Mathematics for Chemistry
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Module – 06
Lecture – 27
Homogeneous and non-homogeneous equations

So in today's class I will be discussing some more solutions of second order differential equations, I will be talking mostly about homogeneous equations and I will very little bit about non homogeneous equations.

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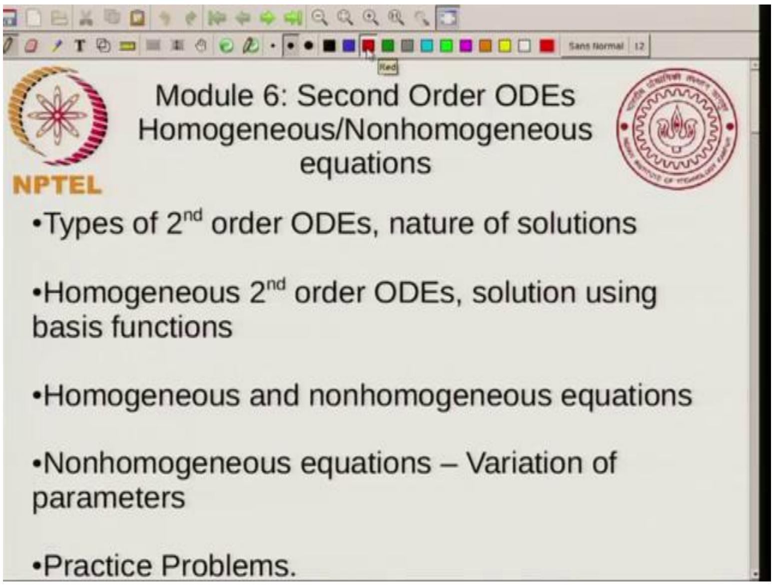

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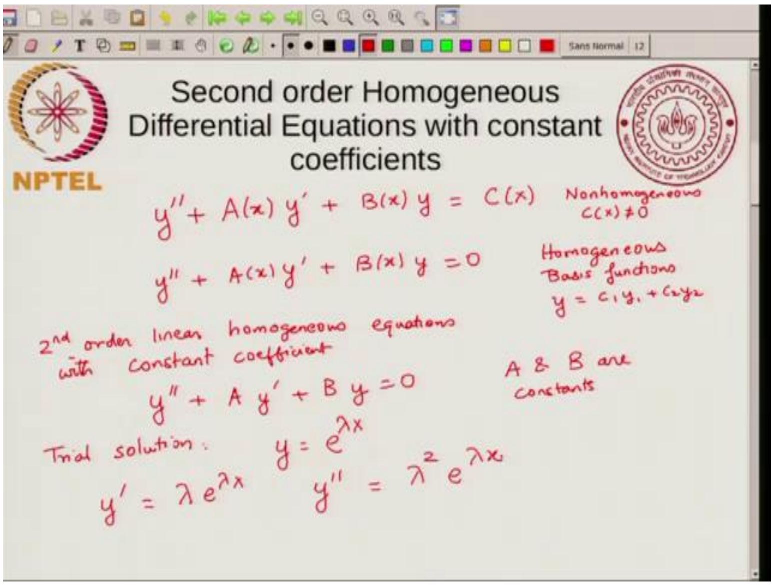

So, let us just remind ourselves what we know about second order differential equations. So, we said that we can write as a general second order differential equation second order linear differential equation as in this form A of x y prime plus B of x y is equal to C of x this is if C of x is not equal to 0, this is called non homogeneous C of x not equal to 0 if you have C of x equal to 0 then you have something like this y double prime plus a of x y prime plus B of x y equal to 0, this is homogeneous and we saw that for homogeneous equations you have basis functions. So, you can write you can write your general solution as a linear combination of basic functions C 1 y 1 plus C 2 y 2 where y 1 and y 2 our solutions or solutions of the differential equation.

