Mathematics for Chemistry
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Module - 06
Lecture - 29
Nonhomogeneous equations - variation of parameters

So in today's class, I am going to talk about non homogeneous second order differential equations and how you can write a general solution of a non homogeneous second order differential equation. We have already seen how to solve homogeneous differential equations or at least get solutions of homogeneous differential equations if one solution is known how you can get the second solution and so on. So, today I will talk about non homogeneous equations.

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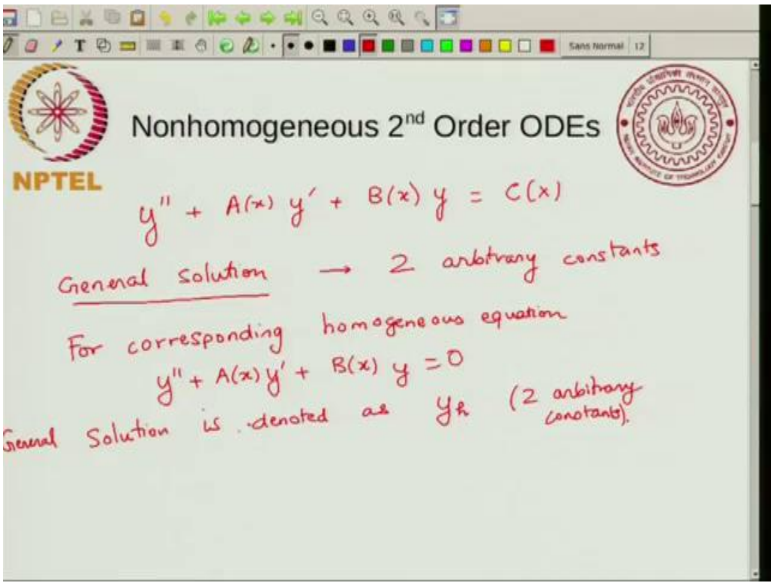

So, just to remind ourselves, this non homogeneous second order differential equation looks like this y prime plus B of x y equal to C of x this is the general non homogeneous second order differential equation and we want the general solution. So, a general solution means we will have 2 arbitrary constants.

So, you will have 2 arbitrary constants and this is what we want to get. So, we want to get a general solution of this equation now for the homogeneous case. So, for corresponding homogeneous equation and I will just; so the corresponding homogenous equation is will look like y double prime plus A of x y prime plus B of x y equal to 0. So, for this homogenous equation the solutions have denoted as y h of x and or simply y h and what I should say is that this is the general solution the general solution is denoted as y h.

So, I will use y h as the general solution of the corresponding homogenous equation. So, y h this will have 2 arbitrary constants we saw, how we can get y h using linear combination of basic functions, now what we want to do is to write a general solution for this non homogeneous equation and that will also involve 2 arbitrary constants.

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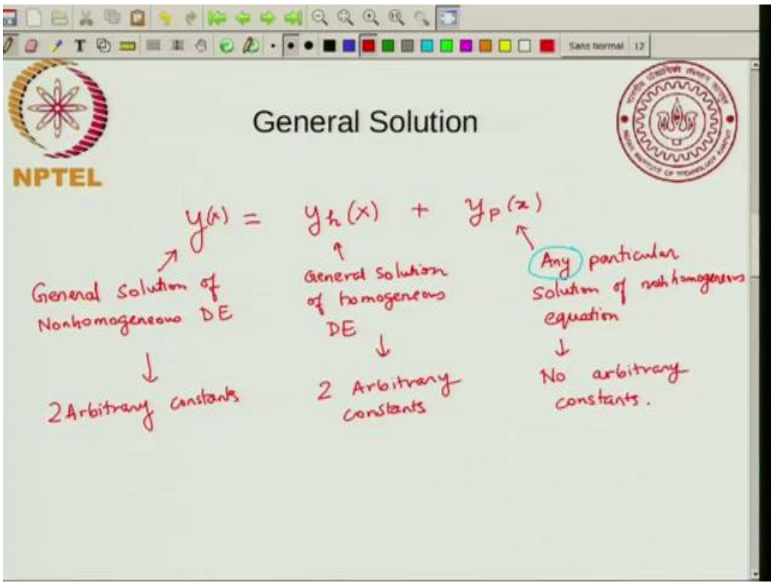

