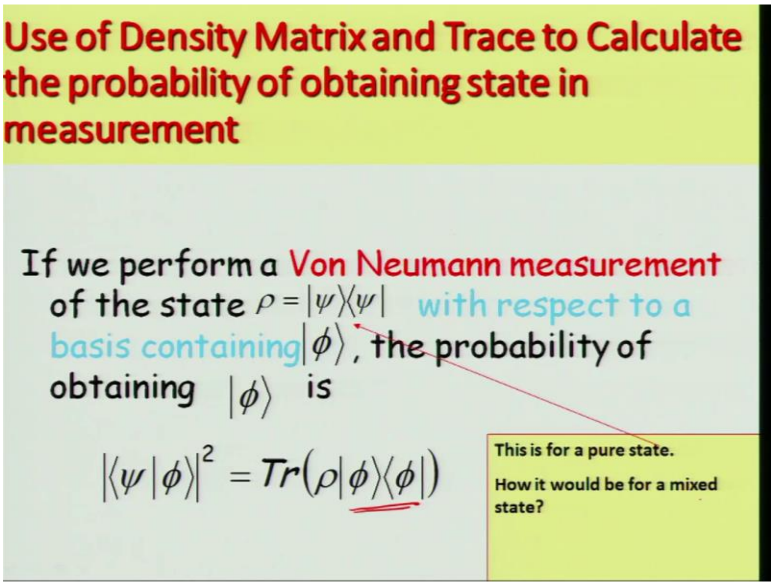

So, we can use density matrices and trace to calculate the probability of obtaining state in measurement. So, if we perform a Von Neumann measurement of the state rho with respect to a basis containing get phi.

The probability of obtaining get phi is given by bracket of psi rho square, which is the probability and that happens to be the trace of rho times in a product of phi. Now this is for a pure state, this is the definition for a pure state question could be how it would be for a mixed state and as we have discussed in the last class it will be the same.

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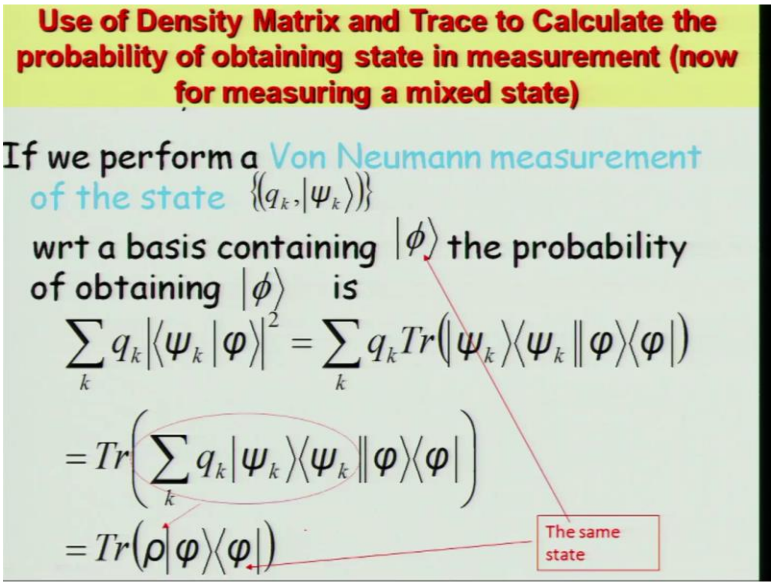

So, this we have looked at in the last class this is just to remind you that essentially it is the same thing. So, if we use the density matrix and trace to calculate the probability of obtaining the state in the measurement, we basically for measuring the making the measurement for a mixed state, it will be the same he will always get the same state. So, if you perform a Von Neumann measurement of the state with respect to the basis containing phi get, the probability obtaining the phi get is going to be given by same probability which is the outer products square time it is a weightage and we can get the trace reduce to the same function same form as we had for the pure state.

Which means that this is a is one of the biggest achievements of density matrix is that it enables us to look at mixed state in the same mixed state and provide information which is otherwise so difficult to get.

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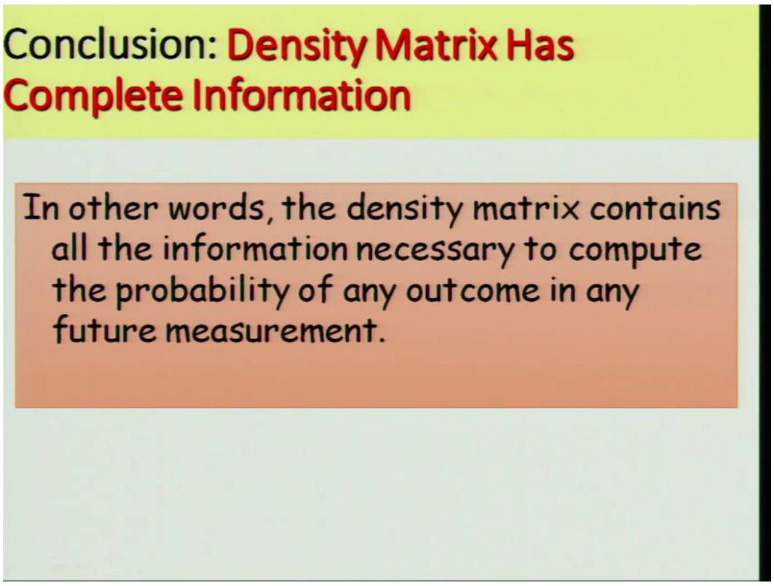

Conclusions that we have achieved as a result of discussions that we did in the last lecture and now is that the density matrix has the complete information and otherwise the density matrix contains the information necessary to compute the probability of any outcome in any future experiments.

