Mathematics for Chemistry
Prof. Madhav Ranganathan
Department of Chemistry
Indian Institute of Technology, Kanpur

Module - 06
Lecture - 27
Homogeneous 2nd order ODEs, solution using basis functions

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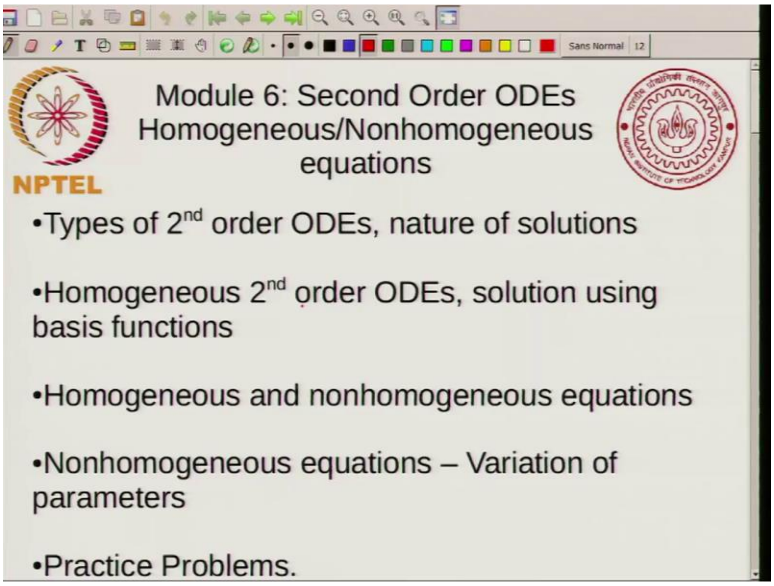

So we talked about the different types of second order differential equations in the last class and we talked about what kind of solutions you can have. Now in this class I will be focusing on homogenous second order ODEs and writing solutions in terms of basis functions. We already saw a bit of this in the last class and what I will do today is actually formalize some of the things that you saw in the last class.

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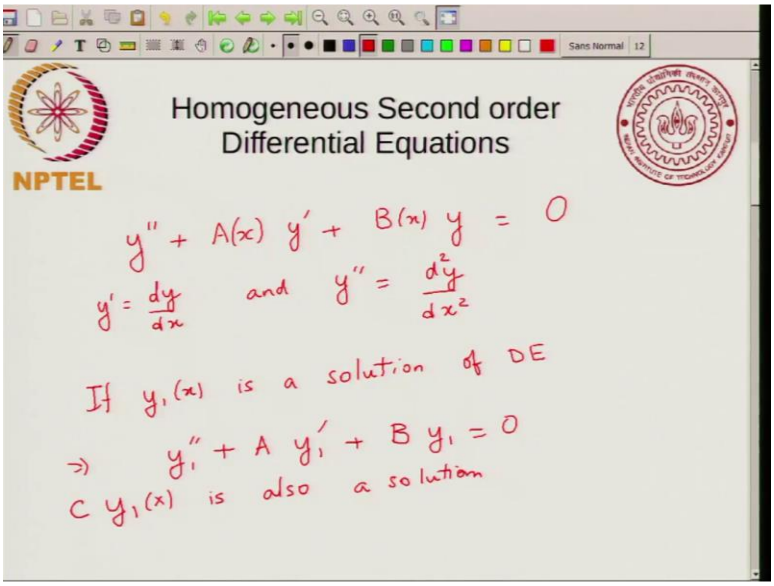

So, let us get back to our homogeneous second order differential equation. So, I will write it in the following form I will write y double prime plus A of x y prime plus B of x y equal to 0. So, this is our homogeneous second order differential equation and what you can see is that suppose you have this differential equation and let me remind you that y prime equal to d y by d x and y double prime equal to d square y by d x square. So, suppose you have a differential equation like this and let us say if y 1 of x is a solution of differential equation. So, if y 1 of x is a solution of this differential equation; that means, this implies that y 1 double prime plus A y 1 prime plus B y 1 equal to 0 and I have suppressed the dependence on x, I have all these are functions of x, but I am not writing the explicit dependence on x. So, if y 1 is a solution then it must satisfy this that is why it is a solution then you can see that a constant times y 1 of x is also a solution; that means, that means I can multiply y 1 by a constant and I will get another solution which is also a valid solution and you can see this because. So, suppose I had c y 1.

