Mathematics for Chemistry
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Module - 06
Lecture - 30
Practice Problems

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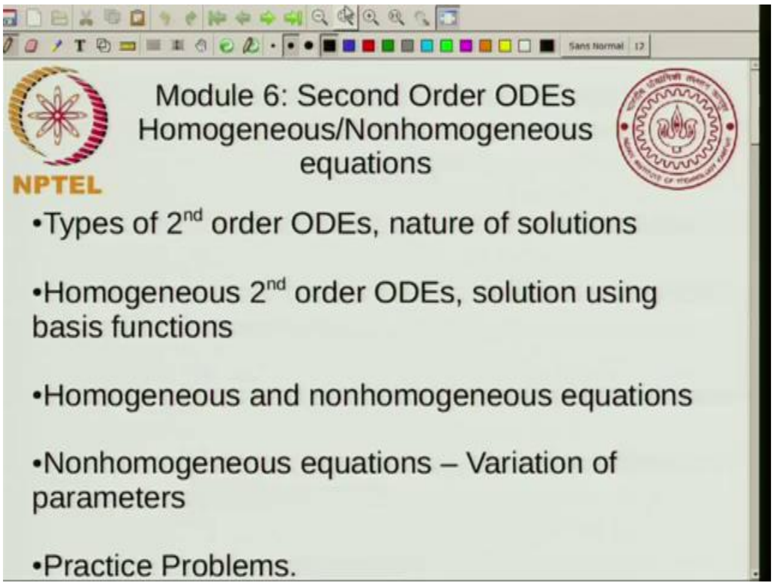

So, in the last lecture of module 6, I will be doing some practice problems on solving second order differential equations.

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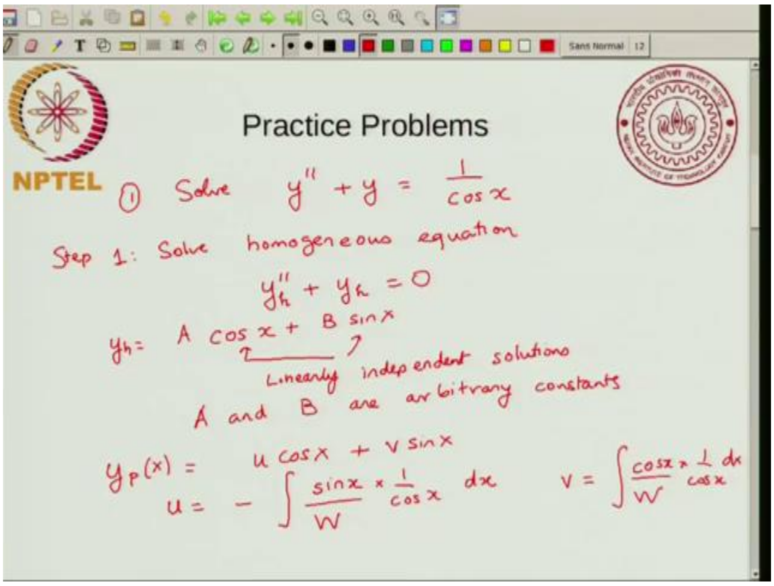

So, let us will we look at a couple of problems and one of them related to forced oscillations and resonance, but first I will take a simple problem. So, first problem is solve y double prime plus y is equal to 1 by cosine. So, I have a non homogeneous equation the right hand side is 1 by cosine x.

Now we can solve this by the method of Wronskians, but we need to solve the homogeneous equation. So, step one solve homogeneous equation. So, the homogeneous equation is y double prime or I will say y h double prime plus y h equal to 0. So, this is your homogeneous equation where I just put right hand side equal to 0 and I replace this by y h just to indicate that it is a homogeneous equation and you can see this, this is I can write the solution as cos x. So, y h is equal to A cos x plus B sin x. So, if you take cos x the second derivative will just give me the function back with a negative sin. So, if I add it, I will get 0 is similarly, if you take sin x you will get the second derivative will again give you the function back. So, cos x and sin x are linearly independent solutions. So, cos x and sin x are linearly independent solutions.