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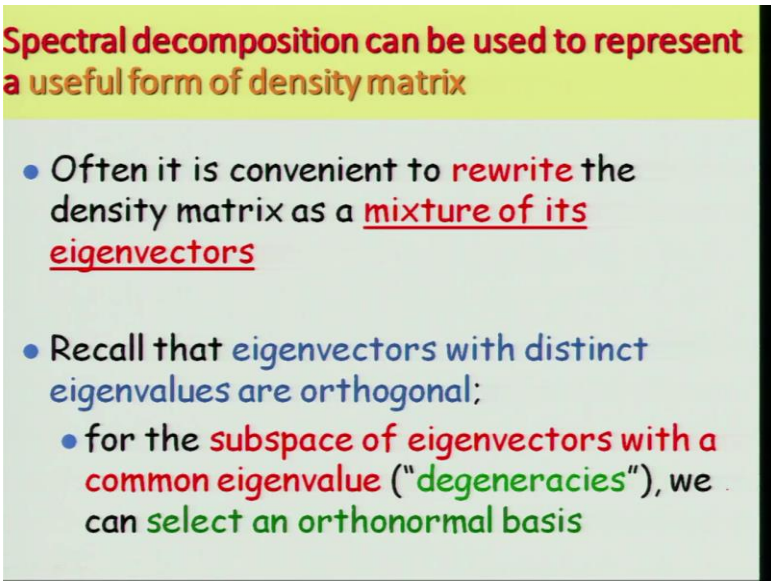

So, spectral decomposition can be used to representing useful form of the density matrix; often it is convenient to rewrite the density matrix as a mixture of eigenvectors. Eigenvectors with distinct eigenvalues are orthonormal that is one of the fundamental requirements of quantum mechanics.

For the subspace of eigenvectors with a common eigenvalue, which are degeneracies we can select an orthonormal cases.

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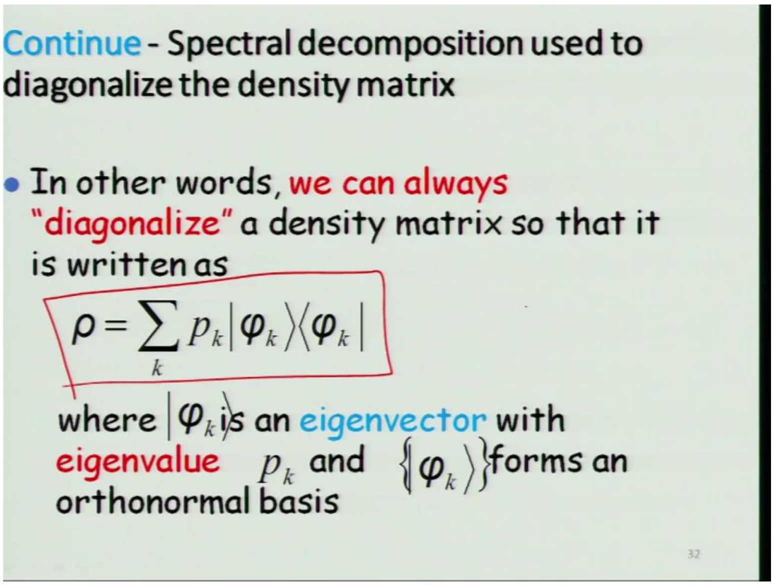

In other words we can always diagonalize a density matrix. So, that it is written in terms of this form where phi k get is the eigenvector with eigenvalues of P k and phi k set forms an orthonormal basis. So, this is basically the definition of density matrix which we have been following. It is a very useful definition because this is one of the ways to get information of states which I have otherwise not possible as we have been discussing.

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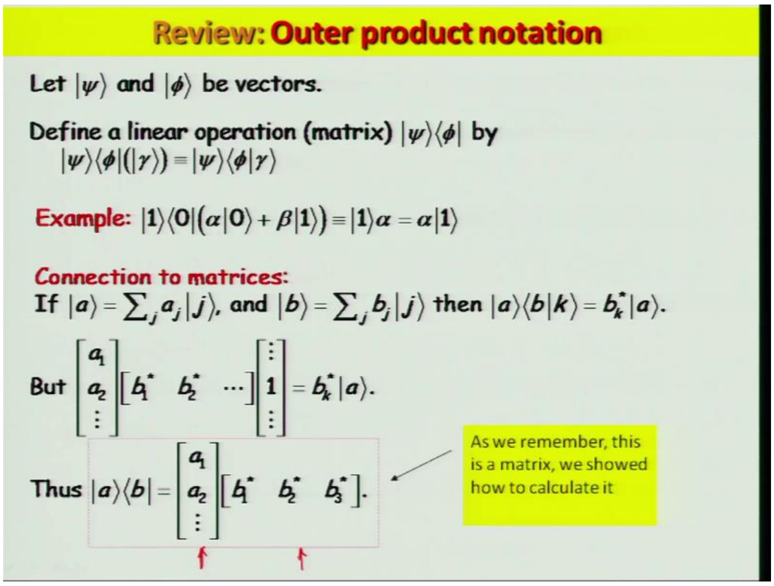

So, we have been using the notation of the matrix notation as well as the vector notations interchangeably. So, it is important to have a small review of the product notations here. So, this is an outer product notation where we show how the wave function psi get and phi get which have vectors, and if we define a linear operation matrix by this form which is more commonly also known as the outer product notation, then we get his particular form where you can write it out, and in the simplest form we can essentially get a projection or get the contribution of the particular state in the super position.

And this can be connected to the matrices by utilizing the fact that these are connected by using their coefficients and that can be written in this kind of a form in bra versus the ket forming the rho versus the column matrices; so the rho is the ket and the bra is the column and by utilizing these principles we can actually write out that the outer product of 2 vectors a and b is simply the multiplication of a column matrix with a rho matrix. The bra being the rho matrix which is the, and get being the column matrix and the bra being the one which is complex conjugate notation.

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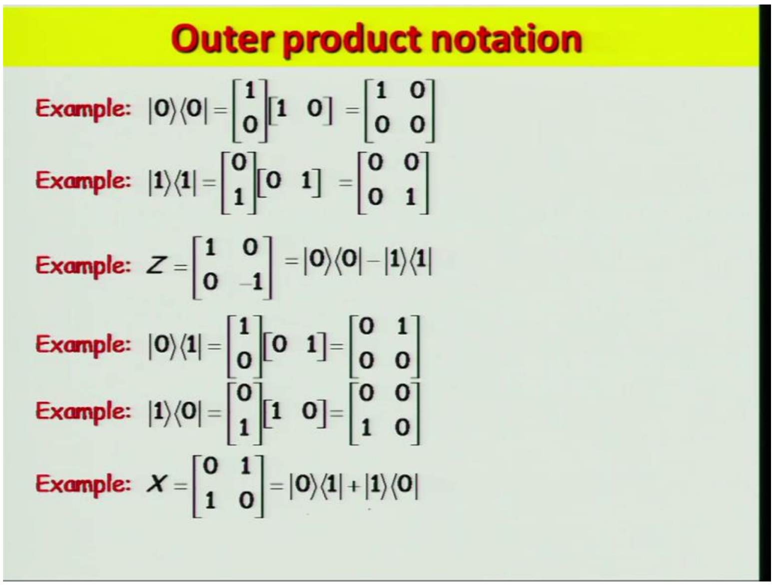

So, when we use this outer product notation we are able to get these different examples where we can see that the 0 0 is a diagonal matrix of 1 0 versus 1 1 is a diagonal matrix of 0 1, and we can have operations which would essentially mean the gate.

