Mathematics for Chemistry
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Module - 08
Lecture - 38
Prosperities of Bessel Functions, Circular Boundary Problems

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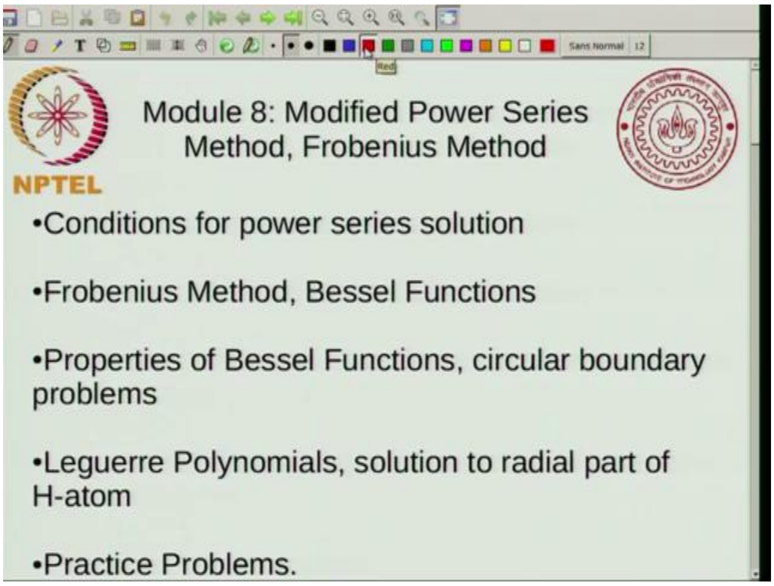

So today we will discuss some properties of Bessel functions we have already seen that Bessel functions are solutions of Bessel differential equation and we particularly looked at those functions for integer values of v the parameter that appears in the Bessel equation.

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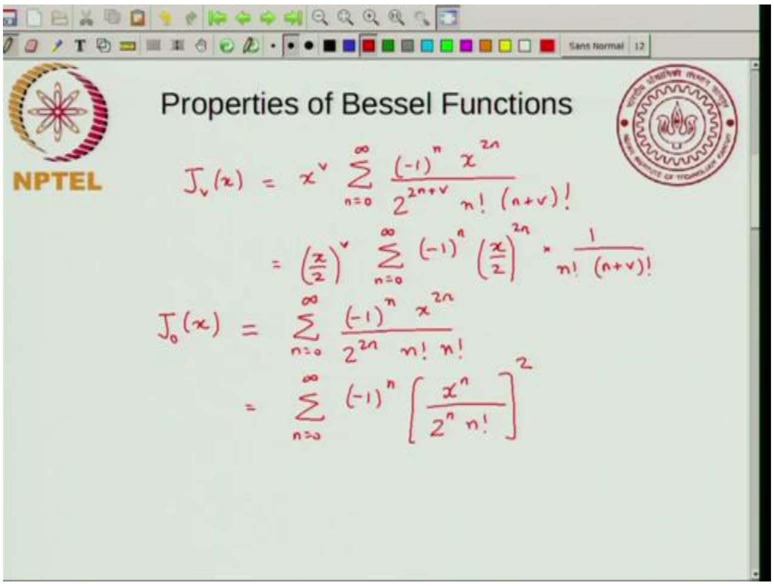

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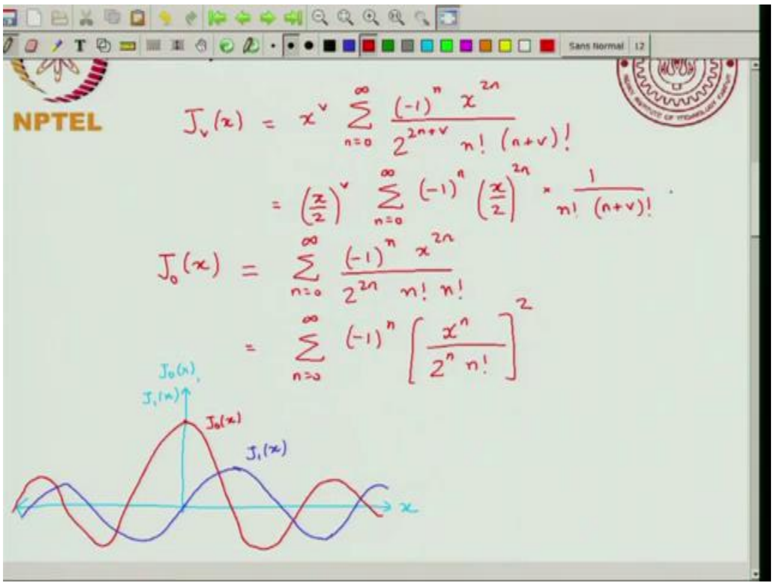