So, homogeneous differential equations have this nice feature now which is not therefore, non homogeneous equations. So, you cannot write this for non homogeneous equations and you can you can easily verify this for certain non homogeneous equations. So, today I want to first start the discussion of a special type of homogeneous linear second order differential equation it is those with constant coefficients. So, what do you mean by constant coefficients. Solves second order linear homogeneous equations with constant coefficients. So, all you do for constant coefficients is that you will say this A of x and B of x are constants these are these are in general functions of x, but if these functions are constants then you will get second order homogeneous equation with constant coefficients. So, you will have y double prime plus a times y prime plus B times y equal to 0, A and B are constants. So, A and B are constants and; obviously, y is a function of x.

Now, how do you solve, how do you solve such a differential equation? Now again there are you can do this using a the system of differential equations you can convert it into 2 to first order differential equations and then you can use matrix methods alternatively you can directly do this in a using trial solutions. So, you are trial solution y equal to e raise to lambda x and I did not put a constant before this because you can put any constant and it will still be a solution. So, since it is a homogeneous equation I can always multiplied by constant and get another solution, but so, I will just put y equal to e to the lambda x I will not bother putting the constant in front. So, if y equal to e to the lambda x then you know that y prime equal to lambda times e to the lambda x and y double prime equal to lambda square e to the lambda x and now if you substitute these in the differential equation.

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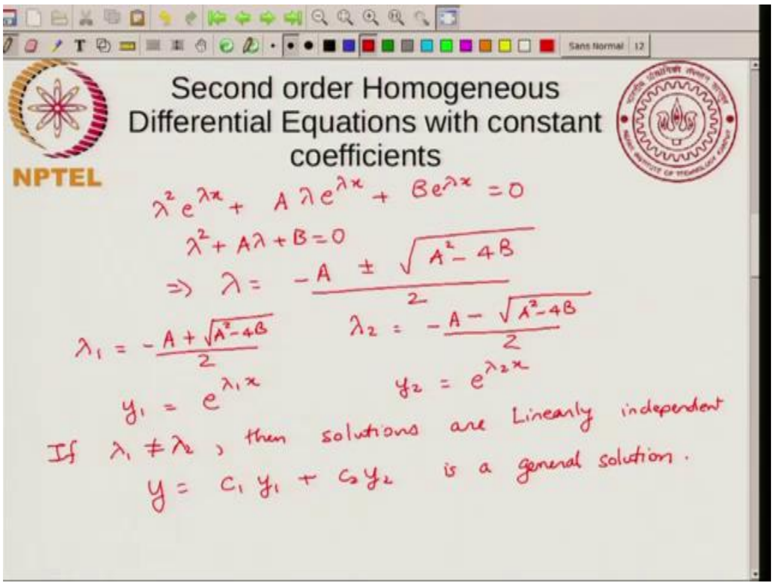

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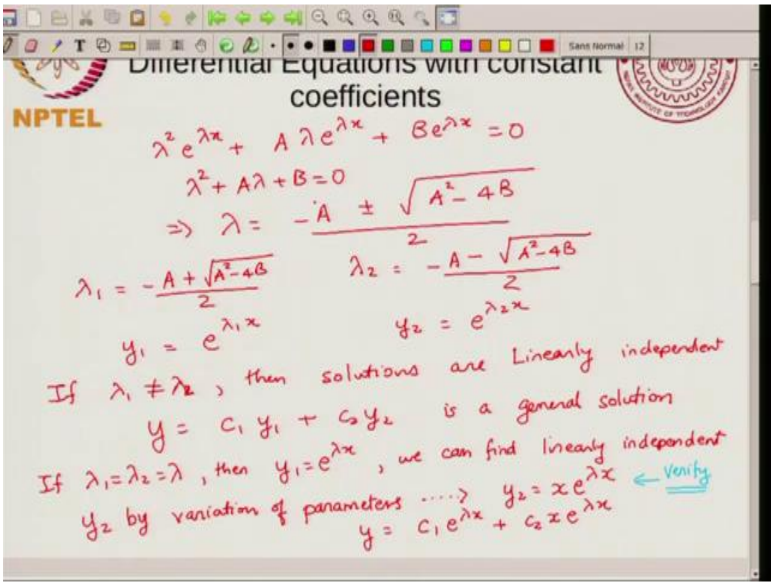