So for instances, Z gate is nothing but in operation where we have the outer product 0 0 of and minus of outer product 1 1 would give rise to the j ket; the outer product of 0 and 1 would give rise to the half diagonal element and similarly the one with 1 and 0 would give the other of diagonal element and that way we can also have the X ket which can be the super position of the 0 1 and 1 0. So, the X ket could be the super position of 0 1 and 1 0 inner products.

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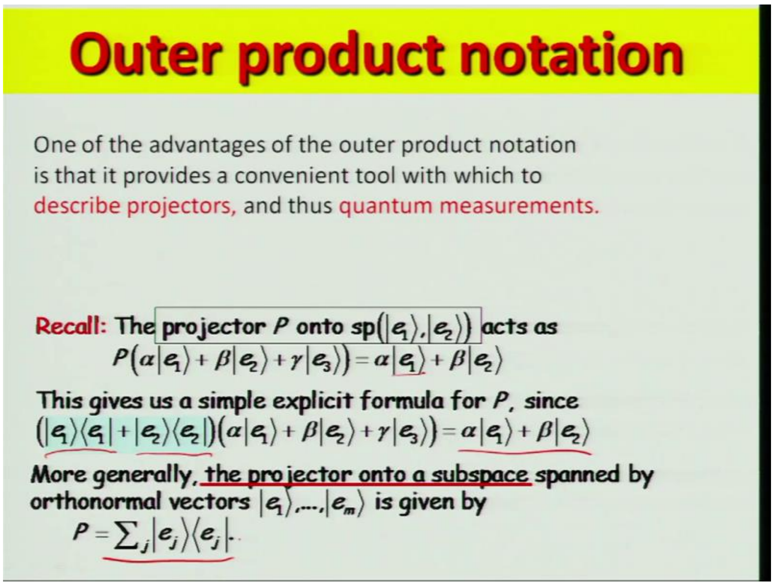

One of the advantages of the auto product notation is that it provides a convenient tool with which to describe the projectors, and thus the quantum measurements.

So, if we recall that the projector P on to space e 1 e 2 acts as the P applied on to alpha e 1 plus beta e 2 plus gamma e 3, it will be essentially alpha e 1 and beta e 2. This gives a simple exhibit it formula for P, since we know that an outer product form of the 2 would give rise to e 1 e 1 and e 2 e 2 coming together in the outer product from a super position of that would give rise to the projection operator. More generally the projector on to a sub space span by the orthonormal vectors is given by this projector, where which is generally they outer product of the 2 vectors e j and e; e j summed over all of them.

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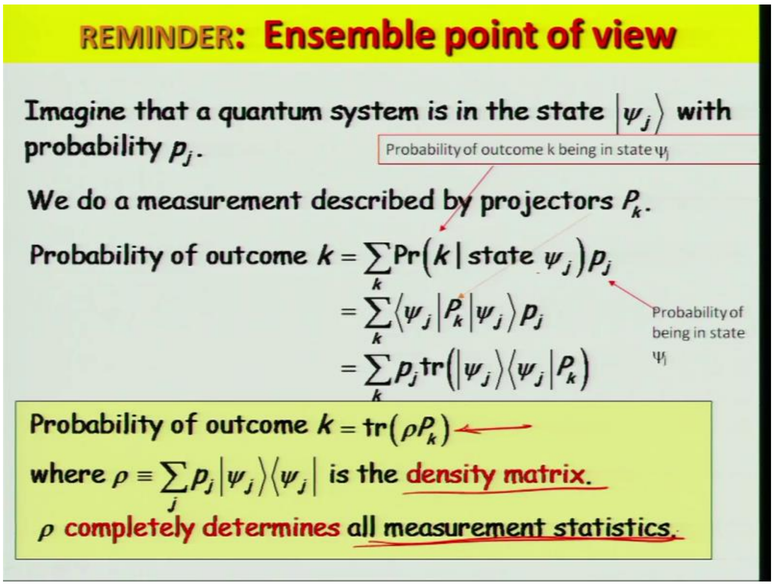

We should also revise the ensemble point of view in this particular context which can be looked at in this way.

Once we imagine a quantum system in the state psi of j with probability P of j; we can do a measurement which is described by projectors P of k. The probability of the outcome of k being in state psi j can be written in terms of this particular form probability of k times state psi of j P j with probability of being in that states with P j can be written in this particular form, so this is in the in terms of the projectors. So that can be rewritten in terms of the trace of the property and then we have the very important aspect where we have the trace rho times the projector P k, where rho is again P j psi in outer product of psi j is and this is the density matrix. So, rho completely determines all measurement statistics.

So, that is one of the reasons once again the measurement statistics so important and so understanding when we do ensemble measurements, because rho basically gives rise to all the results that we are interested. So, this is a high lighting result and so this is something which should be remembered.

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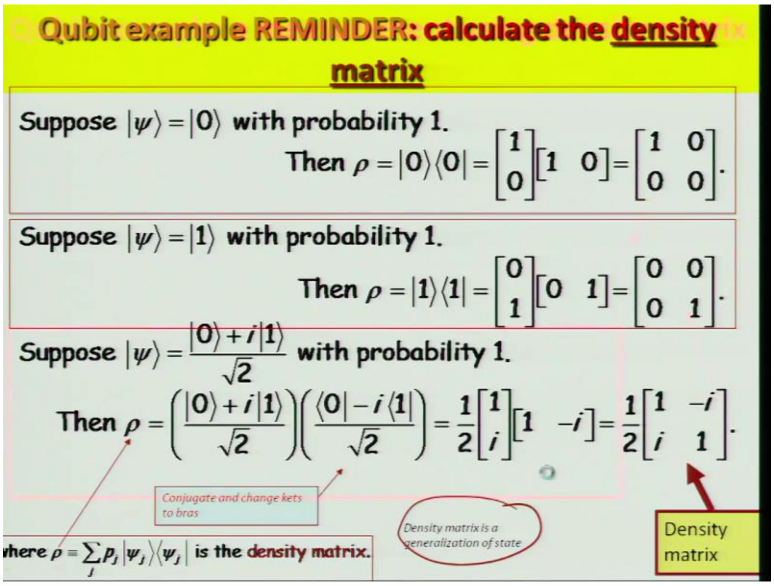