So, these are Bessel functions, if you remember we defined J v of x in the following form as x raise to v sum over n equal to 0 to infinity then you had minus 1 raise to n x raise to 2 n divided by 2 raise to 2 n plus v n factorial n plus v factorial. So, this was how the Bessel functions were defined for integer v and what is important in this you can write this in a slightly different form x by 2 raise to v sum over n equal to 0 to infinity and I will write it as minus 1 raise to n and then I will write x by 2 raise to 2 n times 1 over n factorial n plus v factorial.

Now, what is important to note about these Bessel functions is that it is an infinite series it goes from 0 to infinity it is not a polynomial. So, Bessel function is an infinite series, but it is a convergent series it converges. So, for example, if you look at v equal to 0, so J 0 of x this is the infinite series for v equal to 0. So, when v equal to 0 the; is x raise to v term will drop out. So, what you be left with is sum over n equal to 0 to infinity minus 1 raise to n x raise to 2 n divided by 2 raise to 2 n n factorial n factorial. So, we see n factorial square 2 raise to n square. So, I can write this in this form sum over n equal to 0 to infinity minus 1 raise to n and then I have x raised to n divided by 2 raise to n n factorial the whole thing square.

So, it is not obvious what this is, but this is a convergent series and this is the expression for J 0, similarly you can write for J 1 and so, on now I will just illustrate. So, what these Bessel functions look like? So, suppose you this is clearly a function of x and since it involves only even powers of x it is an even function of x if I plot J of x as a function of x. So, if I plot J 0 of x as a function of x then clearly I will use the red color. So, when x equal to 0 when x is equal to 0 then this becomes basically 1. So, x raise to 2 n that is equal to 0, so when x equal to 0, So, you just get you do not have clearly the only term that contributes is the n equal to 0 term and the n equal to 0 term if you put n equal to 0 here. So, this will become 1 here you have 1 here you have 2 raise to 0 which is 1, 1 factorial 1 factorial. So, when x equal to 0 then this goes to 1 and the behavior of this as s x changes.

So, it looks I will just I will just show what the behavior looks like. So, it looks something like this and it is symmetric about the x a. So, it looks like this. So, it has oscillations like sins and cosine's. So, it keeps oscillating this way and if you look at J 1 of x, now J 1 of x it will look it looks slightly different I will just this is. So, if I also want to show J 1 of x on the same graph. So, this is my J 0. Now if I want to show J 1 of x I will use a different color and in this case if you look at J 1 of x now in that case what you have is you have a slightly different function and in this case you have x raise to 1 leading term of x raise to 1. So, this goes to 0 when x equal to 0 and it has a it has a slightly different functional form I will just draw this, but it is also oscillating it also it also shows similar oscillations and it and this is actually odd function. So, because of this leading x raise to 1 in front of it is an odd function. So, it looks like this.

So, this is x raise to 1. So, this is J 1 of x. So, the point is J all the even v's correspond to even functions odd v's correspond to odd functions and remember it is an infinite series I cannot write this in a closed form I cannot write any of these in a closed form all right. Now there are some more interesting properties of these of these Bessel functions, so this J v of x which has this long expression.

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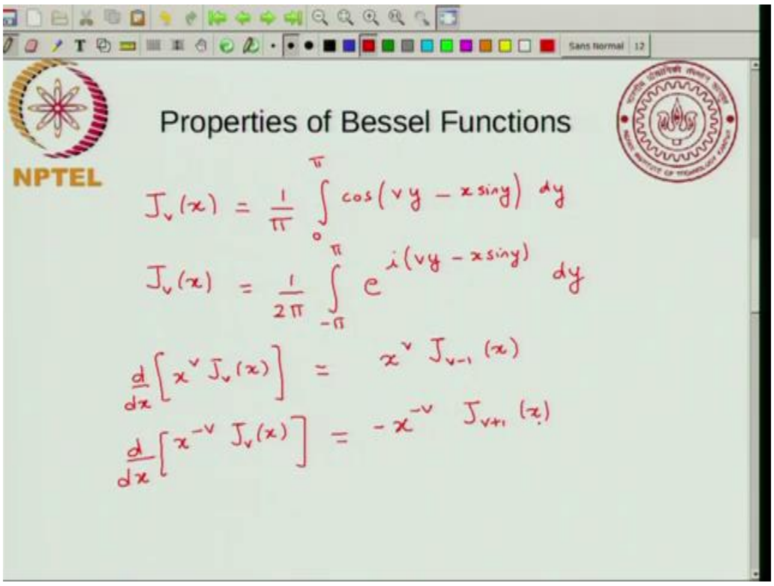

