Mathematics for Chemistry
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Module - 02
Lecture - 02
Linear independence, basis, dimensionality

We have talked about vectors and scalars, we have talked about vector space and we have talked about vector products.

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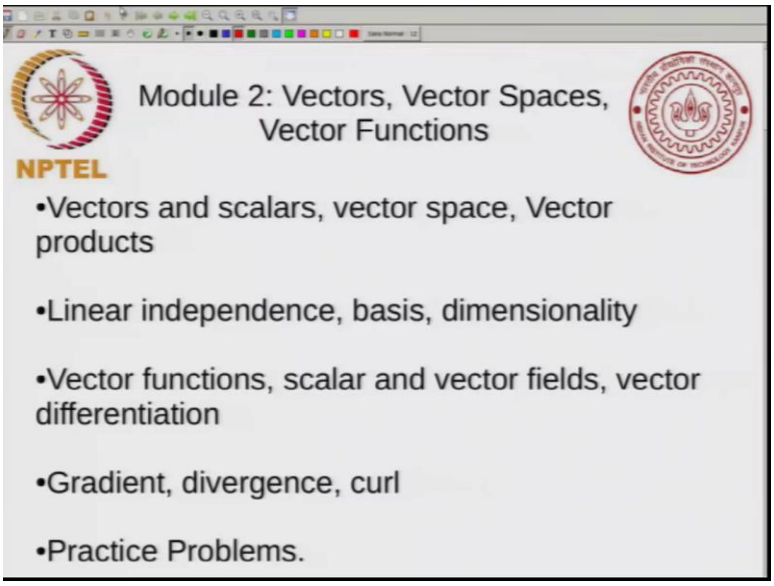

Now, the next topic that will be the topic for today's lecture will be linear independence, basis and dimensionality.

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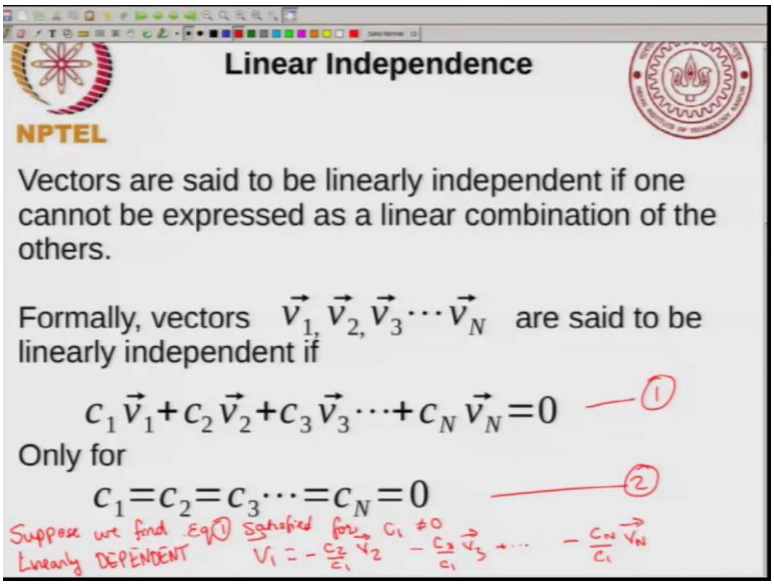

So, what is linear independence? So, vectors are said to be linearly independent if one cannot be expressed as a linear combination of the others. So, if you take a set of vectors then you say that if you can express one as a combination of the others then you say that they are linearly dependent and if you cannot do that you say that they are linearly independent.

So, formally the way you define linear independence is that if you have a set of vectors V 1, V 2, V 3 up to V N you say that these vectors are linearly independent if the statement C 1 V 1 plus C 2 V 2 plus up to C N V N equal to 0 is only valid when C 1 equal to C 2 equal to C 3 equal to C N equal to 0. So, C 1, C 2 etcetera are real numbers; C 1, C 2 up to C N these are real numbers and this is always true whenever you have C 1 equal to C 2 equal to C 3 and up to C N equal to 0 then this first relation will hold. So, this first relation holds whenever the second relation holds.