So, let us now connect it to the qubit case. So, we can calculate the density matrix in this form; suppose we say that the pure state of psi 0 with the full probability, then our probability density matrix is outer product of zeros which give rise to this particular density matrix; if we have the wave function representing the pure state 1 then we have the other case where it is rho 1. So, if we have the wave function which is a combination of say 0 and i of i, i of 1 with root 2 with probability 1 then the rho would be a conjugate of these 2 times where we change the bras and the kets we get the negative sin

Then we will be having a density matrix which we look like this. So, density matrix is generalization of state and it can be looked at for any particular state that we are looking at. So, we can calculate density matrix respective of how the particular state looks like: pure states as well as states which have different contributions all give rise to values which can be represented by density matrices.

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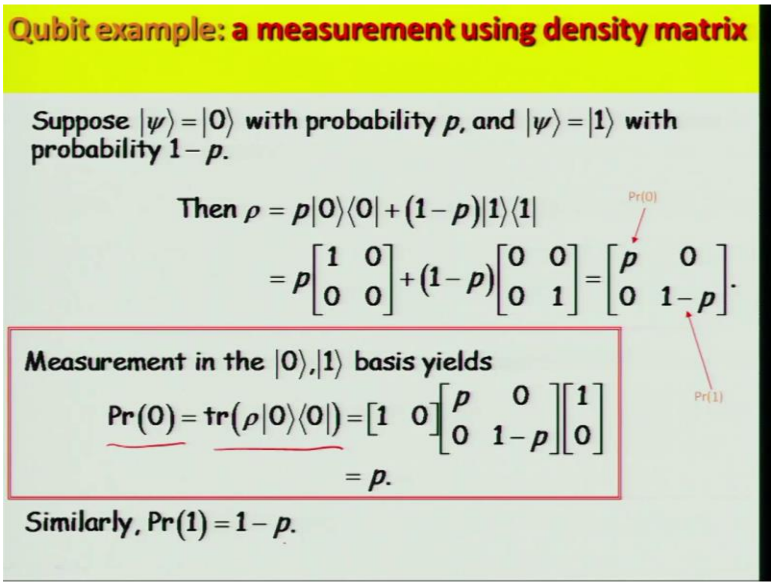

Now, if we let the probability of measuring the pure states 0 with P and pure state 1 with probability 1 minus P, then we can find that the density matrix would be represented in terms of the probabilities of these 2 and that trace has to be 1 so the probability of state 0 is along the diagonals with P and 1 minus P.

So, the measurement in the 0 1 basis leads to the probability which can be looked at with respect to this and so the probability of 0 simply is the trace of rho with respect to the outer product of 0 states and we get the correctly the value is P as our probability and similarly when we do the same with for 1 we get 1 minus P as expected.

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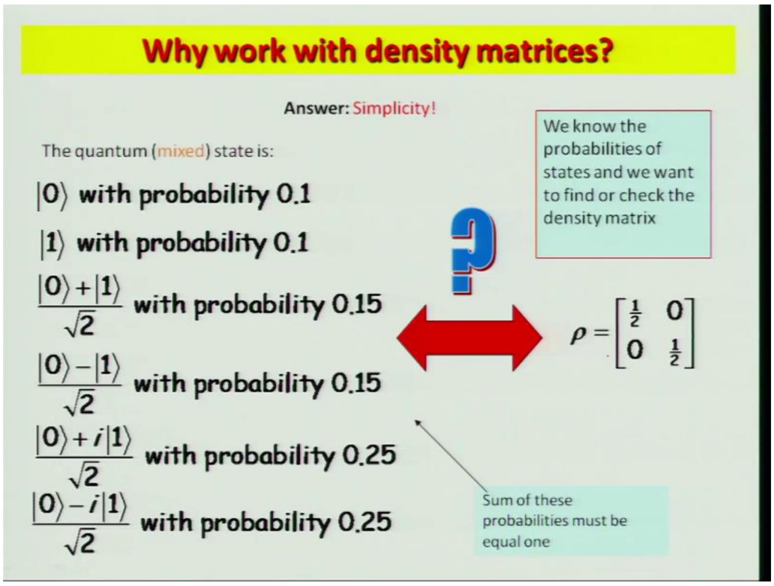

So, the advantages I have been mentioning is the simplicity of the overall process the quantum states which are mixed that can be looked at as a result of density matrix notation. If we have an arbitry some of states which are with all these probabilities as long as the some of these probabilities has to be one as we know.

We can have a condition where we can said them up and you can write they out the density matrix; we know that the probabilities of the states have been one to find out check the density matrix, we can always write it down with each of the probabilities for each of them to find out the density matrix and as long as it is only 2 states involve it will be a 2 by 2 density matrix and so on and so forth.

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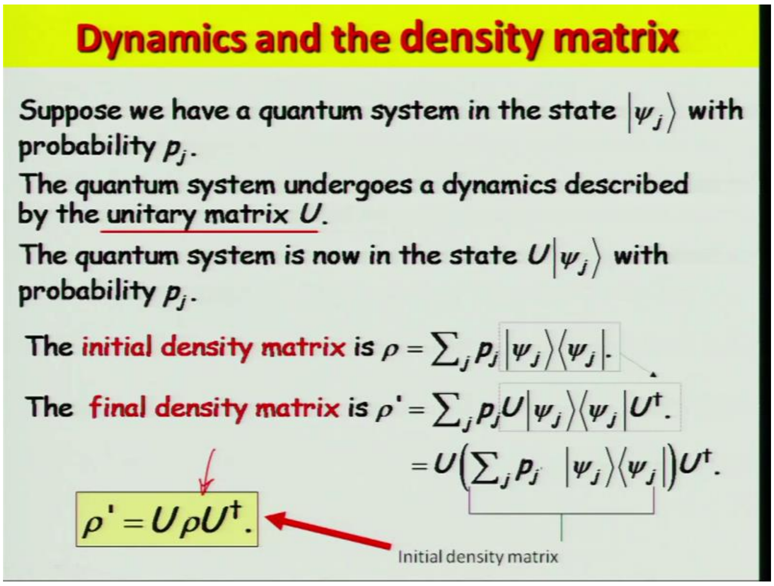

So, we can also look at the dynamics of the density matrix, if we have a quantum state in state psi j with probability P j the quantum system under goes the dynamics described by the unitary matrix U. The quantum system is now in the state U psi j with probability P j. The initial density matrix is as we have discussed P j psi j outer product. The final density matrix is rho’s prime with an unitary operator applying on both sides and since the unitary operation is communicative it will come out and will we can get back to this form which gives rise to the operator; the unitary operator acting on the rho to give rise to the final result.

