**Mathematics for Chemistry**
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**Module - 08**
**Lecture - 37**
**Frobenius Method, Bessel Functions**

So, now, that we have seen how to use the, how to formulate the Frobenius method, let us look at applying the Frobenius method and we will applied to a very well known equation called the Bessel equation.

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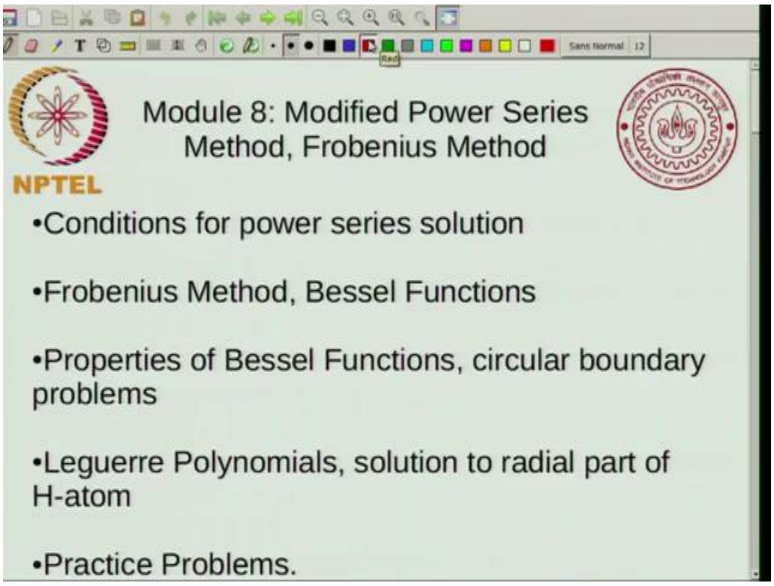


So, let us go to the Frobenius method for the Bessel equation, and this will lead us to naturally to something called Bessel functions.

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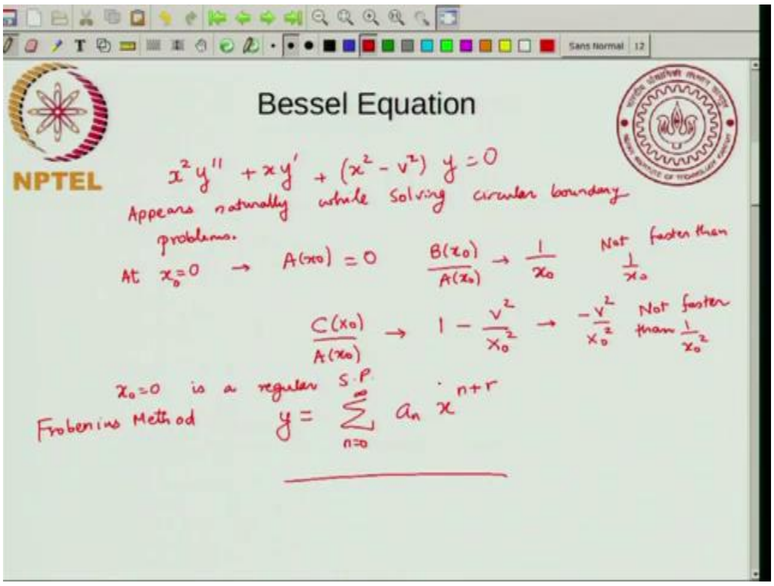


So, what is the Bessel equation? So, the bessel differential equation is has the following form x square y double prime, plus x y prime plus x square minus called this v square, y equal to 0; and v is some constant we will just say that right now we will just take v as a constant has some real number.

Now, this Bessel equation, so this I will mention right now that it appears naturally while solving circular boundary problems. So, problem with usually plane problems with circular boundary this appears naturally. So, suppose we have a drum a vibrating drum with the perimeter of the drum is circular then you will see Bessel functions appearing very naturally in the solution of that problem.