So, the second relation implies the first, but if the vectors are linearly independent then there is no other solution, there is no solution with nonzero C 1 C 2 with any one of one or more of these not equal to 0. So, that is what it means to say that the vectors are linearly independent. So, again the formal way of stating is that if you have a statement like C 1 V 1 plus C 2 V 2 plus C 3 V 3 up to C N V N equal to 0 that necessarily implies C 1 equal to C 2 equal to C 3 up to C N equal to 0 if the vectors are linearly independent.

If they are linearly dependent you can have another set of C 1's where all of them are nonzero which still satisfy the first equation. So, under the way now suppose you have a set, suppose we find equation 1 satisfied for C 1 not equal to 0. So, you find C 1 not equal to 0 and maybe some of the others are also not equal to 0, maybe some of the others are also not equal to 0. So, if you find equation 1 satisfied for let us say C 1 not equal to 0 then I can take V 1 to the other side and I can write it (Refer Time: 03:12) as a following, I can write V 1 is equal to C 2 and I can divide by C 1 because it is not equal to 0 C 1 minus, minus C 3 by C 1 V 3 and so on, minus C N by C 1 V 3. Why I am writing this is that C 2 by C 1 is now a real number as a C 3 by C 1 and so on.

So, what I did is I wrote V 1 as a combination as a linear combination. So, this is called a linear combination where you multiply each vector by a real number and you add them up. So, I wrote V 1 as a linear combination of V 2 V 3 up to V n. So, whenever you have vectors that are not linearly this is a case when they are linearly dependent, dependent. So, when vectors are linearly independent then you can write 1 as a linear combination of the other which is the same as this formal statement of linear independence.

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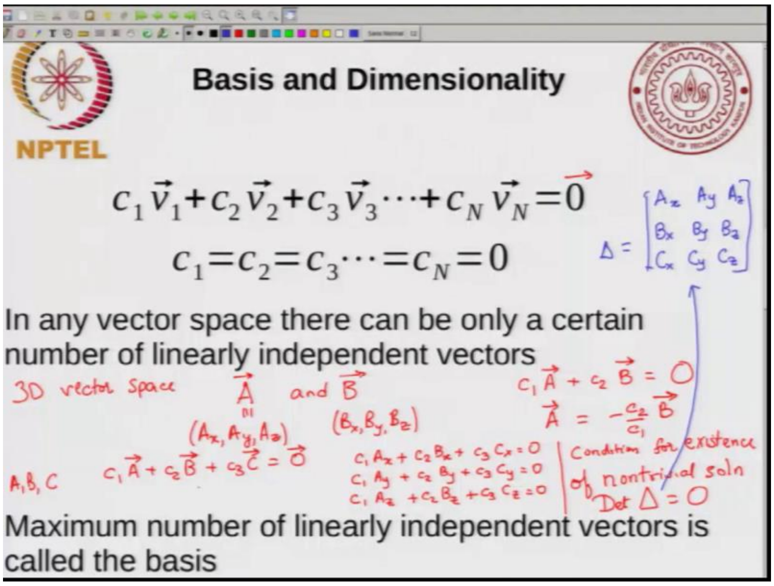

So, just to emphasize again we said that these vectors V 1, V 2, V 3 up to V N are linearly independent if the first equation implies the second. So, now, in any vector space there are always a maximum number of vectors that are linearly independent and we will come to that in a minute. But, let us take an example suppose we take the usual 3 dimensional vector space. Now if you take vectors let us say, if you take any 2 vectors let us say A and B. So, if you take 2 vectors then you ask; what is the condition that these vectors are linearly independent? Now A will have certain components let us say A x, A y, A z and B has components B x, B y, B z.