Then you get what you get when you substitute this in this. So, y double prime is lambda square e to the lambda x plus A lambda e to the lambda x plus B e to the lambda x equal to 0 and since this is true for all x, I can cancel this e to the lambda x term I get a relation lambda square plus A lambda plus B equal to 0 and A and B are constants. So, this implies lambda equal to negative a plus minus square root of A square minus 4 B divided by 2.

So, it is a quadratic equation and these are the solutions. So, what I can write? I can write let us say I call it lambda 1 as minus a plus square root of a square minus 4 B divided by 2 and if I write lambda 2 as minus A minus square root of A square minus 4 B by 2 then my I have 2 solutions I have 2 solutions. So, I have y 1 is equal to e to the lambda 1 x and y 2 equal to e to the lambda 2 x.

Now if lambda 1 and lambda 2 are distinct then these 2 solutions are linearly independent. So, if lambda 1 not equal to lambda 2 then solutions are linearly independent and if they are linearly independent then you can write general solution y is equal to C 1 y 1 plus C 2 y 2 is the general solution. Now this is very nice because we converted our differential equation into a simple quadratic equation and we know the roots of the quadratic equation and we can immediately find the solutions. If they are not linearly independent if these if lambda 1 is equal to lambda 2 what you do well it is not too hard you can you can find one solution, but you learnt in the last class that once you have one solution you can find the second solution by variation of parameters.

So, the second solution is, if I just have 1, y 1 I can find a linearly independent solution by using method of variation of parameters. So, I will just write. So, if lambda 1 equal to lambda 2 equal to lambda then y 1 equal to e to the lambda x and we can find linearly independent y 2 by variation of parameters what we did in the last class with those you and so on. So, if you go ahead and do this then what you will get is that you will find that this will lead to y 2 is equal to x e to the lambda x. So, the general solution y is equal to C 1 e to the lambda x plus C 2 x e to the lambda x these 2 are linearly independent solutions and that is what you can do and you can easily verify this.

So, I will just I will just say that you can verify this verify this using the methods that we did in the last class. So, you write your y 2 as u times y 1 and then you substitute in the differential equation and after all this cancellation and everything you will find that u u prime equal to 0 and. So, u will be equal to constant capital u and. So, and an after that since u is a constant your small u just becomes equal to x. So, and. So, I can write y 2 x x e to the lambda x what this shows is that if you have a homogeneous equation with constant coefficients.

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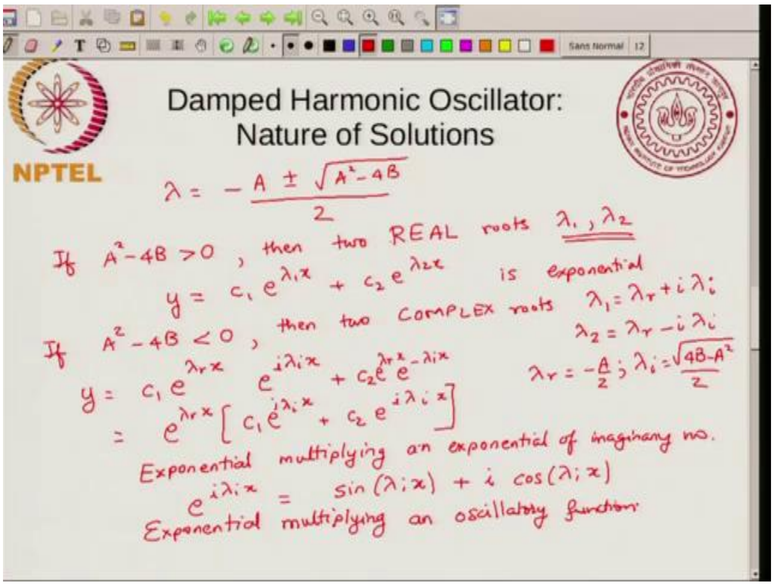