Now so, we will also I mean this also appears when you use cylindrical polar coordinates. So, when you use cylindrical polar coordinates you will naturally see Bessel functions. Now let us take this equation, now what we see is that you are at x equal to 0 at x 0 equal to 0. So, this is A of x 0 equal to 0, B of x 0 divided by A of x 0, this goes as 1 over x 0. So, this is not faster than 1 over x 0. So, it is not faster than 1 over x 0. So, that means, its 1 over x 0 to a power that is not greater than 1.

Similarly, you have C of x 0 divided by A of x 0. So, this goes as 1 minus v square over x 0 square, now the leading term as x 0 goes to 0, this the second term will become very large. So, this goes as minus v square by x 0 square and it is not faster than 1 by x 0 square. So, basically in this case you have the usual conditions, you have the conditions for a regular singular point so; that means, x 0 equal to 0, is a regular singular point. If it is a regular singular point then I can write my solution. So, I can use Frobenius method and write y is equal to sum over n equal to 0 to infinity, a n x raised to n plus r. So, I can write this as a solution.

Now, once you write this as a solution then you can go ahead and you can take it the first derivative and the second derivative and then substitute in that equation. So, let us do that.

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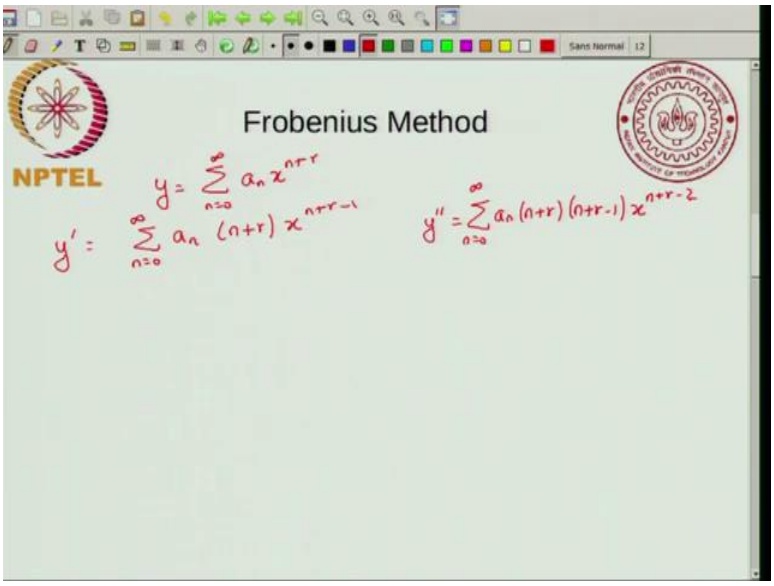

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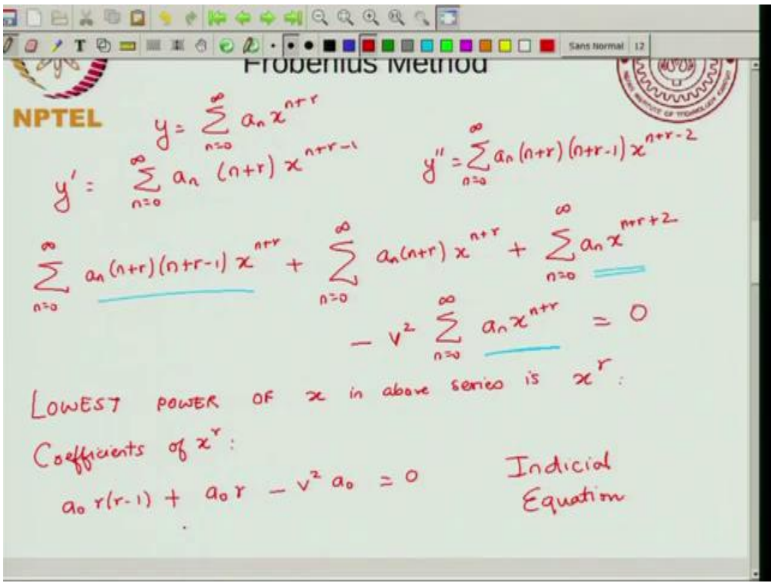


So, let us look at the first derivative. So, y is equal to sum over n equal to 0 to infinity, a n x raised to n minus r. Now what about y prime is equal to sum over n equal to 0 to infinity, a n into n plus r, x raised to n plus r, r minus 1. What about y double prime is equal to sum over n equal to 0 to infinity, a n, n plus r n plus r minus 1, x raised to n plus r minus 2. So, this is what you start off with notice that instead of in the previous case we just add n x raised to n minus 1 here we have n plus r, x raised to n plus r minus 1 and so on.