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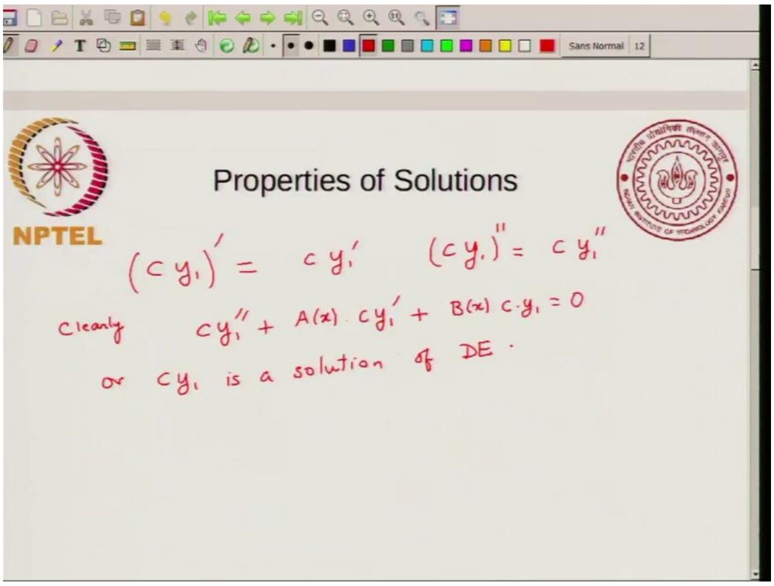

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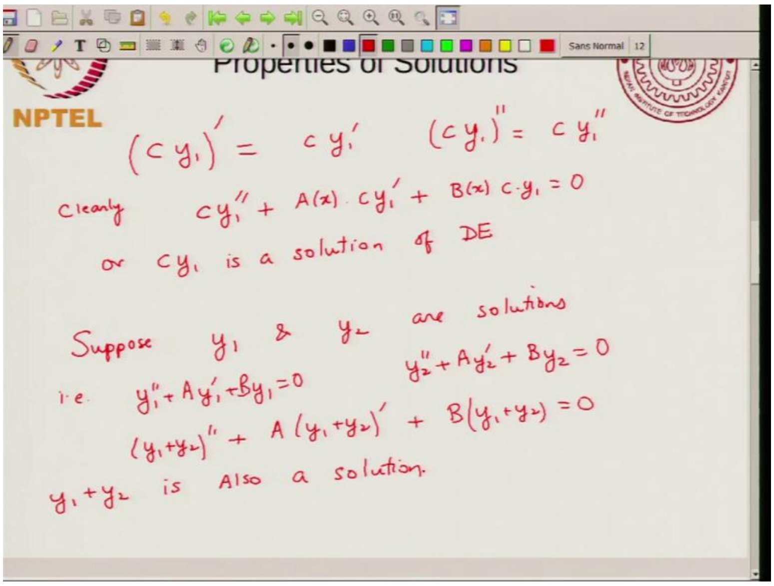

Now, c y 1 the whole thing prime the derivative of this is nothing, but c times the y 1 prime and you have c y 1 double prime is nothing, but c times y 1 double prime.

So, now if I take this differential equation and instead of y 1, I place it by c times y 1 then where I have y 1 double prime, I will have c times y 1 double prime where I have y 1 prime I will have c times y 1 prime and where I have y 1, I will have c times y 1. So, if I make this substitution then clearly c times y 1 double prime plus A of x into c times y 1 prime plus B of x into c times y 1 equal to 0 or c y 1 is a solution of differential equation.

So, what; that means, is that if y 1 is a solution then any constant multiplied by y 1 is also a solution so; that means, you can have many solutions with different constants.

Now, next suppose y 1 and y 2 are solutions that if that is. So, if both these are solutions; that means, you will have y 1 double prime plus A y 1 y 1 prime plus B y 1 equal to 0 and if y 2 is a solution. So, you have y 2 double prime plus A y 2 plus A y 2 prime plus B y 2 equal to 0. So, you have both these conditions.

Now, you can see that if I just add them then I will get y 1 plus y 2 double prime plus A times y 1 plus y 2 prime plus B times y 1 plus y 2 equal to 0 or in other words y 1 plus y 2 is also a solution. So, basically if you have 2 functions y 1 and y 2 which are solutions of the differential equations of this homogeneous differential equation you add them up you will get another solution and we already saw that you will multiply any of them by a constant you get another solution. So, this leads us to the important idea that that my solution.

