Mathematics for Chemistry

Prof. Madhav Ranganathan
Department of Chemistry
Indian Institute of Technology, Kanpur

Module - 08
Lecture - 36
Conditions for Power Series Solution

We now start module 8 which is the last week of this course, and the last module of this course and in this week we will be talking about the modified power series method and the Frobenius method.

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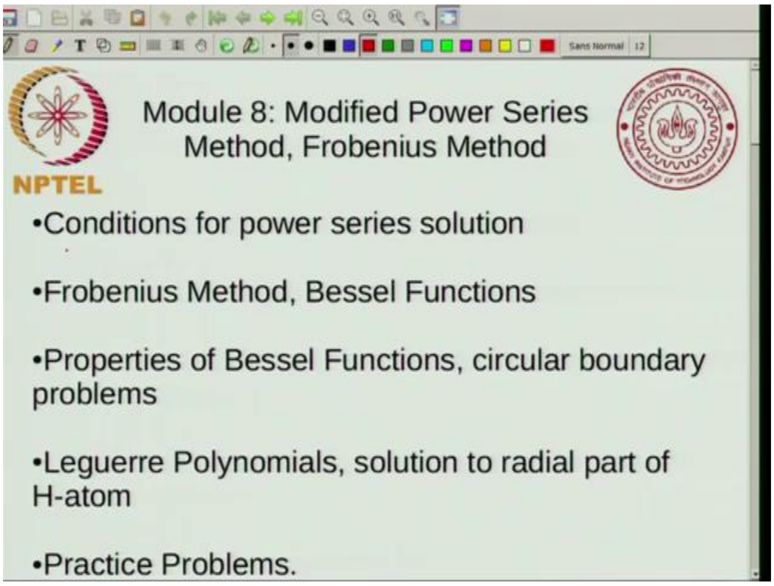

So, in each of the lectures in the first lecture we will take, we will look at the conditions for a power series solution, the mathematical conditions under which differential equation can be solved using the power series method. Then we will discuss the Frobenius method and through the example of bessel functions, we will look at some properties of bessel functions and then how they appear naturally in circular boundary problems, and then we will talk about Leguerre polynomials which appear in the solution of the radial part of the hydrogen atom, and then we will end up with some practice problems.

So, now let us go to the first part today which is conditions for power series solution. So, now, suppose I have a differential equation of the following form.

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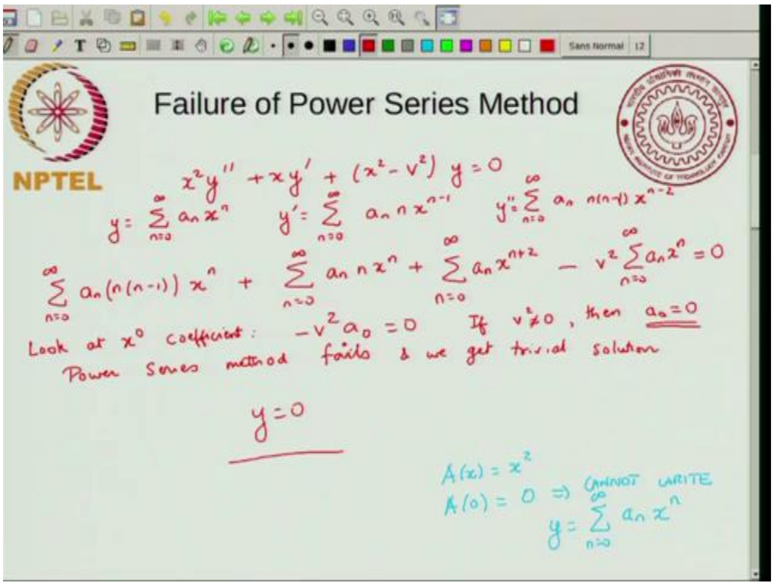