Now, when you substitute this in the differential equation, so you go back to your differential equation x square y double prime, plus x y prime plus x square minus v square y equal to 0. So, when you substitute this. So, x square y double prime that will look like sum over n equal to 0 to infinity, a n, n plus r, n plus r minus 1, x raised to n plus r plus sum over n equal to 0 to infinity. Now notice I had n plus r minus 2 and then I have x square y double prime. So, then it becomes n plus r.

Similarly, x y prime will become y prime multiplied by x. So, I have a n, n plus r, x raised to n plus r. And then I have x square minus nu square, so x square minus v square. So, the x square term will look like sum over n equal to 0 to infinity, a n x raised to n plus r plus 2, and minus v square sum over n equal to 0 to infinity, a n x raised to n plus r equal to 0. So, this is our equation now the lowest power of x in this equation, remember n goes from 0 to infinity. So, the smallest power of x, so if n equal to 0 then you have x raised to r, if n equal to 0 your x raised to r if n equal to 0 here you start from x raised to r plus 2, and if n equal to 0 here you have x raised to. So, the lowest power of x in above series is x raised to r.

Now, you look at coefficient of x raised to r. So, if you look at the coefficients of x raised to r, then what you have is in this sum you have to have n equal to 0. So, if n equal to 0 in that sum then what you will get is a 0 in to r, r minus 1 n equal to 0, and then you have x raised to r. The second term in this case you have n equal to 0, so what you have is a 0 times r. In the third term if you if even if you have n equal to 0 you will have x raised to r plus 2. So, the third term will not contribute. So, this term will not contribute to x raised to r coefficient, what about this last term. So, last term will contribute to x raised to r, so that will be minus v square, a 0 x raised to r; and this has to be equal to 0 this is what is called here called a indicial equation, and this indicial equation you can just cancel a 0 across this indicial equation, and you get your indicial equation can be written the following form. So, if I just cancel a 0 then I just get r square minus r, plus r minus v square equal to 0, and what I will get is r square minus v square equal to 0.

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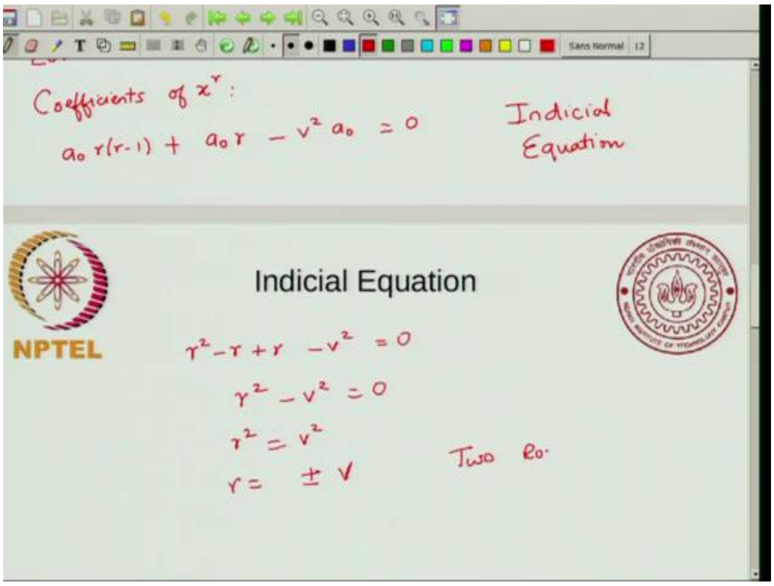


So, there are 2 roots so; that means r square equal to v square, r equal to plus or minus v, so 2 roots.