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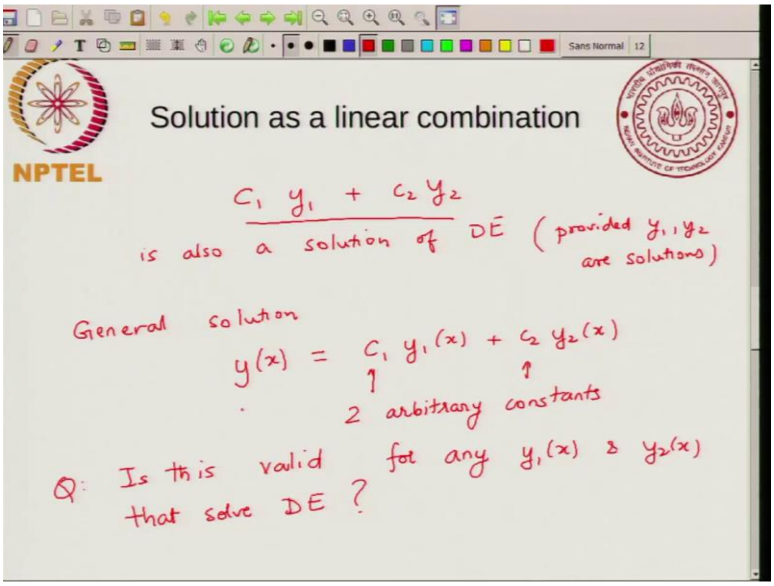

If I take c 1 y 1 plus c 2 y 2 is this will also be a solution of DE that is provided y 1 y 2 are solutions. So, what; that means, is that if I have one solution y 1 if I have another solution y 2 then I can write a third solution which is a linear combination of these 2 and in fact, in fact you can say that the general solution y of x can be written as a linear combination of y 1 of x plus c 2 y 2 of x. So, you can write a general solution as a linear combination of as these 2 solutions. So, if you have 2 solutions you can write a general solution as the linear combination and this you notice has 2 arbitrary constants. So, we already learnt that if we have a second order differential equation the general solution will have 2 arbitrary constants. So, if you know 2 solutions y 1 and y 2 you can write a general solution and what I mean by these 2 solutions do not have any arbitrary constants. So, if y 1 and y 2 do not have any arbitrary constants then I can write a general solution as a linear combination of these and I will have my 2 arbitrary constants.

So, the question is this valid for any y 1 of x and y 2 of x that solve DE. So, that means, can you use any 2 solutions and write the general solution as a linear combination of those 2 solutions and the answer is no. So, you can take any 2 solutions you can take a linear combination you will get a solution, but in order for this solution to be a general solution in order to get a general solution the important thing is that y 1 and y 2 should be linearly independent.

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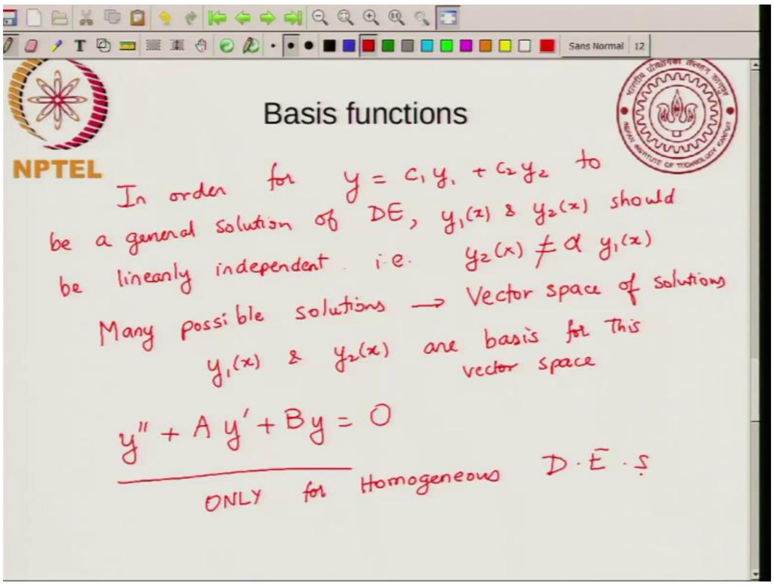

So, in order for y equal to c 1 y 1 plus c 2 y 2 to be a general solution of DE and when I say a general solution of DE I what I what I imagine is that any other solution of the DE can be written by choosing c 1 and c 2. So, you can write any other solution by choosing c 1 and c 2 appropriately. So, in order for this to be a general solution of the of the differential equation y 1 of x and y 2 of x should be linearly independent or in other words y 2 of x is not equal to some constant alpha times y 1 of x. So, y 2 should not be proportional. So, these 2 are functions and 2 functions are linearly independent if you cannot write 1 as a multiple of the other then you say that they are linearly independent.