Then you know the solutions you can always find 2 linearly independent solutions now let us look at the nature of the solutions. So, the solutions we said that you have lambda is equal to negative a plus minus square root of a square minus 4 B divided by 2. So, if a square minus 4 B is greater than 0 strictly greater than 0 then we have 2 real roots 2 real roots; that means, lambda 1 and lambda 2 are real these are real numbers then the solution y is equal to C 1 e to the lambda 1 x plus C 2 e 2 the lambda 2 x is exponential. So, it is basically a sum of exponential functions. So, it involves 2 exponential functions and it is a sum of these 2 exponential functions and you know lambda 1 can be positive or negative if it is positive then it is exponentially increasing if it is negative it is exponentially decreasing.

Now if a square minus 4 B is less than 0 then 2 complex roots, I will just call the 2 roots as lambda 1 equal to lambda real plus I times lambda imaginary. So, lambda is lambda is the imaginary part which is basically this term the second part of this of this solution lambda real is basically minus A by 2 and similarly lambda 2 will be exactly the same lambda real minus I times lambda imaginary where I will just emphasize again. So, lambda real is equal to minus A by 2 and lambda imaginary is equal to 4 B minus a square root of that divided by 2. So, since a square minus 4 B is less than 0. So, for B minus A square is greater than 0, so the imaginary path is just this that is multiplied by the by I, so the 2 complex roots have this feature. So, then I can write my y as C 1 e to the now lambda 1 will have a lambda real plus I lambda imaginary.

So, I will just take the lambda real times x and then I have e to the i lambda i times x and in the second case for my second function again I will have C to e to the lambda r x and e to the minus lambda i x. So, I can write this as e to the lambda r x C 1 e to the lambda i times lambda i x plus C 2 e to the i times lambda i x and again I want to emphasize that both lambda r and lambda i are real quantities. So, this is basically an exponential multiplying a multiplying and exponential of imaginary of imaginary number and you know that exponential of imaginary number can be written as a linear combination of sines and cosines. So, you know that e to the i lambda i x can be written as sign of lambda i x plus i times cosine of lambda i x. So, this is a general property of an imaginary number. So, you can write it as a sum of sines and cosines. So, essentially what you have here is these functions look like a combination of sines and cosines.

So, what you have is an exponential function multiplying an oscillating function. So, you have an exponential function multiplying an oscillating function. So, you have an exponential multiplying and oscillatory function.

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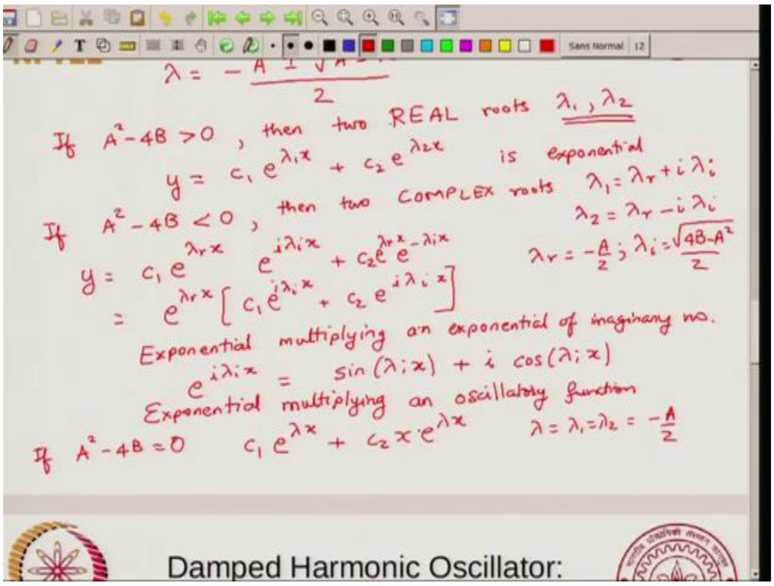