So, these are 2 linearly independent solutions. So, the general solution is written as a linear combination of these 2. So, A and B are arbitrary constraints. So, we are supposed to did determine a and B and then to determine that you will use the boundary conditions, but this is the solution of the homogenous equation. Now the solution of the non homogeneous equation we will write y p of x is equal to u cosine of x plus v sin of x and you know the way to calculate u and v. So, you know how to calculate u and v. So, u is equal to minus integral. So, the second function is sin x and sin x into the right hand side right hand side is 1 by cos x and this should be divided by w times d x and v is basically integral of cosine of x into 1 by cosine x d x divided by w. So, these are what u and v are.

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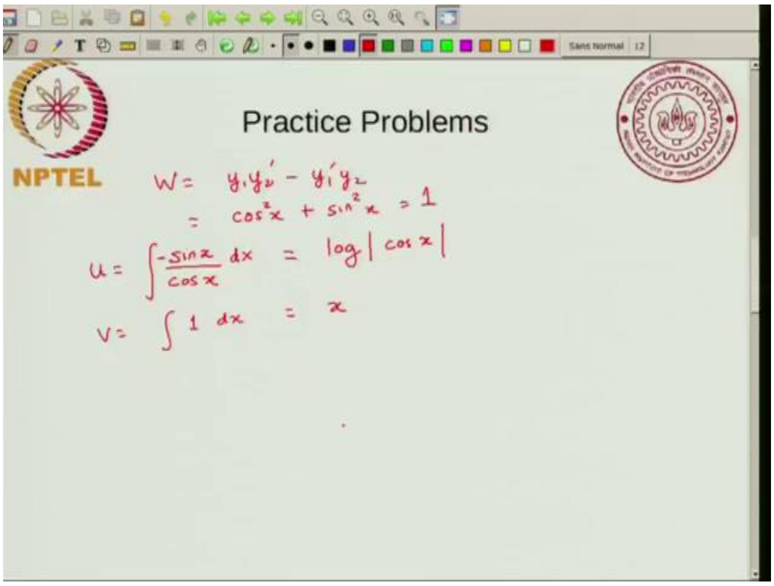

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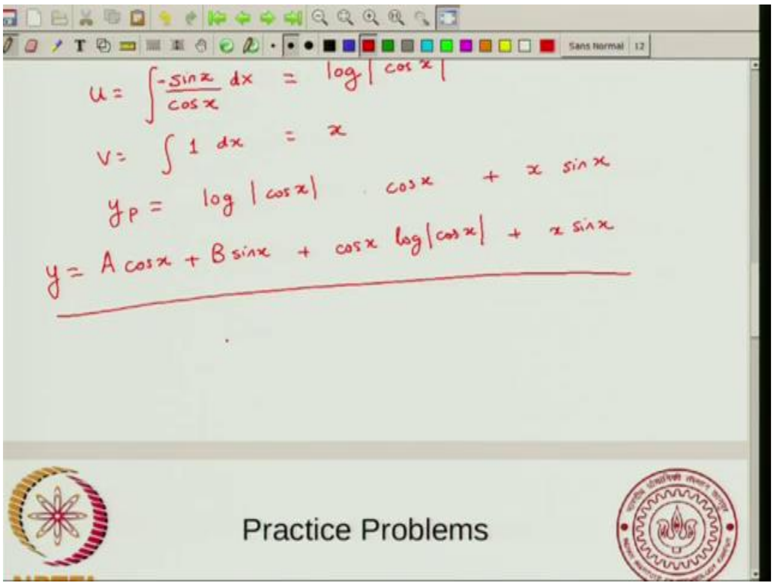

Now, let us calculate what w is. So, what is w? So, if you have sin x and cos x then w is equal to y 1 y 2 prime minus y 1 prime y 2. So, this is y 1 is cosine of x. So, cosine x y 2 prime is y 2 prime is derivative of sin which is cosine x. So, you get cos square x and then now you will have minus sin x into. So, you will get minus sin x into minus sin x. So, we will get you will get plus sin square x this is equal to 1. So, w equal to 1, so now, I can write u is equal to integral of what I had here was sin x divided by cos cosine of x sin x divided by cosine of x and there is a minus sin.

So, I will put take the minus here d x this is nothing, but because the derivative of cosine x is minus sin x. So, I can write this as logarithm of absolute value of cosine of x. So, this is logarithm of absolute value of cosine of x what about v of x v of x is just now the cosine x will cancel you just have 1 d x. So, v equal to integral 1 d x equal to x. So, I can write my y p y p is equal to log of cosine x into cosine x plus x sin x. So, my general solution y is equal to A cosine of x plus B sin of x plus cosine of x into log of cosine of x plus x sin x.