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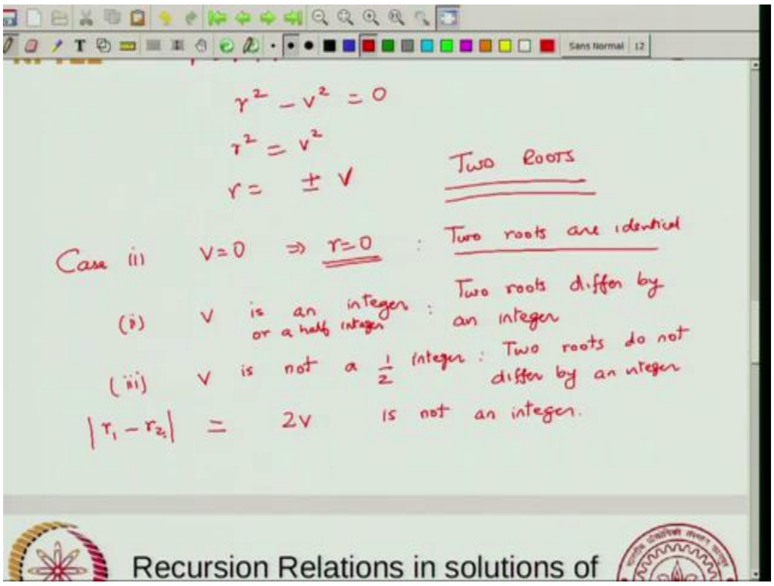


So, now, you have 2 root plus v and minus v, now we have to look at the various cases. So, suppose we look at the case; so case 1 v equal to 0. So, then r implies r equal to 0. So, roots are identical, case 2 v is an integer, now 2 roots now in this case 2 roots differ by an integer, 2 roots differ by an integer.

Now, what you wanted do is your apply the Frobenius method and you want to calculate 1 root, and then and then you can either use variation of parameters or you can directly calculate the second root and check if it is linearly independent. Third case, so v is not a half integer. So, I should mention that v is an integer or a half integer. So, in either case whether v is an integer or a half integer, then you have the difference between the 2 roots is an integer.

Now, v is not a half integer then 2 root do not differ by an integer; that means, you are 2 v which is you know if you take the r 1 minus r 2 this is equal to 2 v, is not an integer and in this case you are solution your 2 r 1 and r 2. So, you have to linearly independent solutions corresponding to r 1 which is v, and r 2 which is minus v. So, these are the 3 cases and you know you know each of these leads to various kinds of solutions and actually each of these cases has many physically relevant problems.

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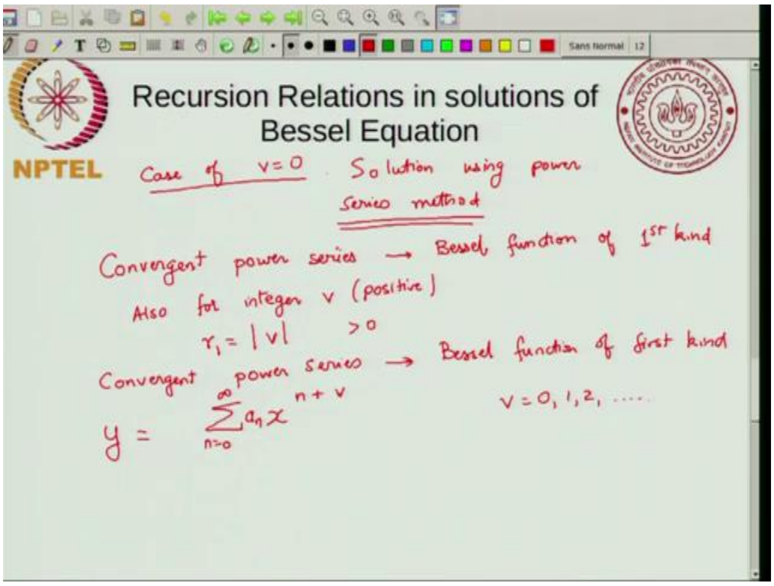


Now, one particular case which leads to a very well known function is, so the case of v equal to 0; now when v equal to 0, then basically you have only one root so you can find one solution using the power series method. So, now, the solution using power series method, we will see the properties of this now I will just mention one properties of this. So, the solution using power series method this turns out to be a convergent power series I will explain briefly what are convergence power series, and it is referred to as Bessel function. So, this convergence power series is referred to as Bessel function of first kind. So, there are various kinds of Bessel functions, this is called the Bessel function of first kind and you have we call we popularly this referred to this as the Bessel function. So, how do you understand this? So, the case when v equal to 0, so that is referred to as the bessel function of first kind.