Now A and y being linearly independent means you will write that you will raise a C 1 A plus C 2 B equal to. So, you check if they are linearly independent then this is satisfied by some C 1 and C 2 which are both not 0. So, if this is satisfied for some for some C 1 and C 2 which is not 0 then I can write A as minus C 2 by C 1 B so; that means, A is proportional to B. So, A is a vector that is in the same direction as B it has a slightly different length. So, if you take 2 vectors then when you say that they are linearly dependent; that means, they are proportional to each other. So, if you take only 2 vectors then they are linearly they are linearly dependent if they are proportional to each other. So, this is the condition for 2 vectors.

Now what if you have 3 vectors suppose you have A, B and C, then what we say is that the condition for linear independence is that this C 1 A plus C 2 B plus C 3 C equal to 0 and I should probably emphasize that this is actually a 0 vector. So, in all these cases this left hand side is a vector. So, right hand side is also a vector. So, it is a 0 vector. So, this equal to 0 for some C 1, C 2, C 3 not equal to 0. So, if they are linearly dependent then you have some C 1, C 2, C 3 which is not equal to 0 for which this is valid and how do you find the C 1, C 2, C 3, I mean how do you state this condition. So, what you will say is that if I write the x y and z components of this I will get 3 equations. So, C 1 A x plus C 2 B x plus C 3 C x equal to 0 and then I will get C 1 A y plus C 2 B y plus C 3 C y equal to 0 and C 1 A z plus C 2 B z plus C 3 C z equal to 0. So, these are the conditions.

So, you have to have some C 1, C 2, C 3 which are not all 0 satisfying these 3 equations and you might remember from theory of these are all homogeneous linear equations. So, so if you think of C 1, C 2, C 3 as variables these are homogeneous linear equations. Now all these are satisfied for C 1 equal to C 2 equal to C 3 equal to 0, but the condition for non trivial solution as a solution where C 1, C 2, C 3 are not all 0. So, that condition for existence of non trivial solution and this you might be familiar. So, we say the determinant involving A x, B x, C x, A y, B y, C y, A z, B z, C z that should be equal to 0.

So, the condition for existence of non trivial solution I will just write determinant of delta equal to 0 and what is this delta. So, delta is this matrix equal to A x, A y, A z; B x, B y, B z; C x, C y, C z. So, if the determinant of this matrix delta is 0 then you say A B and C are linearly dependent. So, if this determinant is not equal to 0 then A, B, C are linearly independent. So, in this way you can that there are very easy ways to check whether 2 or 3 vectors are linearly dependent or independent, and you can do this for all vector spaces you can check linear dependence or independence.

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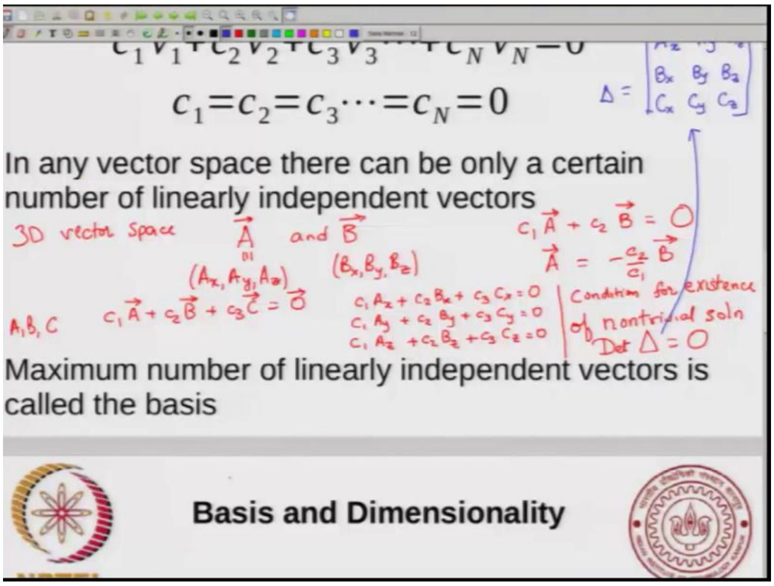

Now, the maximum number of linearly independent vectors is called the basis. So, if instead of having just A B and C if you had a fourth vector D in 3 dimensional space then you would have 4 equations, if you had a fourth vector. So, if you had A, B, C and D then what you would have is you will have C 1, C 2, C 3 and C 4 and you would have 4 unknowns and 3 equations and obviously, you can always find a solution because you can always adjust the fourth variable so that it satisfies these equations. So, what I want to say is that in 3D space you can have a maximum of 3 linearly independent vectors and we will see this in a few minutes.