So, this is the general solution and you can see that there are 2 arbitrary constants in this solution and you can verify that when you substitute this in the in the non homogeneous differential equation it is a solution of this differential equation. So, this is an illustration of the method of the Wronskian to solve this non homogeneous second order differential equation.

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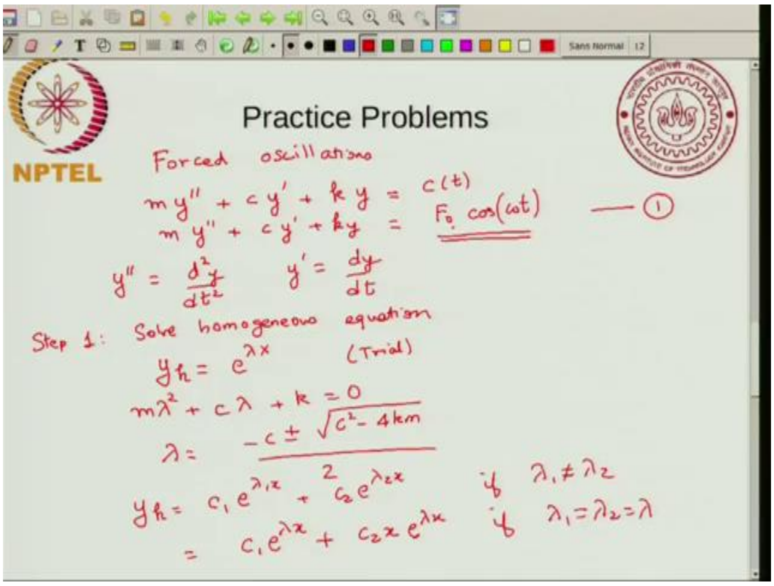

The next problem that I want to do is a very well known physical problem and that is of forced oscillations since. So, if you remember we had the damped. So, I will just write in terms of. So, if you had m y double prime plus C times y prime plus k times y. So, if you had this equal to 0 that would be a damped harmonic oscillator second order differential equation with constant coefficients would correspond to a damped harmonic oscillator now if this is equal to some function I will just call it C of t then you would call it is forced oscillator.

So, the right hand side is called the forcing term. So, it is not equal to 0. So, it is a non homogeneous equation and let us take the case when this forcing term has the form f 0 cosine of omega t and what I want to a emphasize is that y double prime is equal to d square y by d t square. So, t is a independent variable here y prime equal to d y by d t just to just to make connections with a forced harmonic oscillator. So, my differential equation has this form m times y double prime plus c y prime plus k y equal to this. So, this is my; this is the differential equation that I want to solve I will call it the equation 1. So, what you have to do is to solve this differential equation now. So, the first step is to solve the step one is to solve the homogeneous equation. So, here you will get y h is equal to. So, what we said is you get we wrote the general solution as you start with a trail solution.

So, you try your trial solution that you make is has this form e to the lambda x this is trial and when you put this in the homogenous equation. So, we will get m lambda square plus c lambda plus k equal to 0. So, that is the homogenous equation and this has the general for the solution depends on the values of lambda. So, you get 2 roots. So, lambda is equal to negative c plus minus square root of C square minus 4 k m divided by 2. So, if you are if c square minus 4 k m is less than 0 then you get oscillatory solutions if c square minus 4 k m is greater than 0 then you get only exponential solutions if it is equal to 0 then you are right at the critical point where both the both the values of lambda become concurrent. So, the general solution I can write as y h equal to c 1 e to the lambda 1 x plus c 2 e to the lambda 2 x if lambda 1 not equal to lambda 2 and equal to c 1 e to the lambda x plus c 2 x e to the e to the lambda x if lambda 1 equal to lambda 2 equal to lambda. So, that is what the general solution looks like.

Now, what about the particular solution now? Now if you want to find the particular solution you can just look at this you can just look at this and immediately write the particular solution. So, the since; this is cosine what we said is that you should try a particular solution. So, just you should try a particular solution that is the combination of sin and cosine.