I will just I will just mentioned some of these you do not have to remember any of these, but these are just for your for your information. So, I can write J v of x in an integral form I can write this in the following integral form I can write it as 1 over pi integral 0 to pi cosine of v y minus x sin y d y.

So, I am taking a cosine of a quantity that has x sin y. So, you are integrating over y. So, this is a function of x. So, this is one way to express it alternatively you can write it as in a slightly different way 1 over 2 pi integral minus pi to pi. So, this cosine can be expressed in terms of exponentials and you can you can show that this is equal to e to the; I and you have exactly the same thing v y minus x sin y d y. So, I can write this in this form. So, these integral expressions actually showing that this integral gives you the same as this series is a bit of an exercise you have to work it out you have to do a Taylor expansion and then do the integral, but I would not bother that I will just mention that this is one of the integral expressions. So, these expressions I do not expect you to remember, but these are these are good things to know and in fact, Bessel had originally used some of these expansions to derive some properties of the Bessel functions.

Now, what are the other interesting things about Bessel functions? So, the other interesting thing is that suppose I take d by d x of x raise to v times J v of x. So, I take the derivative with respect to x of this whole thing, I get x raised to v J v minus 1 of x on the other hand if I take a derivative of x raise to minus v J v of x then I get minus x raise to minus v J v plus 1 of x.

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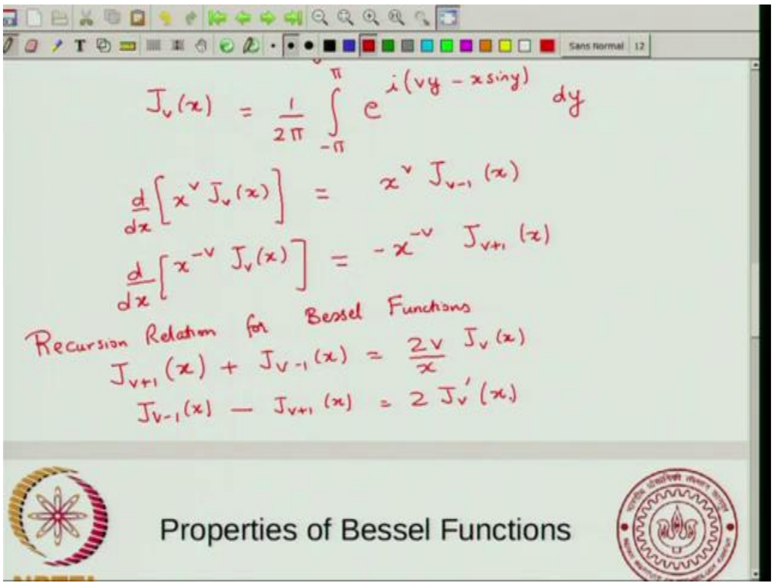

So, these are this is the; these are again interesting properties regarding derivatives of x raise to v times of Bessel function. Using these 2 you can derive a recursion relation (Refer Time: 09:23) for Bessel functions and I will just write the recursion relations and again I do not expect you to remember these, but these are useful things for you to know. So, you can write 2 recursion relations you can write J v plus 1 of x plus J v minus 1 of x equal to 2 v by x divided by x J v of x and the other recursion relation that can be written is J v minus 1 of x minus J of v plus 1 of x this is twice J v prime of x.

So, these are some of the relations that you can write now again as I said I do not expect you to remember the relations for Bessel functions, but the fact that you have these recursion relations is again very interesting and the notice that again once again J of v couples to J of v plus 1 and v minus 1 and this was this turned out to be fairly useful in say the hermit equation when which we used to which we use to analyze the quantum harmonic oscillator some other properties. So, if v is a half integer.