So, now these 2 solutions, what we have right here is you have many possible solutions. So, you can think of a; this as a vector space of solutions and why are you; why are we using the vector space because you can take you can add solutions you can multiply them by a constant and you will still get a solution. So, the set of solutions will form a vector space and y 1 of x and y 2 of x e k can be a basis for this solution basis for this vector space. So, if they are linearly independent if y 1 and y 2 are linearly independent then they can be a basis for this space of solutions.

So, this is a vector space of solutions of the differential equation. So, it is a vector space of functions which solve the differential equation and this vector space of functions will have a basis of 2 functions which are also solutions of the differential equation, but which are linearly independent. So, just to show you again we had the differential equation y double prime plus A times y prime plus B times y equal to 0 and this is a homogeneous linear second order differential equation and I said A and B are functions of x. So, they can be any functions it is still a homogeneous linear second order differential equation and this has a feature that if you know 2 if you have 2 linearly independent solutions y 1 of x and y 2 of x then you can write A j, you can write any solution as a linear combination of those 2 solutions.

So, this is extremely nice because you see a connection between the; what we were talking in terms of basis vectors for a vector space and solutions of a differential equation and remember this is only valid for this is only for homogeneous equations. So, you can only do this if the equation is homogeneous if the equation is not homogeneous you cannot suppose I suppose y 1 is a solution there is no guarantee that c y 1 will be a solution. So, this is not true for any equation. So, for any arbitrary equation you cannot you would not have that c y 1 is also a solution if y 1 is the solution this is only for homogeneous differential equation.

So, now let us illustrate another interesting idea for homogeneous equations. So, sometimes we may find we may find one solution. So, you have a differential equation.

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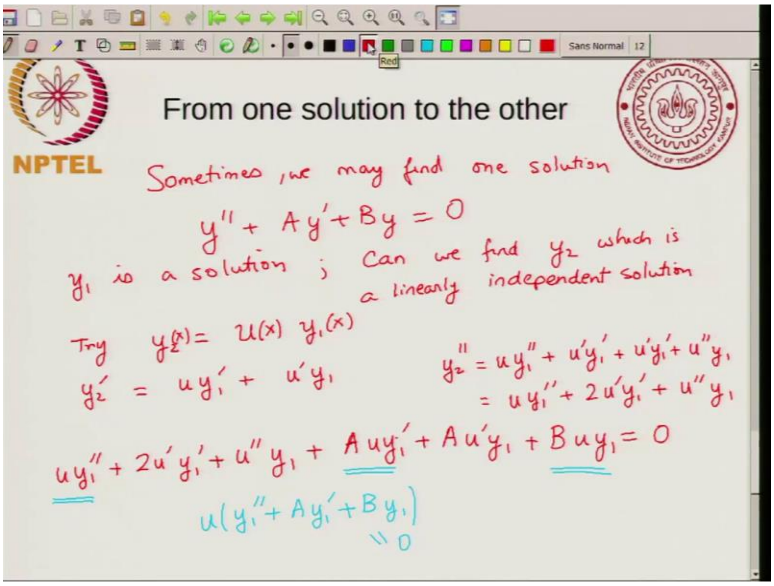

Let me let me just write the differential equation. So, you may find one solution. So, our differential equation is y double prime plus A y prime plus B y equal to 0 and y 1 is a solution. Now the question is can we find y 2 which is a linearly independent solution. So, we want to find another solution you want to find. So, you have one solution you want to find the second linearly independent solution. So, that you can write any solution a general solution as a linear combination of these 2 solutions and the answer is yes, you can find it, the method that you do is called variation of parameters. So, what you say is. So, you try y 2 is equal to u which is a function of x times y one. So, remember all these are functions of x. So, y 2 is a function of x u is a function of x and y 1 is also a function of x.

So, sometime I will not write the explicit dependent on x, but here I want to emphasize that u is a function of x just as y 1 and y 2 are also functions of x. So, suppose you try a solution of this form. So, then what is y 2 prime is u y 1 prime plus u prime y 1 y 2 double prime. So, that is the second derivative of y 2 now again. So, that is the derivative of this of this quantity of the first derivative. So, so if you take the derivative of the first term you will get u y 1 double prime plus u prime y 1 prime and then if you take the derivative of this second term you will get plus u prime y 1 prime plus u double prime y one. So, you have these four terms and you can see that these 2 terms are identical. So, so I can write this as u y 1 prime double prime plus 2 u prime y 1 prime plus u double prime y one. So, all I did was I had y equal to u time y 2 is u times y 1 then y 2 prime I write by the usual product rule and y 2 double prime also I right by the usual product rule and now I substitute in the differential equation because y 2 has to be a solution of this differential equation.