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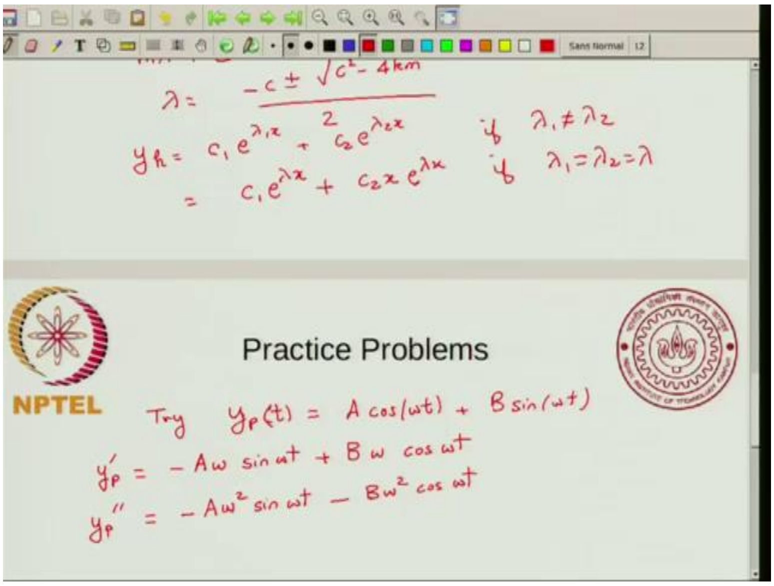

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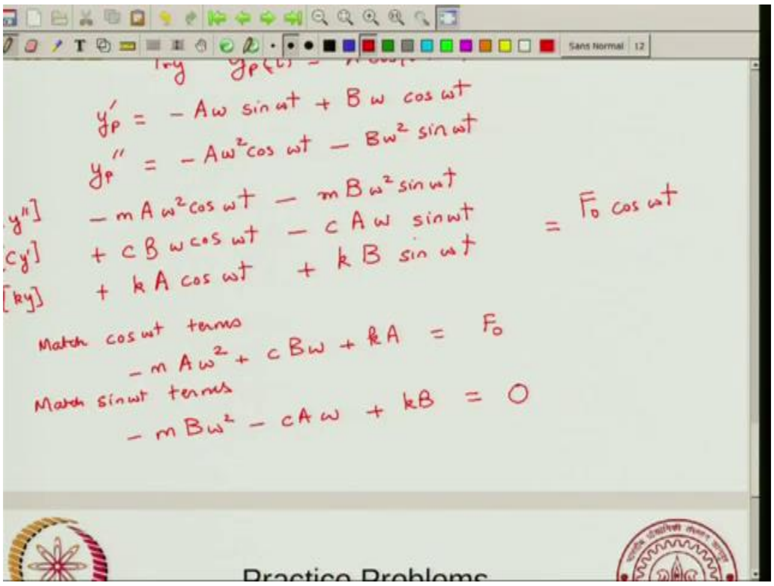

So, trial, so try y p of x is equal to A cos per y p of t rather A cos omega t plus B sin omega t (Refer Time: 11:04) y p of t. So, if you try this solution should I should I use the same A and B no I can use A and B. So, now, if you try this as y p, y p is a particular solution of the same homogenous of the same inhomogeneous equation. So, if you substitute. So, what you will get. So, first thing you will see is that y p prime is equal to minus A omega sin omega t plus B omega cos cosine omega t y p double prime is equal to minus A omega square sin omega t minus B omega square cos omega t.

Now, if you substitute these in your in your differential equation. So, if you substitute y p y p prime and y p double prime these are supposed to satisfy the differential equation one. So, when you substitute in this differential equation one what you will get you will get m times you have m y p double prime. So, you have minus m A omega square sin omega t minus m B omega square cosine omega t. So, that is the first term which was y, y m y double prime then you have c y prime and then k y. So, what is c y prime? So, when you take c y prime I will just collect all the sin and cosine terms separately. So, c times y prime will be minus c A omega sin omega t still you should get cosine and here you should get sin. So, there should be cosine this should be sin cosine and there should be sin.

Now, when you when you take now again I will just collect the sin terms and cosine terms separately, so this will be c B omega, so c times. So, I will get plus c B omega sin omega t minus c A omega c B omega cosine omega t. So, this is from c y prime. So, when I takes c y prime the cosine omega t term looks like c B omega and the sin omega term looks like minus c A omega sin omega t. So, just to remind yourself this is this came from y double prime this came from c y prime then you have k y. So, the third term was k y. So, the k y term will have the form. So, you have k times A cos cosine omega t. So, I will write it here plus k times A cos omega t plus k times B sin omega t and this whole thing should be equal to f 0 cos omega t, this whole thing should be equal to f 0 cosine omega t. So, this is the differential equation and what does what does this tell you this tells you that this should be true for all times.