So, I will just at the way you write the general solution of the non homogenous equation. So, if y is the solution of the non homogenous equation then you can write y as y h y of x I will just put y of x as y h of x plus y p of x, so, y h of x. So, y is a general solution of non homogeneous differential equation this is a general solution of y h is the general solution of homogeneous differential equation and y p is any particular solution of non homo of the non homogeneous equation and when I say it can be any particular solution the word any is very important you can choose any particular solution and you can write the general solution is that.

Now, if you remember when we did homogeneous equations we said that you know you can write you can write different basis for the same equation you can write it in terms of different bases. So, same way I mean there are different ways to write the general solution and therefore, you can choose any particular solution of the non homogenous equation and that will give a; that we give a general solution of the non homogenous equation.

So, just an example of this; so what is important is that we are getting a general solution of the non homogenous differential equation which will have this will have 2 arbitrary constants. Now the general solution of the homogenous differential equation will also have 2 arbitrary constants, but this particular solution will have no arbitrary constants. So, I think it is best if we look at an example just to illustrate this point. So, the 2 arbitrary constants for this for y came entirely from y h.

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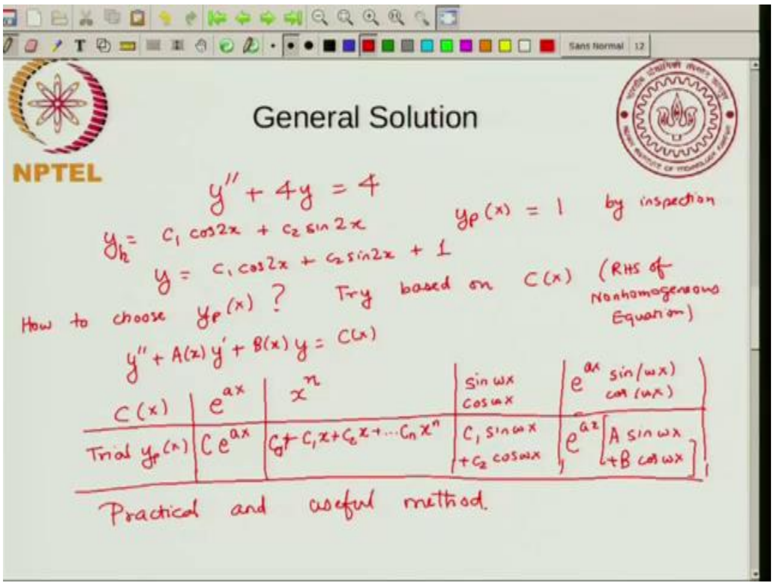

So, let us look at an example now. So, suppose you have the differential equation y double prime plus 4 y to equal to 4 it is a simple differential equation and in this case you can you can see that the general solution of this is given by y is equal to. So, this is y double. So, if you look at the homogeneous equation. So, y h now this is homogeneous equation you have y double prime plus 4 y equal to 0 and the general solution of that will be a sum of cos 2 x and sin 2 x. So, I can write it as C 1 cos 2 x plus C 2 sin 2 x. So, this is my y h, it has 2 arbitrary constants.

Now, you need to find a y p, you need to find one particular solution and you see that there is a 4 on the right. So, if y equal to 1 then this term will become 4 and y double prime of 1 is 0. So, you will get 4 equal to 4. So, particular solution y p of x just choose as one just by inspection we will come later to show you how it is possible to choose this in a more reasonable way, but if you just use this just by inspection we can choose this as the particular solution

So, the general solution y looks like C 1 cos 2 x plus C 2 sin 2 x plus 1 that is the general solution of this of this differential equation and you can verify this. So, suppose you just suppose you just substitute this in this equation what you will get is exactly that you can see that this will clearly satisfy this differential equation. So, what we have shown is that how you can get a general solution of a non homogeneous differential equation from the general solution of the homogeneous differential equation and one particular solution.