So, now v equal to 0 and also for integer v integer v. So, if v is an integer for integer v that is what is v referred to is a positive integer, so the positive root. So, when where you take r 1 equal to v greater than 0. So, we take a absolute value of v. So, now, this power series this also leads to convergent series power series, this is also called Bessel function of first kind. So, the Bessel function of first kind appears whenever you have v equal to 0 or v is a positive integer. In fact, what you want to say is the following you want to say that y is equal to sum over x raised to n plus v, where v equal to 0, 1, 2 etcetera. So, we will see this in a few minutes. So, v equal to 0, 1, 2 etcetera; so then a n I forgot the a n n equal to 0 to infinity. So, this will be the starting point of that power series solution.

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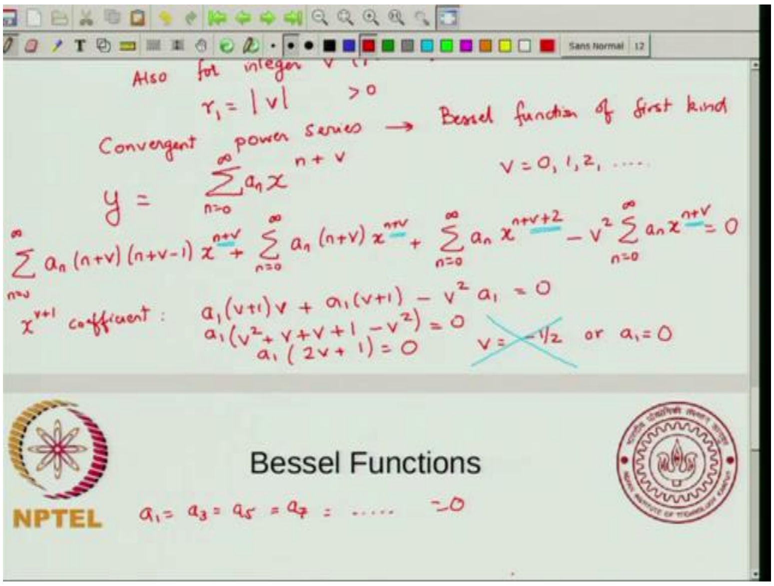


Now, let us take this as a power series solution and now let us substitute let us go back to the power series that we had and see what the recursion relations look like. So, what is recursion relation look like for such a power series, let us go back to what we had here. So, before we had the indicial equation we had we had this condition. Now what we are saying is that r equal to v which is and v is a positive integer. So, we have we have these 2 conditions r equal to v and v equal to v, v is a positive integer.

If you just take the recursion relation that we had and we put that case where r is instead of r we have v, and we say that v is a positive integer. So, let us just go ahead and do that. So, we had a n n plus r, n plus r minus 1, x raised to n plus r and so, if you just go ahead and write that in the following form. So, we have sum over n equal to 0 to infinity a n, n plus v, n plus v minus 1, x raised n plus v plus sum over n equal to 0 to infinity, a n, n plus v x raised to n plus v. So, these come from the first 2 terms and then you have an x square term. So, that looks like sum over n equal to 0 to infinity, a n, x raised to n plus v plus 2 then you have minus v squared term. So, you have minus v square sum over n equal to 0 to infinity, a n x raised to n plus v equal to 0. So, this is the general solution. So, the first thing is we already saw what is the coefficient of x raised to v.