So, therefore, the cosine term should match on the right hand side the sin terms should match on the right hand side. So, if you match the cosine terms cosine omega t terms then you get minus m a omega square plus c B omega plus k A equal to f 0 and if you match the sin omega sin omega t terms then you get minus m B omega square minus c A omega plus k B equal to 0 and what this gives you what this gives you if you look at it carefully this is you want to determine A and B. So, these are 2 linear equations and from these you can determine A and B. So, what you will get A as so I will just write it out explicitly we will determine A and B.

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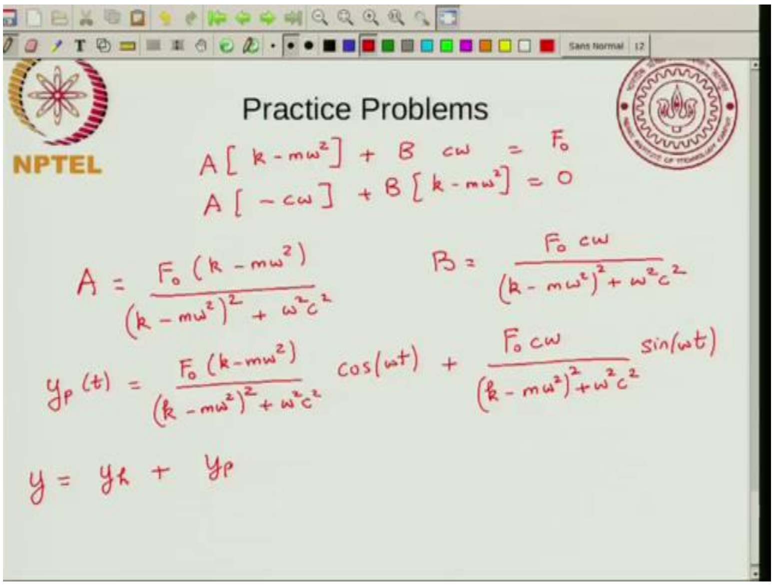

So, let us write this in the following form we will write this as a times k minus m omega square plus B times c omega equal to f 0 and you have a times minus c omega plus B times k minus m omega square equal to 0 these are my 2 equations and you can solve them and you can get the value of A and B.

So, you will have a determinant in the denominator and it is not very hard to see I will just write the final answer. So, A it will look like f 0 times k minus m omega square divided by k minus m omega square the whole square plus omega square c square, this is the determinant of this of this system of the of the matrix that describes the system and you will get B is equal to f 0 c omega divided by k minus you will get the same term in the denominator. So, now, you have the particular solution. So, I can write I have the general solution and the particular solution and I can write the overall solution.

So, the y p of t is equal to f 0 times k minus m omega square divided by k minus m omega square whole square plus omega square c square into cosine of omega t plus f 0 c omega k minus m omega square the whole square plus omega square c square into sin of omega t. So, this is my y p and I can finally, write my general solution which was y is equal to y h plus y p and y h you remember if the roots were distinct then it then it was c 1 e to the lambda 1 x plus c 2 e to the c 1 e to the lambda 1 t plus c 2 e to the lambda 2 t. So, should again this should be ts here not x now this should be t. So, the variables that that is u should be t. So, y h had had these 2 possibilities if lambda 1 is not equal to lambda 2 and if lambda 1 equal to lambda 2 then the second then each case you have lambda. So, e to the lambda t and the other independent solution is t e to the lambda t which should be t t e to the lambda t.

So, we have managed to solve this equation of a forced oscillation and we did this using a trial solution and actually you can analyze the solutions and you can show, what is what is known as the resonance condition when omega square when this term goes to 0 then you will find that the amplitude becomes very large. So, that is the condition of resonance which you can show in this case. So, I will leave that as an exercise to you to actually try to look at various cases of this of this solution, but with this we will conclude the discussion on second order homogeneous and non homogenous equations in the in the next week what I will do is try to find the general method of solving the homogenous homogeneous differential equation.

So, far the homogenous differential equation we have solved for the cases when you know one root then you when you know one basic solution you can solve for the second basic solution by variation of parameters, but what if you do not know any root how did you go about trying to generally solve a homogenous second order linear differential equation and in doing this will come up with. So, the general method that we use called the power series method. So, the next 2 weeks we will be looking at the power series method and its applications.

Thank you.