Now, how do you choose? So, in this case we just saw we and we decided what y p was. So, how do you choose y p in general now the idea here is that you try based on C of x based on C of x which is the right hand side of non homogeneous equation? So, what I mean is if you; your equation looks like y double prime plus a of x y y prime plus b of x y equal to C of x this thing equal to 0 if your homogeneous equation and it and when it is not equal to 0 you have to worry about the non homogeneous equation.

So, your choice of y p depends on what C x's and a general recipe to calculate y p is to choose it based on C or based on C x. So, I will just make a small chart here. So, I will write C x out here and I will write trial y p of x. So, so what I will show you, I will just make a chart with various choices.

So, suppose my C x looks like e to the e to the a x e to the some constant times x then my trial will also be some C e to the a x. So, you just take e to the a x multiplied by some constant that would be my trial function suppose on the other hand is my C x looks like looks like x raised to n then my trial function will be will look like a polynomial in n. So, I take it as I take it as C one or C 0 plus C 1 x plus C 2 x plus up to C n x, raise to n if my C of x is just a power of x I try with these constants and then I substitute in the equation and then I will find out what the values of these constants should be.

Similarly if you have sin omega x or cosine omega x in either case you just take your trial function as C 1 sin omega x plus C 2 cosine omega x and then you substitute in the differential equation and then you find out the values of C 1 and C 2, if you had a product of e to the a x times a x times sin omega x or cosine omega x. Then again your trial function will also look like will look like e to the a x and then you have a sin omega x plus b cosine omega x. So, you try this form can you substitute in this equation and then you calculate your; you calculate the values of A and B. So, what I am showing you here is a recipe of how you choose your particular solution just by looking at looking at what your form of C x's.

Now, this is this works often, but there is also a more and we will see examples this is a very practical method. So, I will emphasize that this is a very practical and useful method then because then when it works quite well; however, a formal method of calculating y p is, what is called variation of parameters. So, this is a formal method of no trial. So, no trial function, so, you do not use any trial function you just take whatever your C of xs and you solve this.

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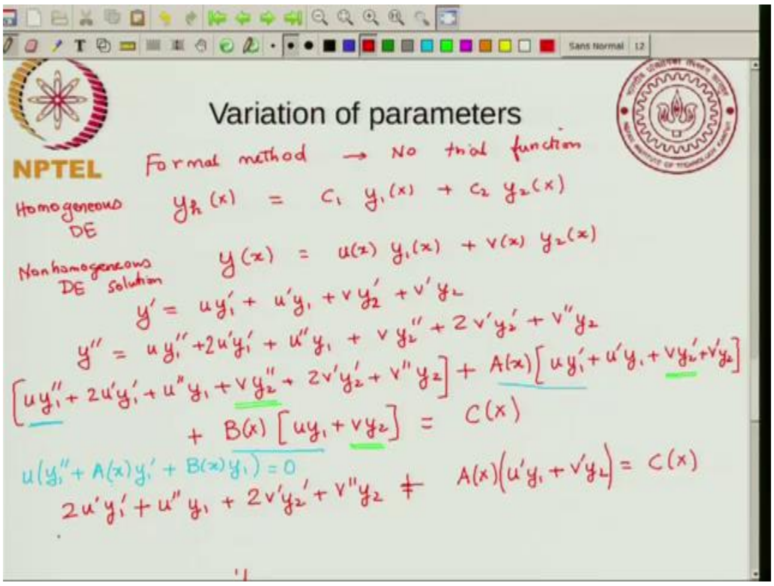

So, let us take an example or let us see what the formal method is, so the formal method is your suppose your y h of x is equal to this is solution of homogeneous equation the corresponding homogeneous equation suppose this solution you wrote it as a form C one y one of x plus C 2 y 2 of x then the non homogeneous equation d e solution what you write it is you write it as y of x is equal to u of x times y 1 of x plus v of x times y 2 of x.