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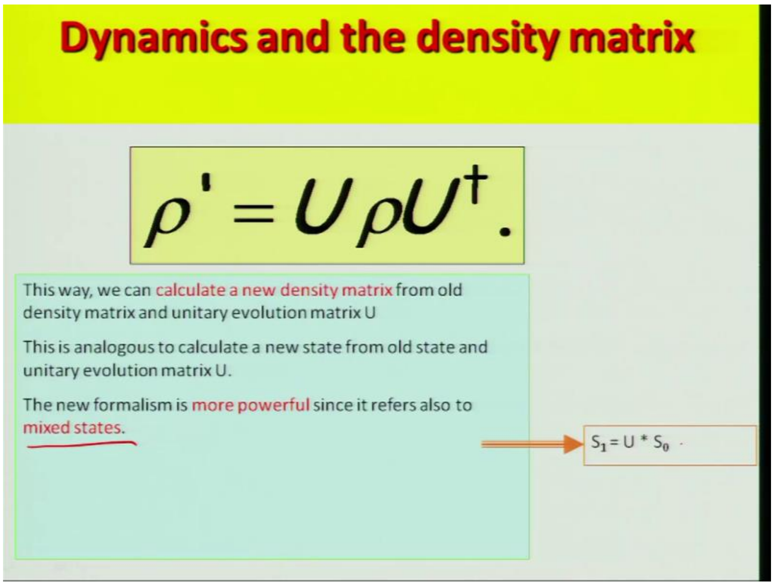

So, the dynamics of the density matrix can be calculated a new density matrix from the old density matrix and unitary evolution of the matrix. This is analogous to calculating a new state from an old state and unitary evolution matrix U this new formalism is more powerful since rho also refers to mixed states, and that is the biggest advantage of the density matrix. And so whenever this particular operation is being done, it can be looked at for any particular state.

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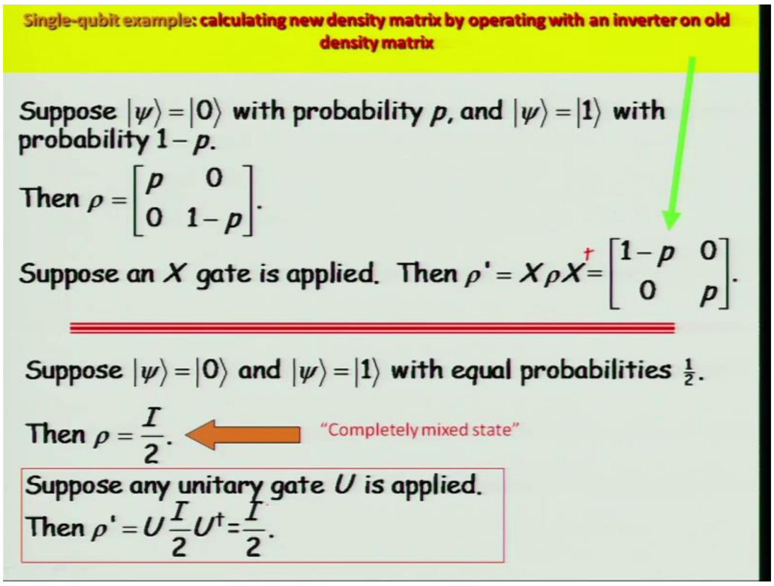

Let us take a single qubit example, where we calculate new density matrix by operating with an inverter on old density matrix. Say you have the pure state 0 with probability P and state 1 with probability 1 minus P, then our density matrix will be P 1 minus P 0 0 as we have discuss. If there is an X ket is applied then rho prime will be x rho X dagger essentially and that will be equivalent to this.

Now if wave function is 0 and 1 with equal probabilities. Then rho is equal to I over 2 which is the completely mixed day and if we have any unitary operation U is mixed, unitary get U is applied then rho prime will again give rise to the I over 2 (Refer Time: 23:22).

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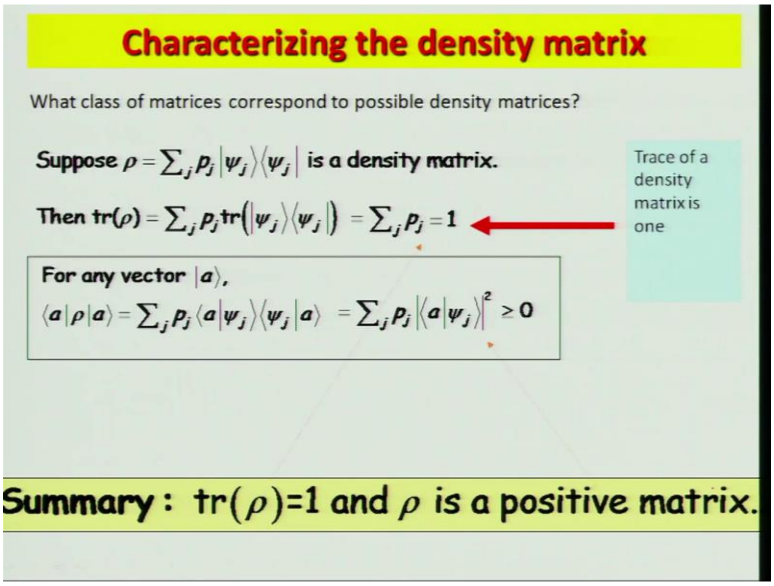

So, what class of matrices correspond to possible density matrix? Suppose we have rho equal to probability times the outer product of psi j and psi j is a density matrix, then the trace of the density matrix should be equivalent to equal to 1, the trace of the density matrix is one. So, for any vector a as long as we can write the density matrix in such a way that this trace is 1, and any projection of that is going to be greater than 0 then it will be occurrence.