So, suppose I have a differential equation that has the form x square y double prime, plus x y prime, plus x square minus v square y equal to 0. Now suppose you apply the power series method to this equation, suppose you just apply the power series method to this to this problem then what you will find is that you will say that y is equal to sum over n equal to 0 to infinity, a n x raised to n, and then you will say y prime equal to sum over n equal to 0 to infinity, a n n x raised to n minus 1 and you will write y double prime equal to sum over n equal to 0 to infinity, a n into n, n minus 1 x raised to n minus 2. And when you substitute in this equation what you will get is the following, you will get sum over n equal to 0 to infinity, you will get a n times n, n minus 1 x raised to n plus sum over n equal to 0 to infinity a n, n x raised to n plus, now you will get 2 terms. So, the first term will be sum over n equal to 0 to infinity, a n x raised to n plus 2 plus v or minus v square is I write as a minus, minus v square sum over n equal to 0 to infinity a n x raised to n, this equal to 0 and now if you look at coefficient of x raised to 0, look at x raised to 0 coefficient.

Now, the first term you have to have if you want x raised to 0 you have to have n equal to 0, but n equal to 0 term is 0. Second term also n equal to 0 term is 0, third term if you want you cannot have x raised to 0 because you would start from x raised to 2. On the fourth term you have minus v square a 0 equal to 0 and in general if v square is not equal to 0. So, if v square not equal to 0 then a 0 equal to 0. Even if v square equal to 0 then you do not get any equation for a 0 that is the point.

So, the point is this equation does not give any coefficient of x raised to 0, but if v square equal to 0 then you get a 0 equal to 0, and when a 0 equal to 0 you can easily show that all the corresponding terms will all go to 0 in this equation. So, if a 0 equal to 0 then the whole power series goes to 0. So, we say that the power series method fails; a 0 equal to 0 then; obviously, you can see from here that all the even powers will be 0 and if you do the same thing you will also get the odd powers will also go to 0. So, the power series fails and we get trivial solution y equal to 0. So, notice since this is a homogeneous equation y equal to 0 is a solution, but it is a trivial solution. So, that is only solution that you will get.

Now, so why did this happen? And the reason this happened is that has to do with the x squared term here, and the fact at the x square term goes to 0 at x equal to 0. So, this is what we want to explore a little more. So, whenever you have a differential equation of the following forms.

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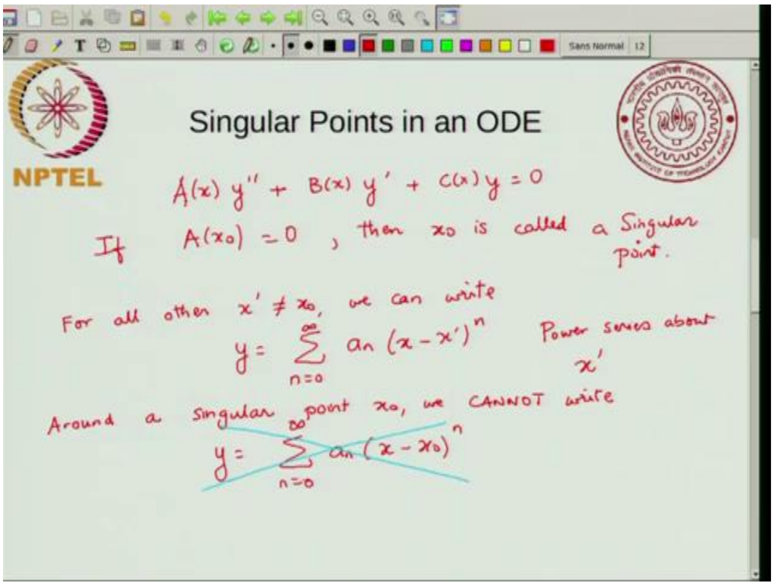