So, this is the case when a square minus 4 B is less than 0. So, a square minus 4 B is greater than 0 then you have only exponential functions if you have a squared minus 4 be less than 0 you have exponential multiplying an oscillatory function and what about of a square minus 4 be equal to 0 a square minus 4 be equal to 0 then the solutions will just be it look like e to the lambda x and you have C 1 e to the lambda x plus C 2 x e to the lambda x.

So, you have only 1 root. So, both the roots become the same because lambda equal to lambda 1 equal to lambda 2 equal to minus a by 2. So, the solutions just look like this. So, lambda is just minus a by 2. So, you have e to the minus a by 2 x and you have C 1 plus C 2 x. So, the other linearly independent solution is just x into the lambda x.

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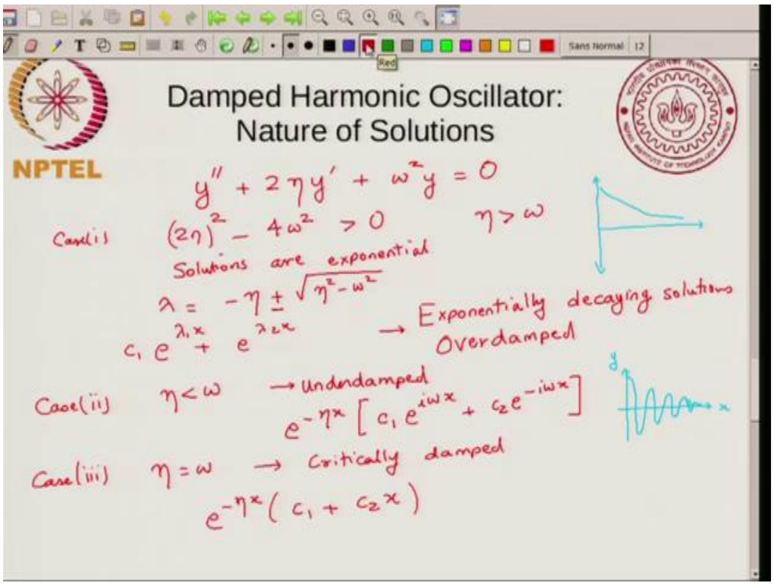

Now, now these ideas that we said here are actually seen in a very practical situation which is called the damped harmonic oscillator. So, what is the damped harmonic oscillator? So, the damped harmonic oscillator the is has satisfies the differential equation y double prime plus 2 eta y prime plus omega square y equal to 0. So, if you did not have this middle term then it would just be a simple harmonic oscillator y double prime plus omega square y equal to 0, but this is a damping term that is there and n I should mention that eta and omega square r both greater than 0.

So, now, now this is exactly of this form that we had before this is exactly of the form of this second order differential equation with constant coefficients. So, it has exactly this form instead of a you have 2 eta and instead of B you have omega square and the reason we put omega square is because we want this quantity to be strictly greater than 0 now there are 3 possible cases. So, case one is when is when the; what you have is you can say that it is square of this. So, 2 eta square minus 4 omega square is let me say greater than 0.

So, in other words this will, so I can cancel the 4 and what I will get is eta square is greater than omega square or eta is greater than omega. So, you have the case. So, if eta is greater than omega then this quantity is greater than 0 and what we said if this quantity is greater than 0 you have exponential solutions. So, the solutions are exponential and you will recall the solutions look like e to the lambda 1 x and e to the lambda 2 x where lambda 1 and lambda 2 have this form I will write it explicitly. So, lambda equal to. So, in this case you get minus eta plus square root of eta square minus omega square and you have. So, you have plus minus this and since eta is greater than omega then this solutions will be purely exponential. So, your solution looks like e to the lambda 1 x plus e to the lambda 2 x.