Summary is that the trace is going to be 1 and rho is a positive matrix then we can consider that to be a density matrix. So, to consider any matrix to be a density matrix this is the first 2 properties that and necessary to be seen.

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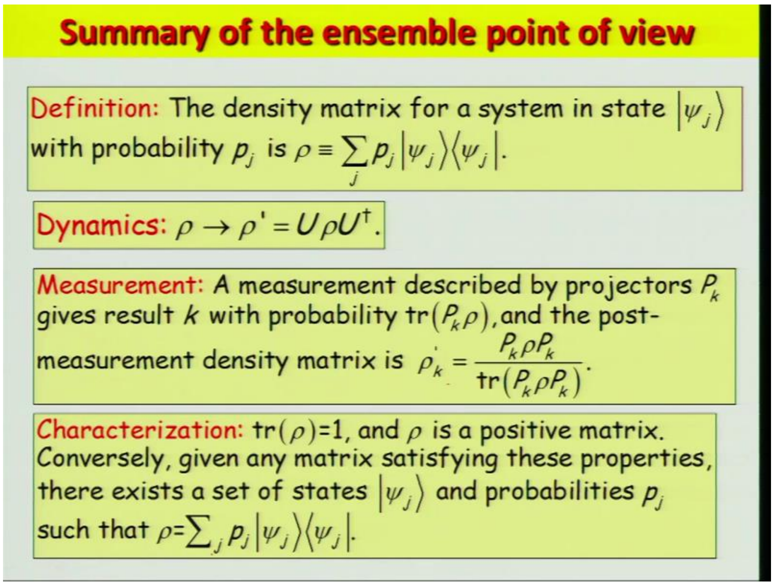

So, in summary the ensemble point of view that we are looking at it has the definition of density matrix for a subsystem, for a system with state psi j is probability P j which is given by density matrix which is the outer product the psi states with probability P j. The dynamics can be seen as any unitary operation can be utilized to see the new state. A measurement can be described by projectors P k which give result k with probability trace of P k rho, and the post measurement density is rho of k prime which is P k rho P k over trace of P k rho P k.

And it can be characterized with the fact that trace rho equal to 1 and rho is a positive matrix. Conversely given any matrix satisfying these properties there exists a set of states psi j ket and probabilities P j such that rho is equal to summation over j P j outer product of psi j’s.

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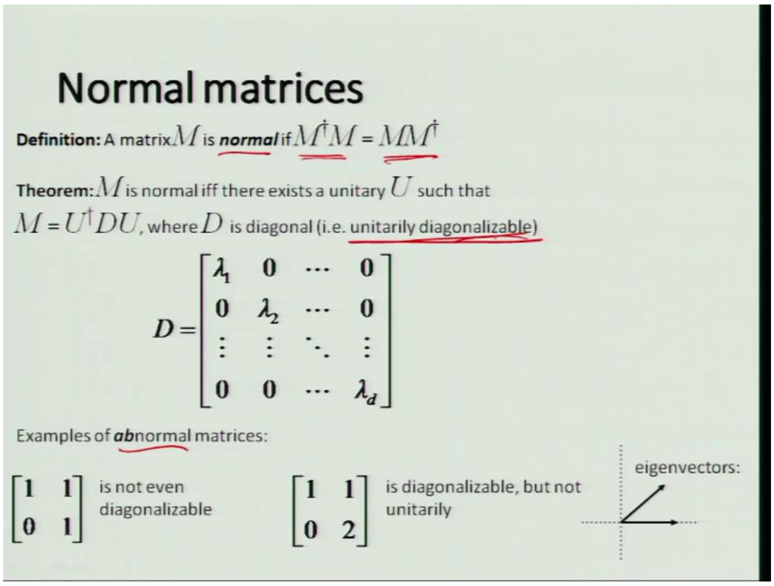

So, in this regard what we have been discussing is a very important property of these normal matrices also which have been implicit; which is that in matrix is normal if it is commutative. So, the concept is if there exist a unitary U such that the diagonal such that there is a matrix M which can be reached by using a unitary operator through a diagonal matrix, that is a unitary diagonalizable then the matrix m is normal. So, examples of abnormal matrices is for example, this particular simple matrix 1 1 0 1 which is not even diagonalizable, and the other which is 1 1 0 2 which is diagonalizable, but not unitary.

So, these are the kinds of abnormal matrices, but generally matrices are normal which is what we use if this is satisfied and if they are unitarily diagonalizable. So, these two properties are to be satisfied for having normal matrices; eigenvectors and eigenfunctions always utilized normal matrices.

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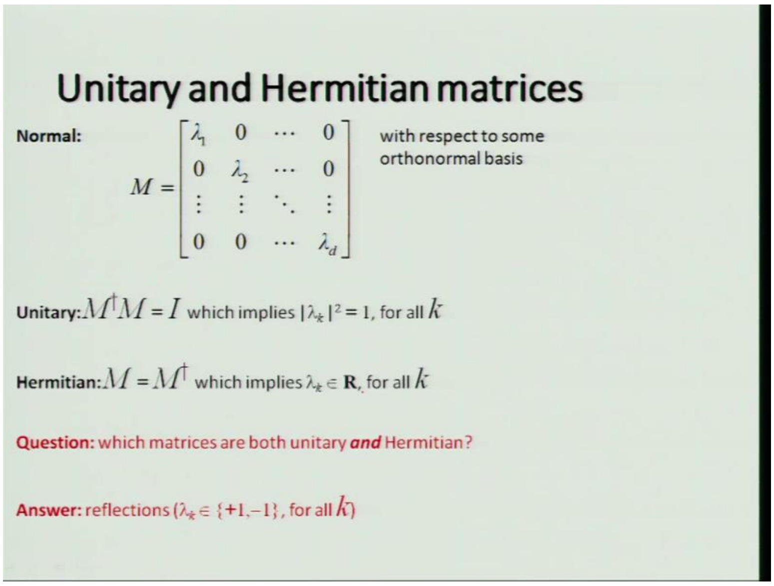

So, for unitary and Hermitian matrices we have normal matrices with respect to some orthonormal basis, there unitary which implies that lambda k square is equal to 1 for all the diagonal square of all of them are equal to 1 for k all k. Summation which means implies that is a the diagonal elements are all real for all k. And there are certain matrices which have both unitary and Hermitian for example, reflections where the diagonal elements are essentially plus 1 and minus 1 for all case; and so that is the way all things are.