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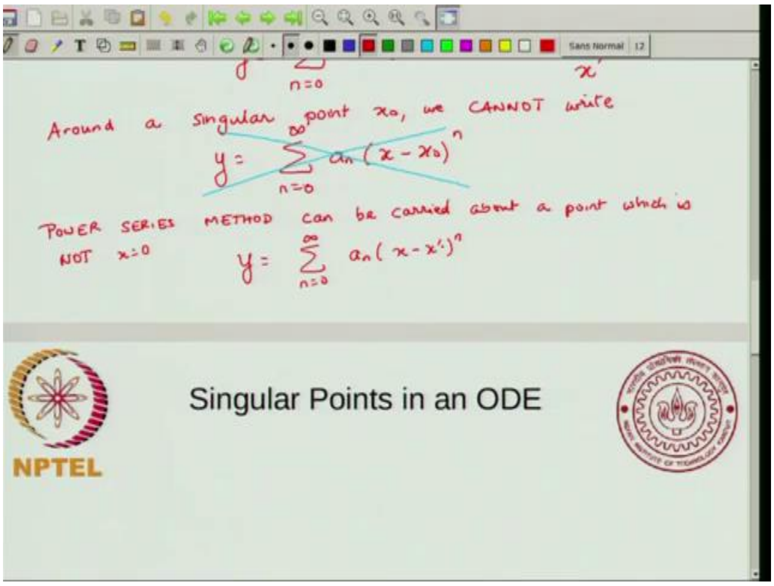

So, suppose you have a differential equation a of x, y double prime, plus b of x, y prime plus c of x, y equal to 0 a homogeneous differential equation; if A of x 0 equal to 0, then x 0 is called a singular point; if A of x 0 is 0 then x is x 0 is called a singular point.

Now what is the significance of the singular point? So, there is a certain significance of this singular point and that is what we will come to. So, I will just write for all other x prime not equal to x 0, we can write y is equal to sum over n equal to 0 to infinity, a n times x minus x prime raised to n. So, this is power series about x prime. So, if x prime is not a singular point then we can write a power series solution in this form, and the power series method will work, but if you have a singular point if. So, around a singular point x 0 we cannot write y is equal to sum over n equal to 0 to infinity, a n, x minus x 0 raised to n. So, this cannot be written. So, this is you cannot write this about a singular point. So, if x 0 is a singular point you cannot write a power series solution about that.

So, now what is a singular point? A singular point is where this your a goes to 0. Now you go back to the equation that we had in the (Refer Time: 08:04) had be just looked at x square y double prime. So, the singular point clearly this is a is equal to x square. So, in this equation you have A of x equal to x square, and a of 0 equal to 0. So, implies cannot write y equal to; you cannot write a power series around x equal to 0 around n equal to 0 to infinity, a n x minus 0 is just x, x raised to n. So, cannot write this. So, this was the reason why the power series method failed in this case, because x equal to 0 was a singular point.

So, the power series around the point x equal to 0 is a singular point. Now this I just I mean you should realize that power series. So, power series need not be around x equal to 0. So, the power series method, the general power series method can be carried out about a point which is not 0, not x equal to 0. So, what that means, is you can write y is equal to sum over n equal to 0 to infinity, a n x minus x prime raised to n; where x prime x prime is some point this is also a power series method only thing instead of taking the instead of taking a power series around x prime equal to 0, you are taking around some other point. So, that is also a valid power series method we in the last case when we are considering the legendre equation and the Hermite equation, we were looking at power series specifically around the point x prime equal to 0, but that need not be the case. You can take power series methods about points which are not I mean about a point which is not the same as the origin.

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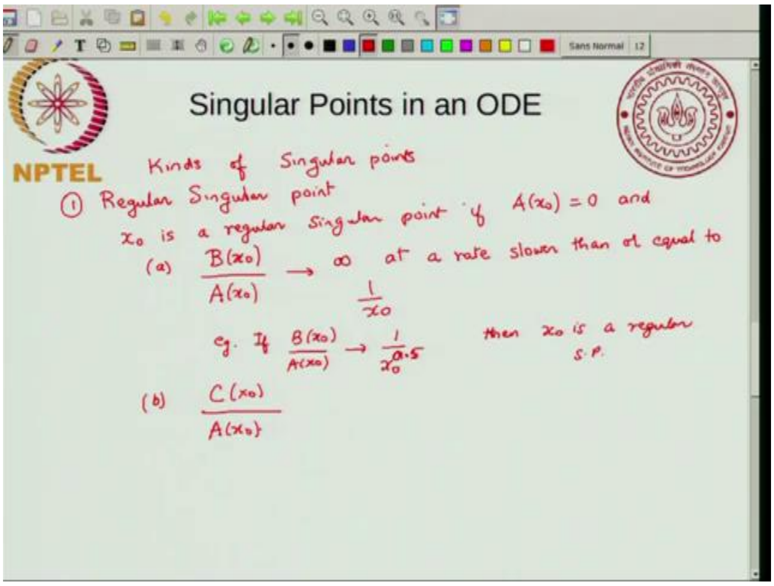