Now, suppose we will look at the coefficient of x raised to v plus 1. So, if you look at the coefficient of x raised to v plus 1. So, now, if we look at the coefficient of x raised to v plus 1, then what we get is the following. So, in this case if you want v plus 1 then n has to be equal to 1. So, you get a 1 into v plus 1 into v, the second term will be a 1, v plus 1; the third time now here you can never have v plus 1 because this start from v plus 2. So, this will not even contribute to v plus 1, and then you have minus now if you want v plus 1 then n has be 1. So, you have v square a 1 is what will have from here, and this equal to 0. So, what you get is v plus 1 v. So, what you will get is v square if you can if you just cancel the a 1, this will v square plus v plus, v plus 1 minus v square equal to 0, and therefore, you get 2 v plus 1 equal to half.

In other words this solution, so x raised to v plus 1 actually gives you. So, when you set the x raised to v plus 1 term to be 0, then you get this relation that v equal to minus half. Oh sorry 2 please true; 2 v plus 1 equal to 0, and you get v equal to minus half. So, this is the condition that you get by setting the x raised to v plus 1 coefficient to 0, now you go back to this equation if you look at the terms. So, either you get I will said this way a 1. So, v equal to minus half or v or a 1 equal to 0. So, those are the 2 possibilities that you can get.

Now, v equal to minus half is not the solution of interest. So, basically you get a 1 equal to 0. So, this is not what we are interested it. So, we get a 1 equal to 0 and now what you get is the following. So, basically if a 1 equal to 0 now if you notice here you have v plus n plus v plus 2, and here everything else is n plus v. So, so; that means, if a 1 equal to 0 then. So, a 1, equal to a 3, equal to a 5, equal to a 7, equal to 0 that is the condition.

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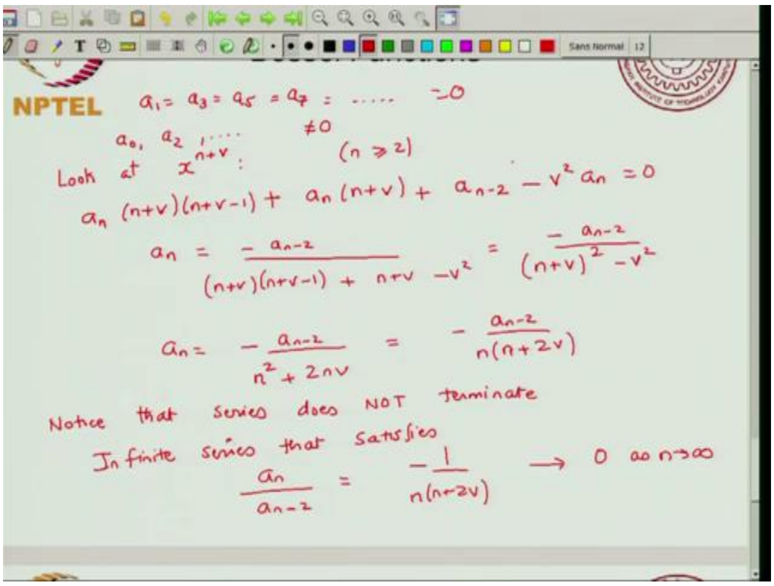

So, if each of these are 0 then you only have the even terms to worry about, we already saw the case that you know we insisting that a 0 is not equal to 0 we got the indicial equation. So, now, you have a 0 now what about. So, a 0, a 2 etcetera not equal to 0 and again if you work out this solution, so we can look at let us say x raised to n plus. So, we look at the coefficient of x raised to n plus v.

So, if you look at the coefficient of x raised to n plus v, then what you will get is the following. So, from the first term you will get a n times, n plus v, n plus v minus 1. From the second term you will get a n times n plus v, from the third term now in this case you can have n plus v what I a m going to say where n is greater than equal to 2. So, if you look at where n is greater than or equal to 2, so then you will get a raised to n minus 2. So, that will be the coefficient of x raised to n plus v. So, if you want n plus v then you (Refer Time: 22:21) have a raised to a of n minus 2 then this will be n minus 2 plus v, so that is plus 2. So, that is n plus v, and then you will have minus v square a n equal to 0.