So, this is the first case now notice that if eta is greater than 0 and if eta is greater than omega then you know that this; the root well this is plus. So, this quantity will always be less than eta. So, this lambda will always be negative. So, what you have is exponentially decaying solutions. So, I can write this as exponentially decaying solutions. So, what you say that the solution is over damped. So, this is called an over damped oscillator in case 2 you have eta less than omega and now we are solutions are now you have exponential multiplied by this exponential of imaginary.

So, exponential multiplied by oscillating. So, these are. So, in this case your solutions are actually you have both these oscillations and you have the exponential functions. So, in this case you say that the solution is under damped and what does this solution look like. So, it look like it look like e to the minus eta x C 1 e to the I omega x plus C 2 e to the minus I omega x you can verify this you can verify this just I just is wrote exactly what you had in this case lambda r and lambda i the third case lambda equal to omega is called as critically damped and here the solution looks like solution looks like e to the minus lambda x C 1 plus C 2 x.

So, basically each of these solutions has different feature in the first case do this exponential functions in the second case you had exponential multiplied by some oscillating functions. So, this might look like if you plot this if u plot y as a function of x y as a function of x then what you will get is you have an exponential minus x. So, you have you have this, but then you have oscillation. So, it looks like it looked like oscillations whose amplitude is constantly decreasing. So, this is what happens in the under damped case in the over damped case you just have you do not see any oscillations you just see you just see exponential. So, since they are negative exponential you will just see something like this and the critically damped is just at the boundary when you are about to get oscillating solutions. So, you do not have oscillating solutions you are just about to get right when you are going from exponential to this exponential multiplied by oscillating solutions.

So, these are well known solutions of the damped harmonic oscillator and we can see them in a very easy way using the solution of the differential equation now next I will just mention another type of equation which you can also solve using very similar tricks.

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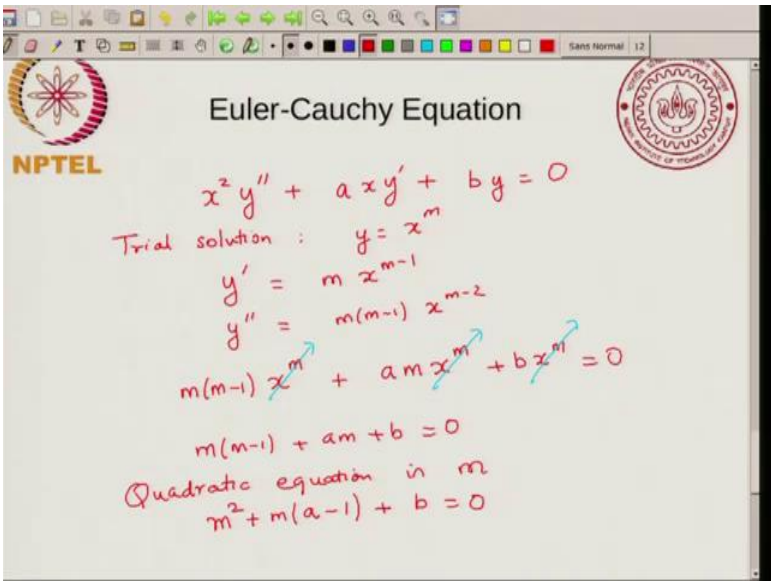

So, this equation has a form. So, this is called the Euler-Cauchy equation and this as a form x square y double prime plus a which is a constant times x y prime plus b times y equal to 0. Now notice what I did was when I have the second derivative I multiplied by x square when I have a first derivative I multiplied by a constant by x. So, that is what the terms look like. So, if you want you can take the x square in the denominator, but you do not need to do that writing it in this form suggests you what the solutions should look like. So, what should the solution look like? Now you know that when you take a second derivative of some power of x then you get power of x reduced by 2.