So,. So, instead of just C one and C 2 being constants for the non homogeneous equation you imagine that you replace it by some function u and this by some function v. So, again once you do this then you have to calculate what is u and v. So, this is just like the variation of parameters that we used in the case of in the case of first order differential equations. So, if you will take this and substitute in the original differential equation what you will get. So, let us just work out what happens when you substitute. So, you will get y prime is equal to u y 1 prime plus u prime y one plus v y 1 y 2 prime plus v prime by 2 and then you calculate y double prime say that will be u y 1 double prime plus write a 2 u prime y one prime plus u double prime y 1 plus v y 2 double prime plus 2 v prime y 2 prime plus v double prime y 2.

So, you have this and when you substitute this in your non homogeneous equation then what you will get is that your instead of y double prime you have you have all these terms. So, you have u y 1 double prime plus 2 u prime y 1 prime plus u double prime y 1 plus v y 2 double prime plus 2 v prime y 2 prime plus v double prime y 2. So, this is your y double prime and then you have you have plus a of x into y prime y prime is nothing, but u y one prime plus u prime y 1 plus v y 2 prime plus v prime y 2.

So, remember this whole thing was your y double prime a of x times y prime and then you have plus b of x times y, y is nothing, but u y 1 plus v y 2 equal to C of x. So, what we did was we just took this trial form of y and we substituted in this differential equation now you have to look at it term by term, but what you will see is the following what you can see immediately remember y 1 and y 2 are solutions of the homogeneous equation. So, y 1 and y 2 satisfy the homogenous differential equation. So, you can see that if you look at this term look at b times y 1. So, there is a u in front then you have y 1 double prime with a u in front and then you have a times y one prime with a u in front.

So, these 3 terms are just u times. So, if I just write this as u times y 1 double prime plus a a of x y 1 plus b of x y 1. So, that is what you get when you add these 3 terms these 3 terms that are underlined in light blue colour and now you can see that the term in the bracket is equal to 0 because y one satisfies the homogeneous equation. So, these 3 terms add up to 0.

Similarly, you can look at you can look at this term and then you can look at the corresponding terms here which looks like this and here you will have a term v y 2 double prime. So, those 2 also add up to 0. So, then what you are left with after this is the following you are left with an equation that looks like this. So, you have 2 u prime y one prime plus u double prime y one plus 2 v prime y 2 prime plus v double prime y 2 this is equal to plus a of x times u prime y one plus v prime y 2 equal to C of x.

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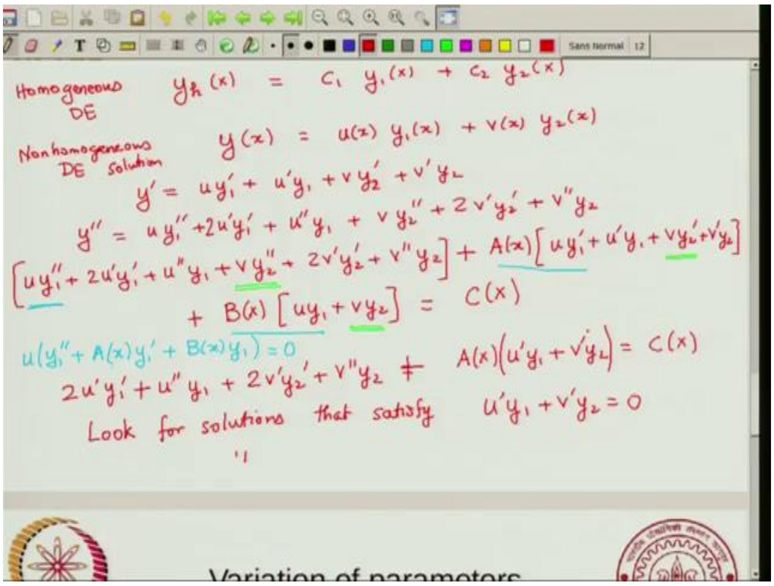

So, notice what we when we did this we got entirely rid of the b term and we are just left with this now actually even this itself is not that straightforward to solve because you have you have u and v and in general you cannot solve this straightforwardly, but you can do one thing you can look for solution. So, here is a trick that we use. So, we look for solutions for solutions that satisfy u prime y 1 plus v prime y 2 equal to 0. So, this term that multiplies a will go to 0.