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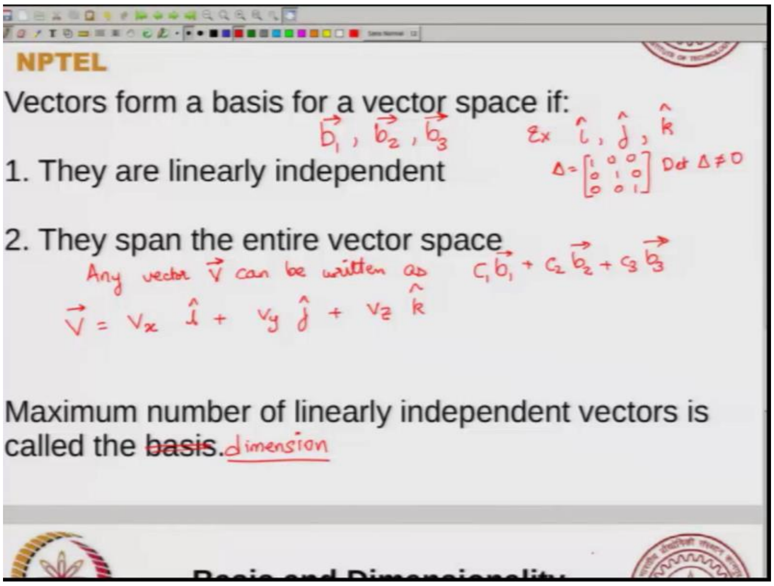

So, this brings us to the idea of basis and dimensionalities. So, vectors if you have a set of vectors they form a basis for a vector space is 2 conditions first they should be linearly independent and they should span the entire vector space. So, suppose you had 3 vectors b 1, b 2 and b 3 these should be linearly independent. So, when you say that they should be linearly independent we already saw the definition of linearly independent, a linear independence and they should span the entire vector space. So, what do you mean by they should span a entire vector space? Any vector V can be written as some linear combination of b 1, b 2, b 3.

So, some linear combinations, so I will just write c 1 b 1 plus c 2 b 2 plus c 3 b 3. So, any vector V any vector, you take any vector you can write it as a linear combination of these 3 basis vectors. So, in 3 dimensional vector space, one example - example is i j k the unit vectors in the x y and z direction these form a basis because they are linearly independent you can show this.

For example, if you take the determinant of the components in this delta that we talked about, so if you have i then A x is 1, A y and A z are 0; j for j B x is 0, B y is 1, B z is 0 and for k vector C x equal to 0, C y equal to 0, C z equal to 1 and you can clearly see that the determinant is equal to 1 and it is not 0. So, the determinant of delta equal to 1, so it is not equal to 0. So, just write delta for this i j k, delta equal to 1 0 0, 0 1 0, 0 0 1 and determinant of delta not equal to 0. So, these are linearly independent and they span the entire vector space because you know that any vector you take, any vector you take, suppose you take any vector V I can write it as; I can write it in terms of a (Refer Time: 13:25) component. So, V x i, V y j plus V z k, so this vector is written as a linear combination, so V x V y V z are scalars. So, you wrote this vector as a linear combination of i j and k so; that means, i j k span the entire space and i j k form a basis in this vector space.

So, they are linearly independent and they span the entire vector space, so any vector in the vector space can be written as a linear combination of i j k. So, that is in 3 dimensions. Now what I said is the maximum number of linearly independent vectors is called the basis of the vector space. So, in any vector space you can have a maximum number of linearly independent vectors and that is called the dimension. So, the maximum number of linearly independent vectors is called the dimension of the vector space.