So, now let us get back to this equation. So, now, the question is we saw that when A of x 0 equal to 0, then x 0 is a singular point and you do not know you cannot use the power series method as it is. So, now, let us look at the different kinds of singular points. So, the first kind of singular point is what is called a regular singular point. So, what is a regular singular point, a regular singular point is where. So, x 0 is a regular singular point if A of x 0 equal to 0 and. So, A of x 0 equal to 0 is necessary for a singular point, now this singular point becomes regular if 2 other conditions are satisfied. So, the first condition is B of x 0 divided by A of x 0. So, since A of x 0 equal to 0, then B of x 0 divided by A of x 0 will go to infinity, but if it goes at a rate slower than or equal to 1 by x 0. So that means, B of x 0 divided by A of x 0 should be 1 should go as asymptotically as x tends to x 0, this should go as 1 by x 0 to the power 1 or less, it should not go to a power greater than 1. So, for example, if B of x 0 divided by A of x 0 goes as 1 over x 0 to the power 1.5 then this is or a 0.5, then x 0 is a regular singular point, this is one condition additional condition. So, this is one condition other condition is C of x 0 divided by A of x 0.

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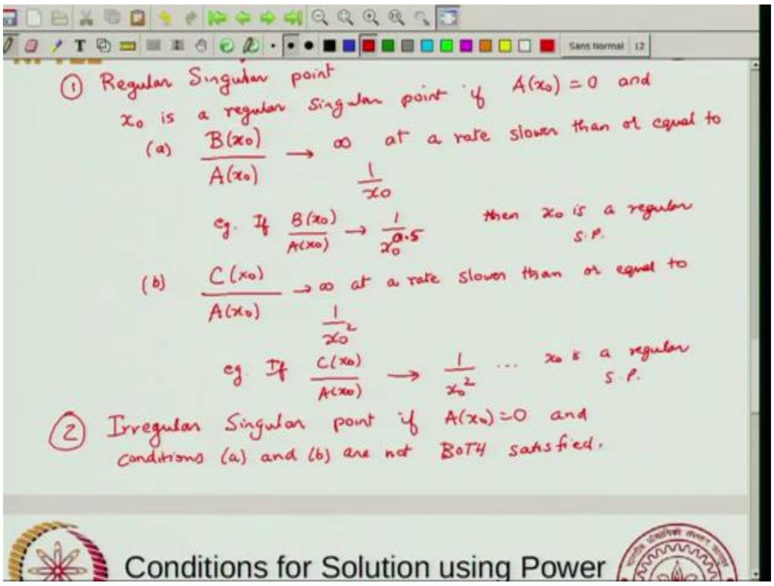

So, let us get back to this differential equation. So, you have C of x is the coefficient of y. So, C of x 0, so C of x evaluated at x equal to x 0, divided by A of x 0 now this will also goes to infinity. So, this goes to infinity at a rate slower than or equal to 1 by x 0 square; that means. So, for example, if C of x 0 divided by A of x 0 this goes as 1 by x 0 square. So, that is slower than or equal to that is equal to 1 by x 0 square, then the x 0 is a regular singular point. So, for all singular points A of x 0 equal to 0, for a regular singular point these 2 additional conditions should be satisfied. Now this is one kind of singular point, now the second so if these are not satisfied. So, I will say an irregular singular point, if A of x 0 equal to 0 and conditions a and b are not both satisfied. So, both of them have to be satisfied. So, both a and b have to be satisfied to be a regular singular point, if either of them or both of them are not satisfied then it is not a regular singular point it is an irregular singular point.