So, this gives you an equation that relates a n equal to minus a n minus 2, divided by n plus v, n plus v minus 1 plus n plus v minus v square, that is equal to minus a n minus 2 divided by n plus v the whole square, this is n plus v whole square. So, whole square minus v square, so that is equal to minus a n minus 2, divided by n square plus 2 n v. So, a equal to minus a n minus 2 divided by n, n plus 2 v. So, this is your recursion relation and what you can do is you can take this recursion relation. So, this recursion relation notice that series does not terminate.

So, you can never have a n equal to 0 for any value of n, you can never have a n equal to 0 series does not termite it is an infinite series. So however, you have one property, that if you take infinite series that satisfies a n divided by n plus 2 or a n minus 2 if you want, a n minus 2 is equal to minus 1 over n, n plus 2 v. So, this tends to 0 as n tends to infinity; that means, as n tends to infinity this ratio of a n to a n minus 2 will go to 0; that means, the series is convergent. So, it is a convergent series.

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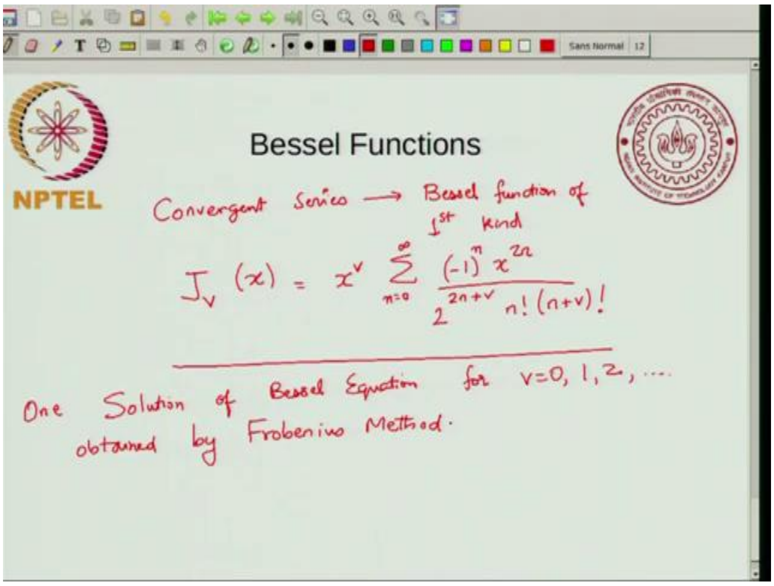


This convergent series is called the Bessel function of first kind and it is denoted by J v of x. So, this J v of x is this convergent series; that means, that means you have an infinite series starting from a 0 and then and then all those things.

So, the a 0 multiplied by that infinite series is called the Bessel function of first kind. So, Bessel function is actually an infinite series. So, what is the infinite series? So, this is J v. So, J v of x is basically x raised to v times sum over minus 1. So, x raised to v was the pre factor in that you had x raised to n plus v right. So, this n plus v I just took the x raised to v outside minus 1. So, this is from n equal to 0 to infinity, minus 1 raised to n, x raised to 2 n divided by. So, only the n is even. So, (Refer Time: 26:24) So, that is why took x raised to 2 n and what I have here is 2 raised to 2 n plus v, and then you have n factorial and you have n plus v factorial.

So, this is what you get. So, basically if you use the power series method in the usual way, you will get this as the convergent series when v is either 0 or a positive integer. So, this is the solution of Bessel equation for v equal to 0, 1, 2 etcetera and this is one solution. So, this is one solution, I will just say this is one solution obtained by Frobenius method.

The other solution is obtained by variation of parameters and this is called Bessel function of a kind sometimes called the Bessel function of second kind and so on, but basically we will just stopped here in the discussion of Bessel functions. So, we just see how the Frobenius method gives you a solution in the form of an infinite series, but we checked that the infinite series is the convergence series and this is a function Bessel function of first kind, it is a very popular function that appears in a lot and physics and engineering and you know even in quantum mechanics, we will I mean we will see some applications of this we will see the circular boundary condition problem where we see, but before that we will see some properties of the bessel function in the next class.

Thank you.