So, you can ask the question what is the dimension. So, in the case of the 3D vector space the dimension is 3 because your number of linearly the maximum number of vectors that can be linearly independent is 3 if you have a fourth vector it will be linearly dependent. So, the basis is also the number of vectors in the basis is also gives you the dimensionality. So, you can either say maximum number of linearly independent vectors or the number of vectors in the basis. So, both are equal to the dimensionality, in 3D space you have 3 basis vectors. So, the dimensionality is 3.

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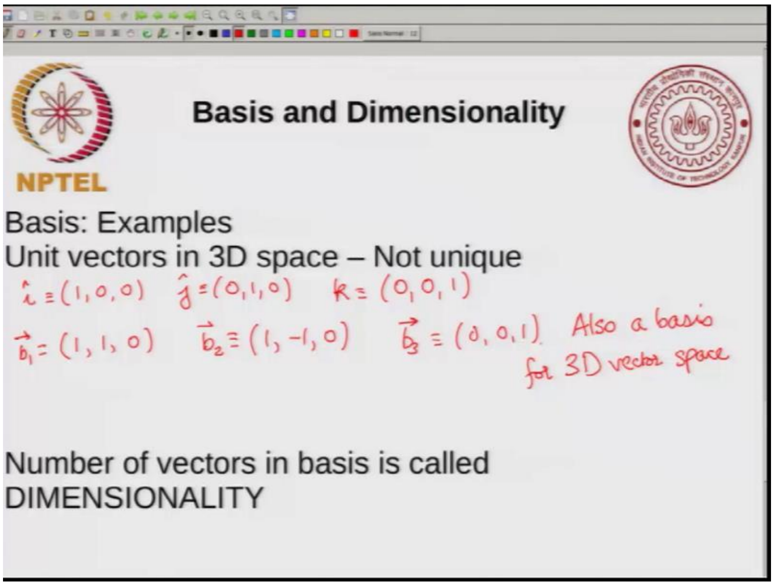

So, let us take examples of basis. So, suppose you have in 3D space, so your I; I can write that as 1, 0, 0, j; I can write as 0, 1, 0, k; I can write it as 0, 0, 1 and what is important is that these are not unique these are not the only basis that you can have, you can have other basis also. So, for example, you can have a vector let us say b 1 equal to 1 1 0, b 2 equal to 1 minus 1 0 and b 3 equal to 0, 0, 1.

So, this is also a basis for 3D vector space. So, the basis is not unique you can have many different basis, but they should still satisfy the conditions that they should be linearly independent and you can write any vectors as a linear combination of these and you can verify that, you can write any vector as a linear combination of these linearly independent vectors.

So, for example, suppose you just take 2 of these vectors b 1 and b 2. Now these are b 1 and b 2 are linearly independent, but I cannot express an arbitrary vector as a linear combination of b 1 and b 2 because whatever linear combination I take of b 1 and b 2 the z component will always be 0. So, 2 just b 1 and b 2 cannot form a basis you need b 1 b 2 and b 3 to form a basis. So, you need the 3 vectors to form the basis and what we said again is that the number of vectors in the basis is called the dimensionality.

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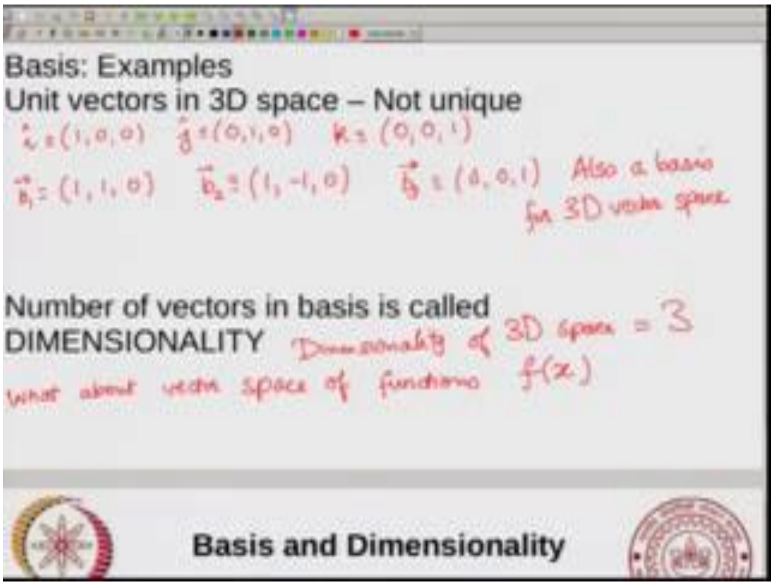