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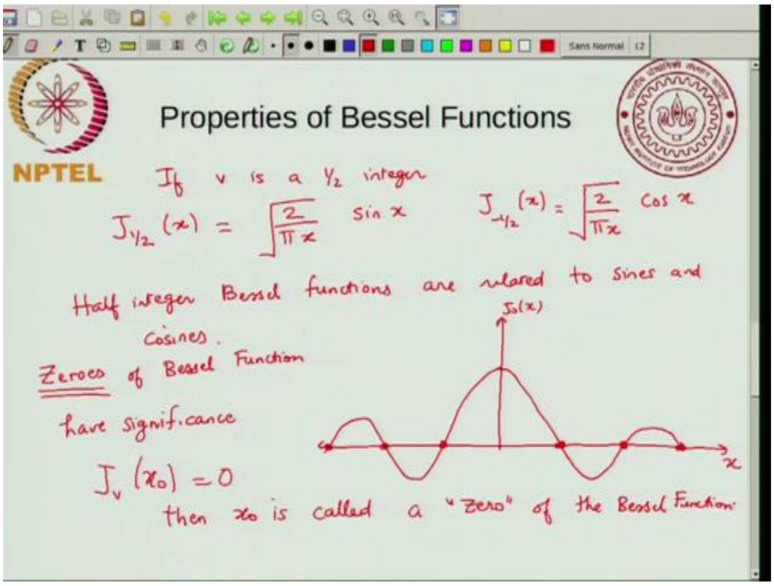

So, for example, J of half J of half of x you can work it out and this; this turns out to have a fairly nice expression. So, it turns out to be square root of 2 by pi x the square root of 2 by a pi is a constant that is added, but you have basically 1 by root x times sin x. So, it has a; very simple connection with sin x. So, it is a sin x divided by square root of x similarly J of minus half of x and you can imagine what this will be this is just 2 by pi x cosine of x and in fact, in fact using this J of half and J of minus half you can derive J of three half J of five half J of minus three half and all those all the other half integer half integer Bessel functions. So, basically the point is that half integer Bessel functions are related to sins and cosines. So, this is one useful thing to remember the other thing is that 0s of Bessel function.

So, these have significance I will just I will just mention that they have significance and we will see we will see in a minute where they will see in a few minutes where they have significance. So, what do you mean by the 0s of the Bessel function. So, suppose you take suppose you take J 0 suppose you take J 0 of x then what you see is you will see a graph that looks like this and these are the points where the where the function crosses 0. So, these are what you mean by the 0s of the Bessel functions. So, these are what you call the 0s of the Bessel function. So, these point these values of x. So, these are. So, at the 0 of x, J v of x 0 equal to 0 then x 0 is called a 0 of the Bessel function and these actually turn out to be very important in lot of applications now next I want to I want to take the Bessel functions and show how they appear very naturally when you are dealing with circular boundary problems. So, what is the circular boundary problem?

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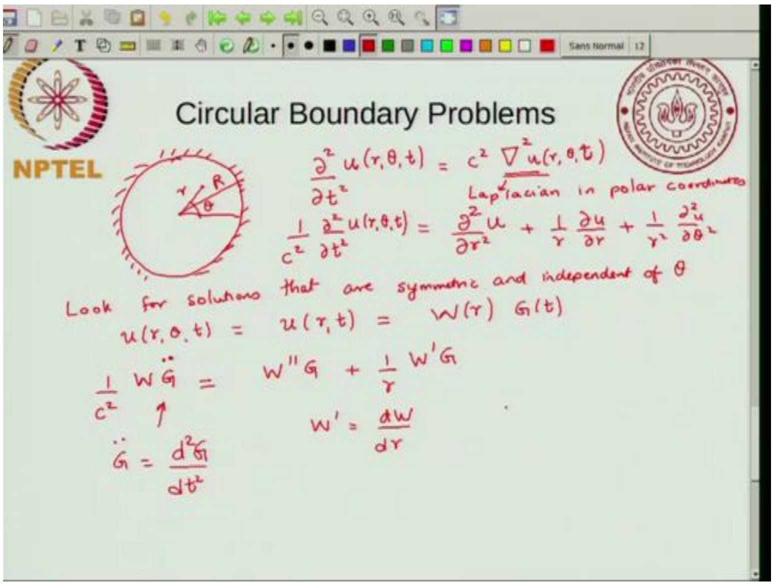