So, now, you are squaring now you are multiplying by expressive you will get back to the same power of x. So, the trial solution and here y equal to x rise to n. So, if you trial solution of y equal to x raise to n then you get y prime is equal to m x raise to m minus 1 and y double prime equal to m m minus 1 x raise to m minus 2 and now if you substitute in this equation.

So, x square times y double prime will give you m m minus 1 into x square into x raise to m minus 2. So, all will get is m, m minus 1 into x raise to m and plus a; now x times y prime is just m times x raise to now you had n minus 1 and multiplied by x j gives you x raise to m and b times y is just b times x raise to m equal to 0 and by doing this you can see that you can cancel the x raise to m. So, just as in the last case you get an equation m m minus 1 plus a m plus b equal to 0 this is a quadratic equation in m in m and I can I can write it explicitly as m square plus m times a minus 1 plus b equal to 0.

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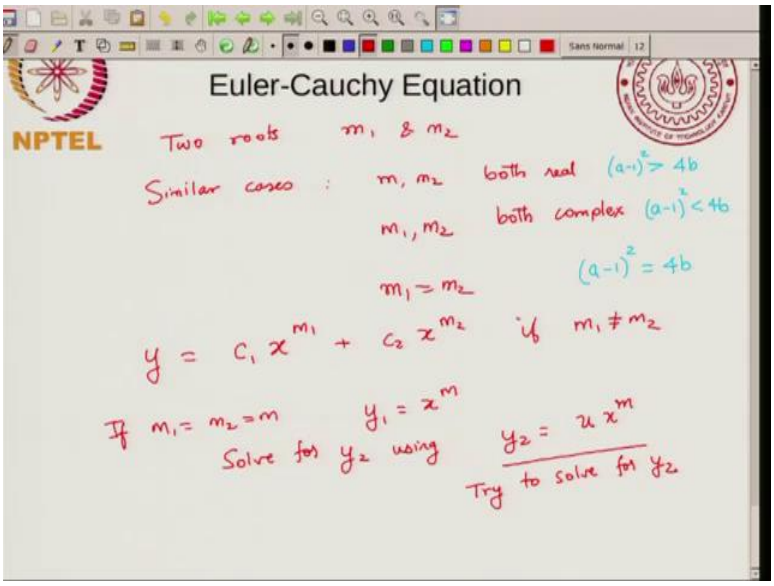

So, that is the explicit quadratic equation in m and whenever you have a quadratic equation you can have 2 roots. So, 2 roots m 1 and m 2 and again you can have the si similar case similar case cases. So, m 1 m 2 can be both real or it can be both complex both complex means they have both the real and imaginary part and you can have or you can have m 1 equal to m 2.

So, these are the 3 cases and I will not do this explicitly, but you can see in which case you would get each of these. So, you would get both real if this B square minus. So, is a minus 1 square is greater than 4 b and there will be both complex if a minus 1 square is less than 4 b and you will get to coincident roots is a minus 1 whole square equal to 4 be. So, these are the conditions in which you will get these solutions and you know what to do if you get. So, the general solution will look like the first 2 cases you can write y is equal to C 1 x raise to m 1 plus C 2 x raise to m 2 if m 1 is not equal to m 2 and if m 1 equal to m 2 then you have to then you find one root and you get the other root by variation of parameters.

So, if m 1 equal to m 2 equal to m then you say y 1 equal to x raise to m and you when you solve for y 2 using y 2 using y 2 equal to u times x raise to n you times y 1 or you times x raise to m. So, you can do you can make the substitution you can put in the differential equation and you can solve for y to I will not do this so, but I will leave it as an exercise for you to try. So, try to solve for y 2 using the same variation of parameters that we had done in the last class. So, I will stop this discussion on homogeneous second order differential equations here. So, in the next class, I will tell you what to do if you have a non homogeneous second order differential equation.

Thank you.