So, we look for solutions that satisfy this identity and now. So, what this gives you is the following 2 equations.

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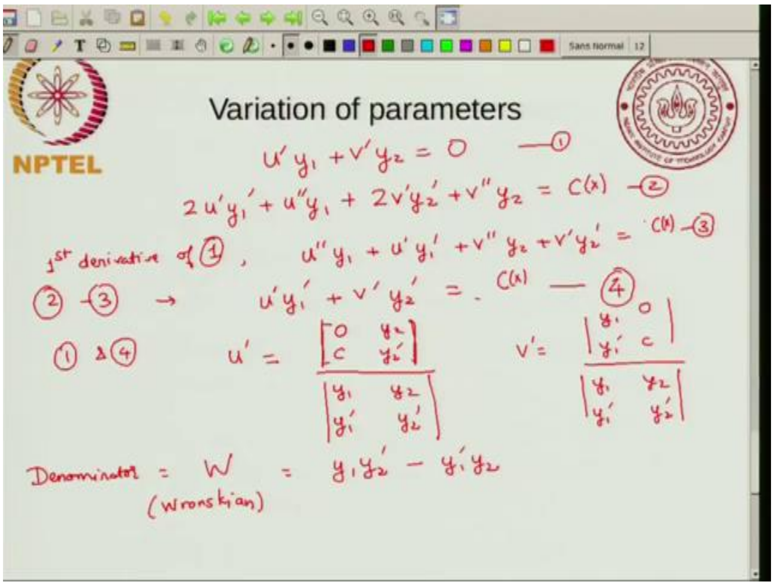

So, the first equation is u prime y 1 plus v prime y 2 equal to 0 and the second equation looks like so. So, now, my second equation will have the form 2 u prime y 1 prime plus u double prime y 1 plus 2 v prime y 2 prime plus v double prime y 2 equal to 0.

So, this may look a little complicated, but actually the solution of this is fairly straightforward now once you give once you impose this condition if you take the first derivative of the of the first equation. So, let me call this equation one and let me call this equation 2. So, if you take first derivative of one then you get u double prime y 1 plus u prime y 1 prime plus v double prime y 2 plus v prime y 2 prime equal to 0 and you can see that in this equation you have you though in the second equation you have u double prime y 1 and instead of single u prime y 1 prime, you have twice u prime y 1 prime.

So, now, if you just subtract the second and third equation, so if you just subtract equation 2 minus 3 then you get u prime y one prime plus v prime y 2 prime equal to 0 and let me call this equation 4, now if you just look at 1 and 4, now this looks like u prime times y 1 and v prime times y 2 u prime times y 1 prime and v prime times times y 2 prime equal to 0.

Now, you can immediately. So, this should be equal to sorry this is not 0 this is; this should be C of x this should be C of x this quantity remember what is equal to C of x. So, and now from the first derivative what we will get on the right hand side is not 0, but we will get d by d x of C c of x. So, finally, what you get is this is equal to d C of x by d x and this is very nice because what if you look at one and 4 they if you if you just look at these 2 then using your matrix methods you can write u prime is equal to in terms of determinant. So, what you will get is 0, I will just call it C prime and then you have y 2 y 2 prime. So, this determinant divided by y 1 y 2 y 1 prime y 2 prime.

So, what I did was I just looked at 1 and 4 as 2, 2 as a system of linear equations for variables u prime and v prime. So, then you immediately get this and you also get v prime is equal to you will get y 1 y 1 prime 0 C prime and you will get y 1 y 2 y 1 prime y 2 prime should just be C of x, this should not be d C by d x it should just be C of x.