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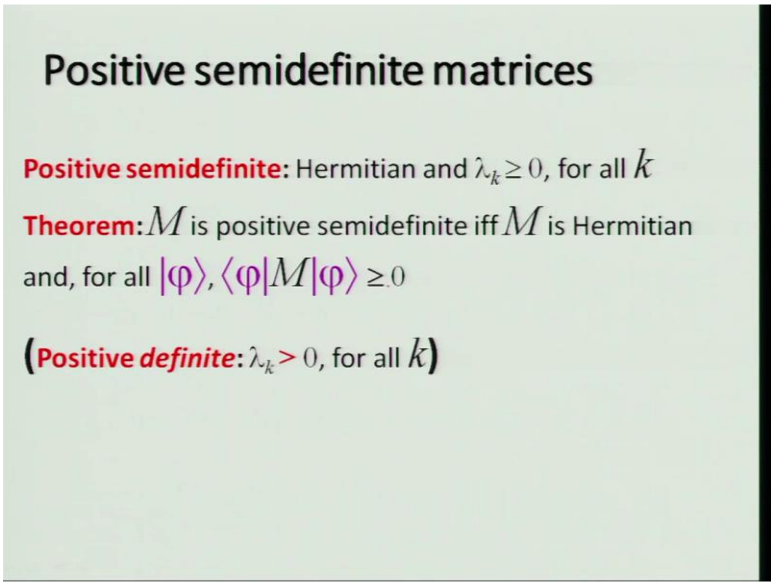

Now, there are these positive semi definite matrices also which we been often used in these kinds of studies in terms of gates and application for the quantum mechanical operations. So, positive semi definite means that they are Hermitian with the diagonal values of greater than 0 for all case. There exists a theorem where then normal matrices positive on matrix M is positive semi definite, if the normal matrix is Hermitian and for all values of ket phi they will be positive. So, positive definite means that the eigenvalues will be greater than 0 for all k.

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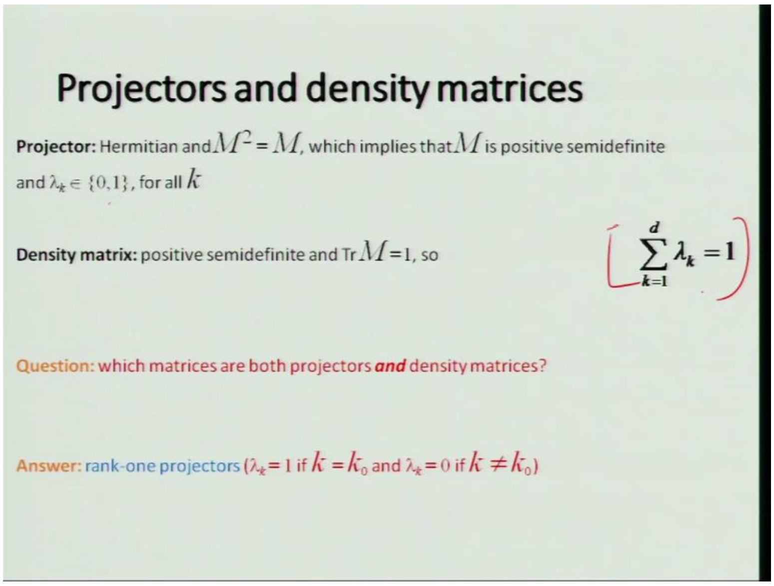

Now, one other very important thing which we have always used at the projectors and density matrices as we have been saying: the projectors are Hermitians and the square of them equivalent to the same, which implies that M is a positive semi definite and the eigenvalues are lies within 0 and 1 for all k. The density matrix itself is a positive semi definite and trace as it is trace of the matrix is 1, so as we know this.

And we both projectors and definite matrices for example, rank 1 projectors where the eigenvalues of 1 if k equal to 0 and it is equal to 0 if k is not equal to 0. So, these are basically the diagonal matrices.

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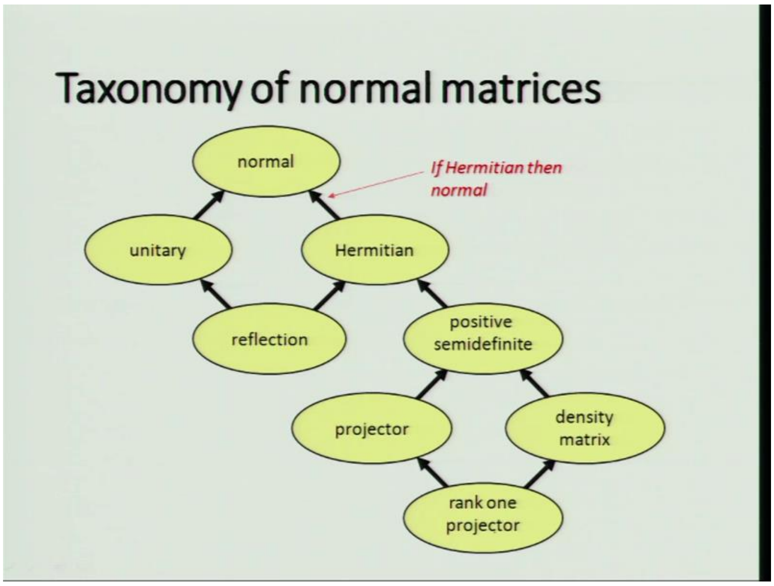

So, in some sense the concepts of the matrices nomenclature of matrices that we have been using; are of the kind where most of the matrices we use for quantum mechanical terms are normal and they are if Hermitian the normal then that is a Hermitian 1. Then there are these reflections which can be unitary and it can lead to normal matrices and there are these rank 1 projectors which can give rise to density matrices, projectors and the positives semi definite.