So, this is so the dimensionality of 3D space by definition equal to 3. So, this is the formal definition of dimensionality and that is why we call it 3D space. Now in the last class I talked about the vector space of functions, of functions of x. So, I said that if you take the space of all functions then that is also a vector space and what is the dimensionality of that vector space.

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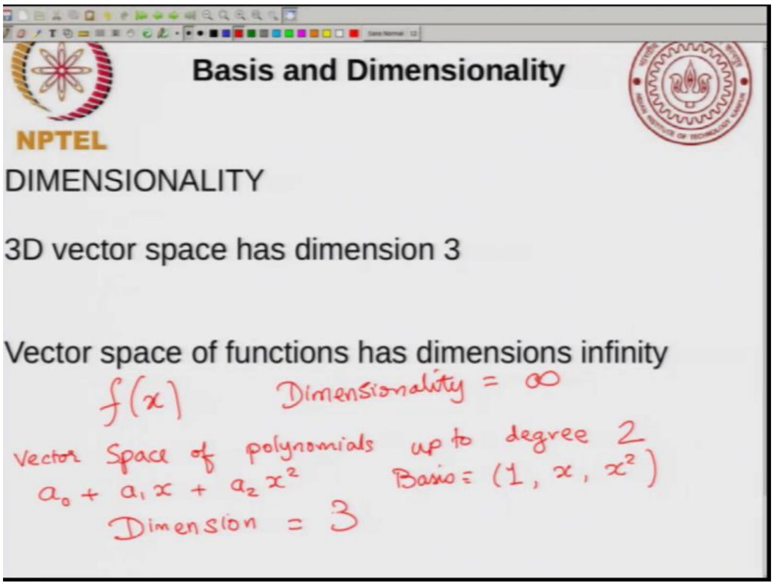

So, vector space of functions. So, this f of x, if you want to write an arbitrary f of x as a linear combination of certain number of f of x because the space of functions is infinite you can have infinitely many different functions. So, actually the dimension of this vector space of functions is infinity. So, dimensionality equal to infinite 3D vector space of course, has dimensionality of 3 and vector space of functions has dimensionality of infinity.

So, now you can look at some other vector spaces. So, for example, you can consider a vector space. So, we said that, so we consider the vector space of polynomials up to degree let us say 2. So, that will be that will be of the form a 0 plus a 1 x plus a 2 x square. So, that is what a typical polynomial will look like.

Now what is the basis for this vector space and you can convince yourselves that basis, one choice of the basis equal to 1, x, x square that is the obvious choice of basis because any polynomial of degree 2 I can write as something into 1 plus something into x plus something into x square. So, that is how we define a polynomial of degree 2. So, basis is 1, x, x square and the dimensionality or the dimension of this equal to 3.

So, in this way you can calculate what are the basis vectors and you can calculate the dimensionality of any vector space and the most important concept in this is that of linear independence. Now actually linear independence is probably one of the most important concepts in vector spaces and often we will be dealing with vector space of functions and you might not take all functions you might consider a certain class of functions and there the dimensionality need not be infinity it can be is lower than infinity. So, it is always keep, always these ideas of linear independence, basis and dimensionality are central to many aspects especially in quantum mechanics.

So, I will stop this lecture with this. In the next class we will talk about vector operators and functions of vectors and vectors and scalar fields.

Thank you.