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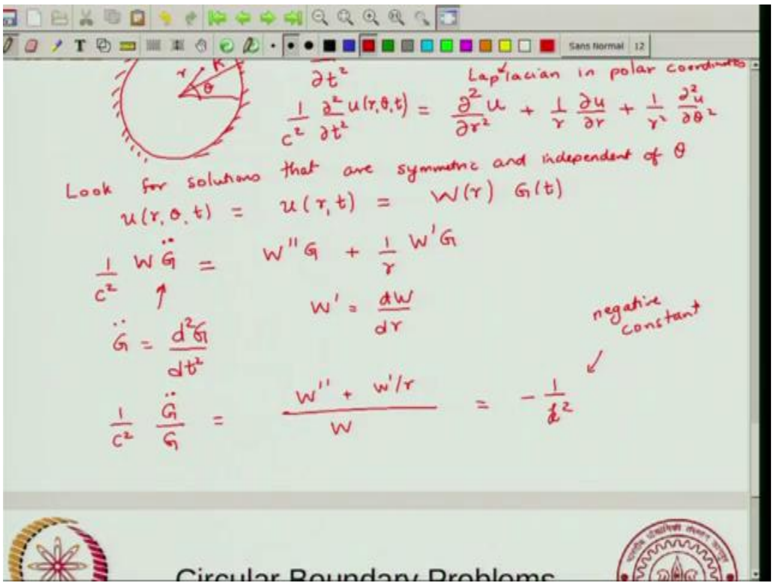

So, you can imagine that you have something like a let say a drum a drop vibrating drum and it is basically clamped at this ends. So, it is the drum is basically clamped at this ends. So, it cannot move here and the drum is allowed to vibrate in between. So, these points are held fixed the boundary is of held fix the boundary the circular boundary and what it makes it make sense to use a r theta coordinate to describe them. So, if you use r and theta to describe any point on the on the on the drum and let us say r is the radius. So, with respect to the center these coordinates. Now the wave equation for this drum has the following form. So, dou square by dou t square of u; u is the amplitude of the wave in general it is a function of r theta and time it is a function of the point in space and time this is equal to c square and you have a gradient square of u and u; u again is a function of r theta and t.

Now, now the point is; what is the gradient look like in polar coordinates? So, gradient is d square by d x square plus d square by d y square. So, in plane polar coordinates, so this takes the form. So, I will write this in a slightly different form I will take the c on this side I will write it as d square by d t square of u of r theta t is equal to; now what I will write here I will write this as. So, if you write this Laplacian in plane polar coordinates. So, Laplacian in polar coordinates, so the Laplacian in polar coordinates has this form d square by d r square of u I am not writing the theta phi r theta t dependence k plus 1 by r dou u by dou r plus 1 by r square dou square u by dou theta square. So, that is what you have on the right hand side and the left hand side you just have d square by d t square.

Now, let us look for solutions that are spherically symmetric or you know not spherically symmetric I should say I should say they have a circular symmetry are symmetric in theta symmetric and independent of theta of theta. So, if you look at u that is independent of theta. So, you say u of r theta t I will write it as u of; I will write this as u of r t and further what I will do is I will say that I can write this as a function only of r times a function only of t, I will write it as separately as a function of r n t you do not need to do this, but this will just make illustrate what we want to say a lot quicker. So, if you do this then the second derivative.

So, I can I can write this as 1 by c square and d square by d t square is just g double dot times w. So, I will write w g double dot. So, double dot implies g double dot equal to d square g by d t square. So, this will be equal to now d square by d r square of w the that us that I will write as w double prime and g plus 1 by r w prime g plus 1 by r square w sorry this term will not be there. So, this term will not be there because w is because your u is independent of theta. So, that is all you will have.

Now, if I divide by w g then I can get where I should mention w prime equal to d w by d r and similarly for w prime w double prime. So, now, what you can do is you can just divide this equation by w g and what you can write this as 1 by c square g double dot by g is equal to w double prime and let me I will just. So, plus w prime by r divided by w, so I just wrote it in this form and what you can see is that this is only of left hand side is only a function of t right hand side is only a function of r. So, that means, if these 2 functions have to be equal. So, this has to be equal to constant and that constant you can show that it should be negative value. So, I will just take that constant as minus 1 by k square. So, this is a constant. So, this is a negative constant and this is just a choice we just call it 2 minus 1 by k square, so then what you can write.