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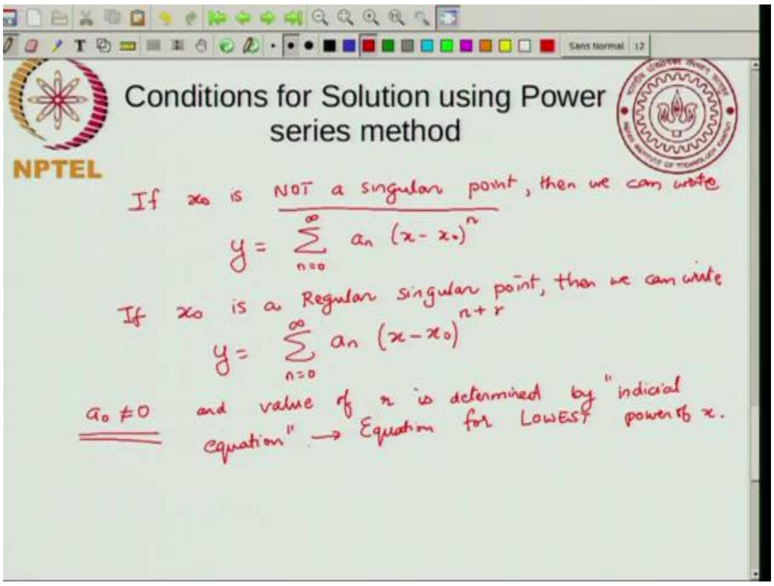

So, there are these 2 kinds of singular points, now whenever you have a singular point.

So, there are these 2 kinds of singular points, now whenever you have a singular point, so if x 0 is not a singular point, then we can write y is equal to. So, if it is not a singular point if it is a. So, point that is not a singular point is called a regular point. So, if it is a regular point then you can write y equal to n equal to 0 to infinity, you can just write the usual power series x minus x 0 raised to n. If x 0 is a regular singular point then we can write y is equal to sum over n equal to 0 to infinity a n, x minus x 0 raised to n plus r, n plus r.

So, r is some quantity, r is some real number and so you can do this and you can do the power series method, and we will illustrate this in the next lecture, but now when you solve this using the power series method. So, you will get a 0 not equal to 0, a 0 should not be equal to 0 because if a 0 is 0 then everything goes to 0, and the value of r. So, r value of r is determined by something called in indicial equation indicial equation. So, indicial equation I will just put it in quotes, we will see this through an example and indicial equation is related to the this is basically equation for lowest power of x, it might be x raised to r if see if you have n equal to 0 there will be x raised to r, if you have n minus 1 you might have a term when it is r minus 1 and so on.

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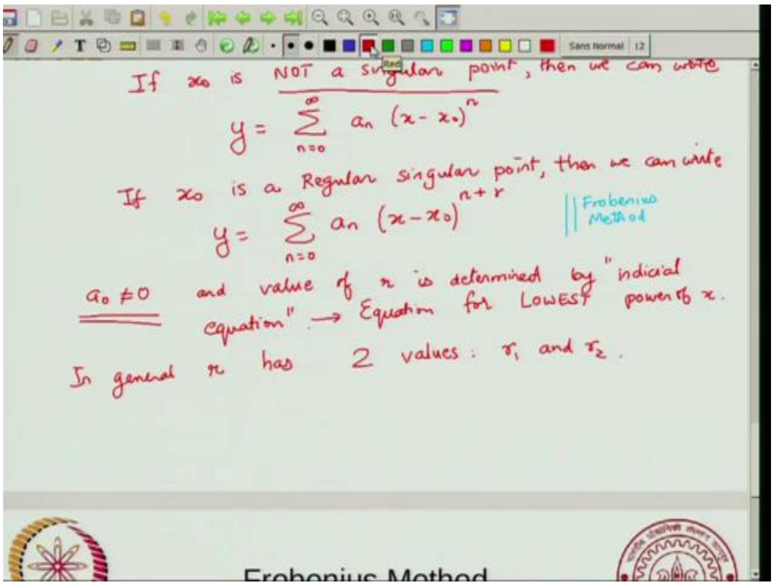

So, it is a lowest power of x, we will see that that is called the indicial equation and in general r has 2 roots 2 roots r 1 and r 2. So, the indicial equation has 2 roots and so r has 2 values, I should say r has 2 values. So, the indicial equation when you solve it you will get 2 values of r, it is like solving a quadratic equation. So, 2 values r 1 and r 2 ok.