So, when you substitute in the differential equation what do you get? So, for y 2 double prime I will put all these terms. So, I will have u y 1 double prime plus 2 u prime y 1 prime plus u double prime y 1 plus a times u y 1 prime plus a times u prime y 1 plus B times u y 1 equal to 0. So, since it has to satisfy the differential equation since we want y 2 to satisfy the differential equation we just substitute y 2 into this equation and you will get and you will come up with this equation, now you look at these terms I will just highlight them in blue. So, I look at this term then I look at this term and I look at this term. So, if I add; if I look at each of these terms as A u.

So, if I write the sum of these terms I have u times y 1 double prime plus a times y 1 prime plus B times y 1 and you can immediately see that the term in the brackets has to be equal to 0. So, therefore, these 3 terms will add up to 0 and if you if you set these 3 terms to 0 then you are left with only the 3 remaining terms. So, what are the 3 remaining terms?

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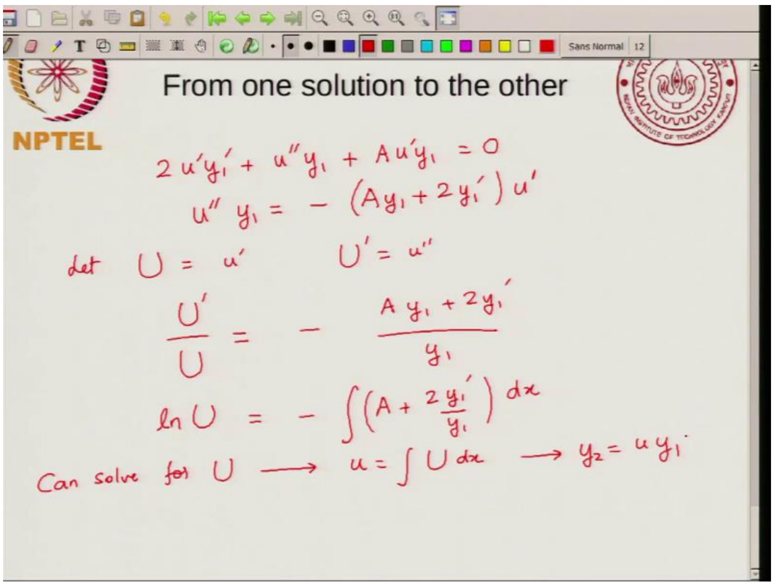

So, you have 2 u prime y 1 prime plus u double prime y 1 and you have plus A u prime y 1 equal to 0 and what I see is that there is no u term in this. So, there is no u term. So, I can just write it as u double prime y 1 is equal to minus of; now I will write it as A y 1 plus 2 y 1 prime into u prime. So, what I can do is I can bring the u double prime to the left hand side and what I will get is; so I will say let capital U is equal to u prime. So, I will just call it capital u and the reason will become obvious because now a capital U prime equal to u double prime. So, then you will get u prime I have write this as u prime by u is equal to minus A y 1 plus 2 y 1 prime divided by y 1.

Now, now you can immediately see the where we are going with this I do not want to do too much further you have a essentially separated u and x. So, remember y 1 is a function of x that you know. So, so we know we know the solution. So, y 1 is a solution that we know and from y 1 we wanted to find y 2 and what we did was we used a trial form of y 2 is u times y 1 and what we said is that when you substitute in this and then you get this relation now you can immediately write this as. So, if you integrate both sides you will get A, you will get the relation that looks like looks like the following.

So, integral of this is natural log of u is equal to minus integral A plus 2 y 1 prime by y 1 d x A is a function of x remember A is a function of x. So, I just integrate it both sides with respect to x and I got this. So, I just wrote these terms explicitly, but the point is that you know y 1. So, we can calculate y 1 prime and you can do this integral and you know A. So, essentially you can find out u. So, we can find out u and then u from capital U, you can find out small u and from small u you can find out y 2. So, I will just write it. So, can solve for u and from u you go to from capital U, you go to small u which is just integral of u d x and then and then from u you can go to y 2 is equal to u y 1.

So, what does this show? This shows that if you have one solution of the homogeneous differential equation you can go to you can get to the second solution. So, what this shows is that you know the power of homogeneous equations is that even if you just have one solution you can get to a second solution and that second solution will be linearly independent because u is not a constant u is some function of x. So, the second solution is by design it is linearly independent and the second solution being linearly independent you can write the general solution as a linear combination of y 1 and y 2. So, what this says is that all you have to do for a second order for a second order homogeneous differential equation is to find one solution and then you can find the general solution of this second order homogeneous differential equation.

So, in the next class, I discuss second order homogeneous differential equation with constant coefficients and then I will tell what are the strategies when your second order differential equation is not homogeneous. So, I will stop here for today.