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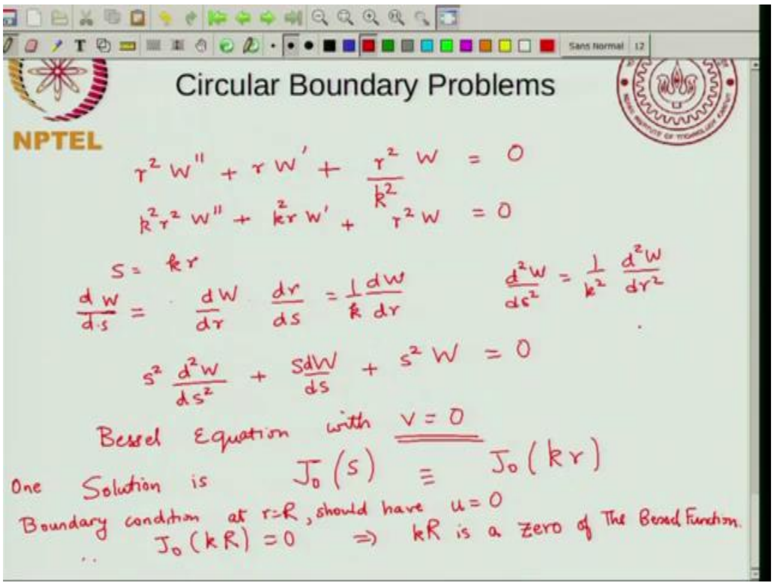

So, now, I am interested mainly in the in the differential equation for w. So, I can write a differential equation for w that has a form r square w double prime plus r w prime plus. So, I multiplied by r square. So, I will get r square by k square w equal to 0 and if I just make a substitution I want it as minus k square not minus 1 by k square. So, minus k square, so, I will take it in this form and then and then I will get oh sorry I will leave it as minus 1 by k square and what I will get is r square divided by k square here. So, that is we will just leave it in this form and then and then we will work it out. So, immediately you can see here that this looks like the Bessel equation there is a k square floating here? So, what we will do is we will just take this, we will just multiply out by k square. So, you will get k square r square w double prime plus k r rather k square r w prime plus and

I will get r square w equal to 0 and why I did this is suppose I make a change of variables suppose I make a change of variables.

Let say I say s equal to k r then I can see that d w by d s or rather d w by d s this is an ordinary differential equation. So, d w by d s is equal to k is I will write it as is equal to d w by d r into d r by d s is equal to d w by d r into, now d r by d s d r by d s is 1 by k and then similarly for the second derivative. So, similarly you can write d square w by d s square is equal to 1 by k square d square w by d r square. So, d w by d r I wrote as w prime and you can you can see this you can see where we are going. So, when you when you do this you and you and you rearrange everything you will get an equation that looks like s square d square w by d s square. So, you substitute this and then do some cancellations and you will end up with a very nice looking equation plus s d w by d s and you will get term that look likes plus s square w equal to 0 and this is nothing where, but Bessel equation with v equal to 0.

So, what is the solution of this equation so; that means, the solution at least one solution. So, one solution is J 0 of s or equivalently it is J 0 of k r now J 0 of k r. So, this is the solution and this is the solution to this it appears naturally in this circular boundary problems and you can see that you can see that you know all that we had was s square w d square w by d s square and s d w by d s and this third term and you can see that even if we had started with a problem that had a theta dependence this would just you just get an additional term, but still you would still be able to express it in terms of Bessel functions. So, the Bessel functions appear very naturally and interestingly the boundary condition boundary condition will be basically at r equal to r we should have should have u equal to 0.

So; that means, that means I can write that J 0 of k r equal to 0 that implies that k r is a 0 of the Bessel function. So, again what I said is that you know you know you have the 0 of the Bessel function appear naturally. So, if you think of what the solutions look like. So, you see if you imagine if you go back to your drum that you had if you imagine this to be your; this I will show it in green. So, this point you have r equal to 0 and you imagine that you have these Bessel functions and they should go to 0. So, right at r equal to 0 it should be it should it should coincide with a 0 and so, this is what it will look like in one direction, but it will look the same way in all directions. So, basically you will have you will have these you will have these waves that have a large amplitude in the middle then they have low amplitude here and then another part with a large amplitude.

So, you might get something that looks like waves that looks something like this it is have a large amplitude here large amplitude here in the middle and this is exactly what a. So, it look from the from the side view it look something like this. So, this is what your oscillations the vibrations of any drum that you take will have these Bessel functions. So, I just wanted to show you where Bessel functions appear very naturally in circular boundary problems similarly if you take a particle in a circular region restricted to a circular region you will again see the Bessel equation appearing.

So, I will stop here. So, in the next class I will talk about laguerre polynomials and how they appear in solution of the hydrogen atom problem.

Thank you.