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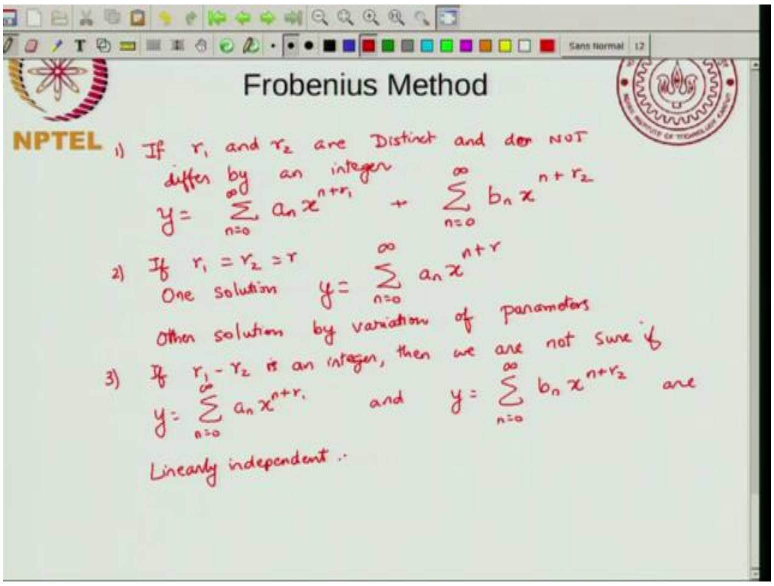

So, what you do with this equation if. So, this method is called the Frobenius method. So, the Frobenius method is where you use this form of the solution, where you use x minus x 0 raised to n plus r. So, that is called a Frobenius method. So, I will just write it here this is called the Frobenius method, and it can only be applied to regular singular points and what we said is that you will get 2 values of r.

So, there are three cases. So, the first case if r 1 and r 2 are distinct.

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They are different from each other and do not differ by an integer; they do not defer by an integer then we can surely right then you can write y is equal to some we can write it as 2 series basically, you can write it as a sum of 2 series sum over n equal to 0 to infinity, a n, x raised to n plus r 1 plus sum over n equal to 0 to infinity, a n, x raised to n plus r 2. So, these 2 are 2 power series and you can write this as a or I should probably use a different alphabet for this is a different power series. So, I will have a coefficients b n, x raised to n plus r 2.

So, if there 2 roots are distinct and do not differ by an integer you can write it in this form. If r 1 equal to r 2 then what you do is equal to r then write one solution y equal to sum over n equal to 0 to infinity, a n x raised to n plus r, and other solution by variation of parameters. So, you can get. So, we have already seen if we have a homogeneous equation then general solution has 2 undetermined constant. So, it has 2 independent solutions.

So, the first solution, so, if we have a one solution then you can get the second solution using variation of parameters now. So, you can do this I would not bother showing what the variation of parameters does, but in general you can do this.

Now, the third case if r 1 minus r 2 is an integer. So, if it is an if r 1 minus r 2 is an integer, then we are not sure if y is equal to sum over n equal to 0 to infinity, a n x raised to n plus r 1 and y equal to sum over n equal to 0 to infinity, b n x raised to n plus r 2 are linearly independent. So, the point is if these 2 differ by an integer, then you are not sure as such I mean. So, you have to actually try it, you actually do it, you substitute it you calculate the values of all the coefficients and then you see if they are linearly independent. If they are linearly independent then you are done, you have 2 independent solutions if they are not linearly independent then you take one of them and you find the second by variation of parameters.

So, this is the basic idea of the Frobenius method, and what we will do I have I made one assumption here which I should just mention it explicitly here. So, here let us say Frobenius method I have illustrated this with illustrated for x 0 equal to 0. So, I have illustrated when x 0 equal to 0, you can do it at any other point then instead of x raised to n plus r 1 I will have x minus x 0 raised to n plus r 1. So, I am just illustrating this part of the Frobenius method for x 0 equal to 0.

So, the Frobenius method is what we will see and now what we will do in the next class is to take the Bessel equation which is this, which is exactly the equation that I show that the beginning of this class, just this equation the first equation where x square y double prime, plus x y prime, plus x square minus v square y equal to 0. So, that is called a Bessel equation. So, we will just take that and we will try to apply the Frobenius method and see what we get. So, with that I will stop today’s lecture.

Thank you.