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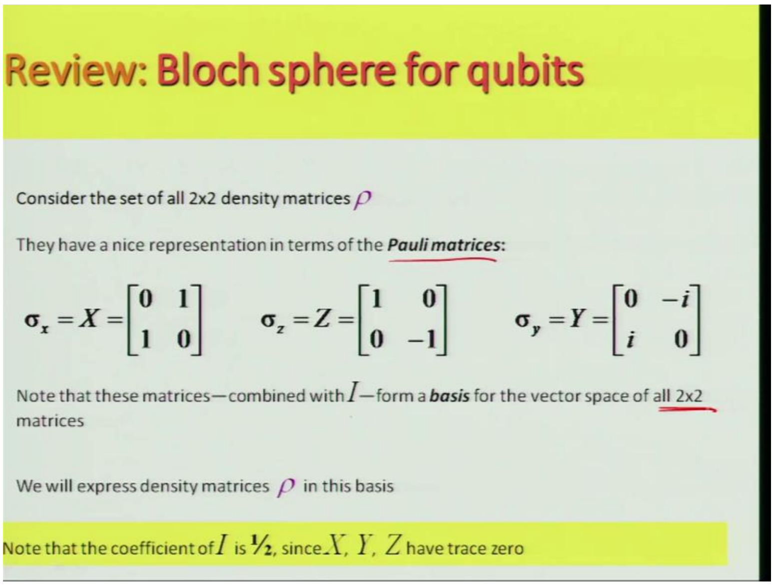

The other very important part which we should refresh is the Bloch sphere for qubits which we have gone through several times, but it is also another place where the density matrix representation can make a big difference. So, if we considers a set of all 2 by 2 matrices of rho, then they have a nice representation in terms of Paullies matrices that we know it and those are the few Pauli matrices which are the gates that have been used all the time, sigma X or X ket which has all the diagonal says 0 and the diagonals as 1. The Z gate say call it which is basically the sigma Z, the diagonals as 0 1 and minus 1 with of diagonal 0 and sigma Y is the Y gate which has diagonals has zeros and out diagonals as complex conjugate of I.

These matrices combined with the identity forms a basis for the vector space of all 2 by 2 matrices and that is why these are very important. And we can express density matrices rho in this basis. Note that the coefficients I is half since X, Y, Z have trace 0. So, all the value of the trace from diagonals suppose to be 1 will have to come from Y.

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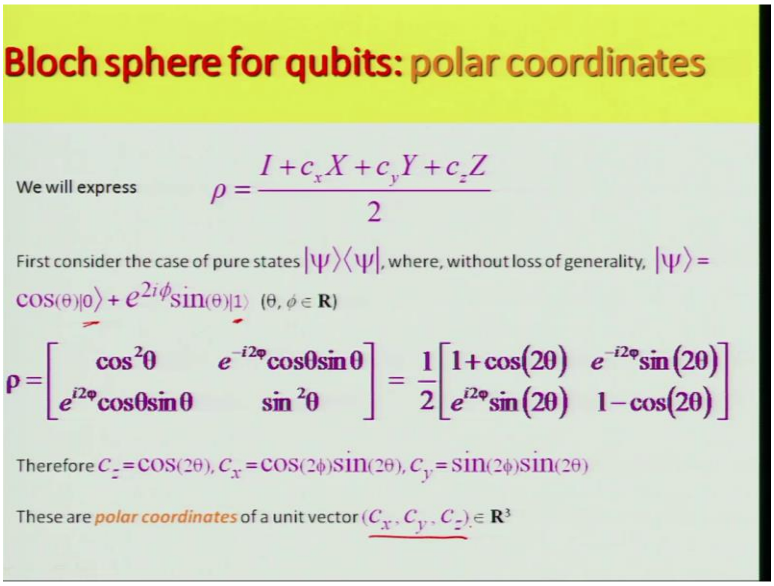

So, we can express the density matrix rho for a Bloch sphere for qubits along this form rho is equal to half with coefficient C x C y C z for X Y Z with I. Let us consider the case of pure state psi where without any loss of generality psi is equal to cosine theta 0 state times 2 power i phi sin theta I, so all these theta and phi are real elements. We consider that then we have a density matrix which looks like this form, which is possible to equivalently write in this form. Therefore, we can get the coefficients C is z C x and C y equivalently in between these 2 forms.

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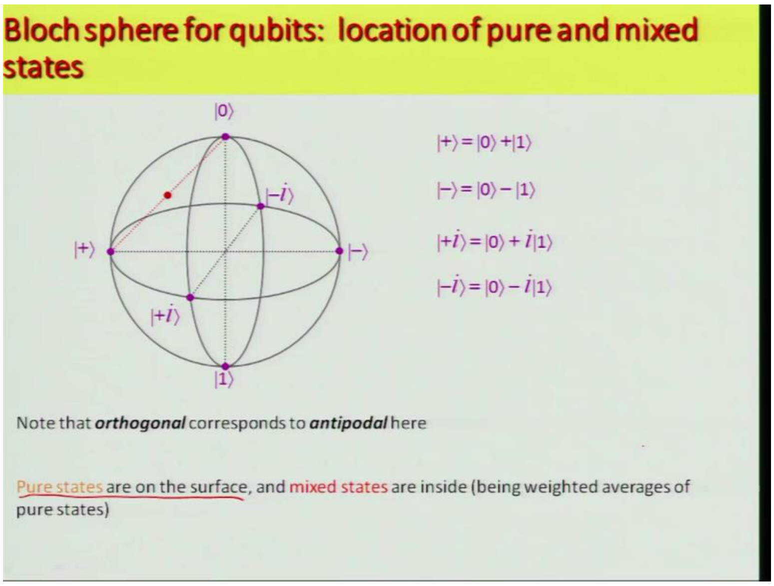

These are the polar coordinates of the unit vector C x C y C z which is in the real form. The Bloch sphere of a qubits location of pure in mixed states can be looked at also in this sense. So, in the Bloch sphere we know that we have a 0 and 1 along the axis and we have these different notations where they you can have the super position of thee states. The orthogonal corresponds to the antipodal here and the pure states are on the surface and the mixed states are inside being weighted average of the pure states.

So, this is how it can be looked at. So the mixed states are the once which are inside this sphere as we show by the red line all the parts of it, whereas the pure state are always on the surface of this sphere. And most of the time when we essentially describe in initial terms how the Bloch sphere is and talk about the discussions about the vector motion and everything it is always with respect to the idea that the vector is pointing on or moving the state is always on the surface of this sphere which is what not true in case of the mixed state; as there being weighted average of the pure states

I think we have covered enough of these concepts where we have linked all the different aspects of the states that we have been; and the gates that we have been using in the terminology of the density matrices. And the next lecture we will be actually utilizing all these terminologies and the connections that we have establish in this class to see how measurements and they understanding that have been utilized in terms of the implementation are much more easily understood in terms of density matrices. See you in the next class.