So, because we took a derivative of one, so what I wanted to emphasize is that is that this should be C of x and so and so I get my u prime and v prime in this form and you can write u as the integral of this whole thing this denominator this denominator.

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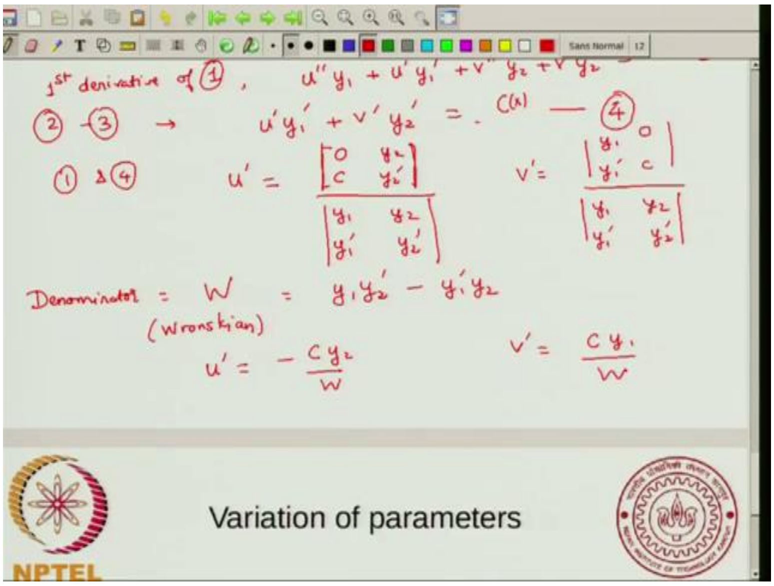

So, the denominator is referred to as w or the Wronskian and this is basically y 1 y 2 prime minus y 1 prime y 2 and so you can write your; you can write your u and v in the following form. So, what you will get is u prime is equal to minus C y 2 by w and v prime is equal to C y 1 by w and so what this variation of parameters gives you it gives you u equal to integral minus C y 2 by w d x and v equal to integral C y 1 by w d x.

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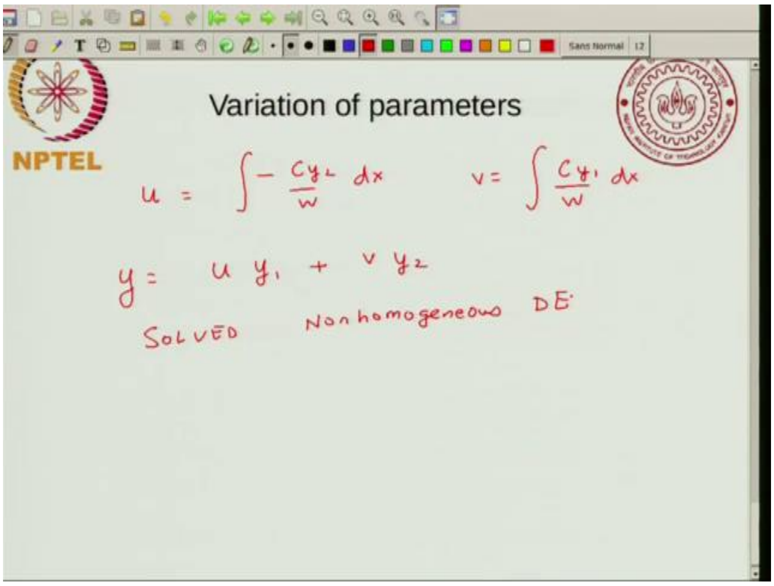

So, we can get; we calculated u and v and this basically so, once we have u and v we can write y is equal to u y 1 plus v y 2. So, we have solved non homogeneous differential equation. So, if you know y 1 and y 2 of the homogeneous equation then you can solve the non homogeneous differential equation using this method of variation of parameters.

So, in the next class, I will discuss some practice problems and then in the next module that is the following week, we will start looking at how to get the solutions of the homogeneous equation, how what is the general method of getting the solutions of the homogeneous equation and this will lead us to something called the power series method.