Mathematics for Chemistry
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Module- 06
Lecture - 26
Types of 2nd order ODEs, nature of solutions

Now we will enter the 6th week of this course. And in this week we will be talking about Second Order Ordinary Differential Equations.

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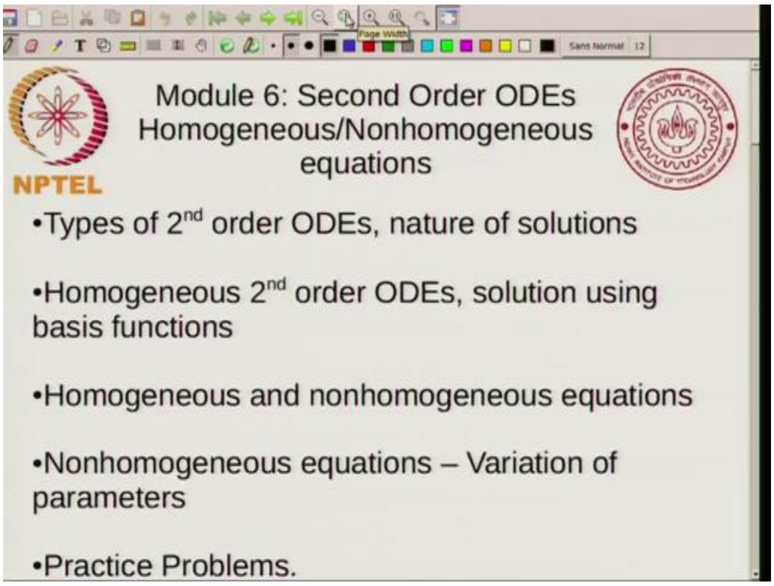

And actually the 6th, 7th and 8th weeks all will be talking about second order differential equations. So, today I will just get started with second order differential equations. And during this week we will be talking about homogeneous and non-homogeneous equations. So, in the first lecture we will talk about the types of second order differential equation and what the solutions look like. Then in the next lecture we will talk about homogeneous second order differential equations, and what are the properties of their solutions, what are the properties of the solutions of homogeneous second order differential equation.

Then in the third lecture we will talk about homogeneous and non-homogeneous equations, and how we can what are the tricks you can use to solve homogeneous equations. And then we will talk about the method of solving homogeneous equations called variation of parameters, and then and then finally we will have some practice problems.

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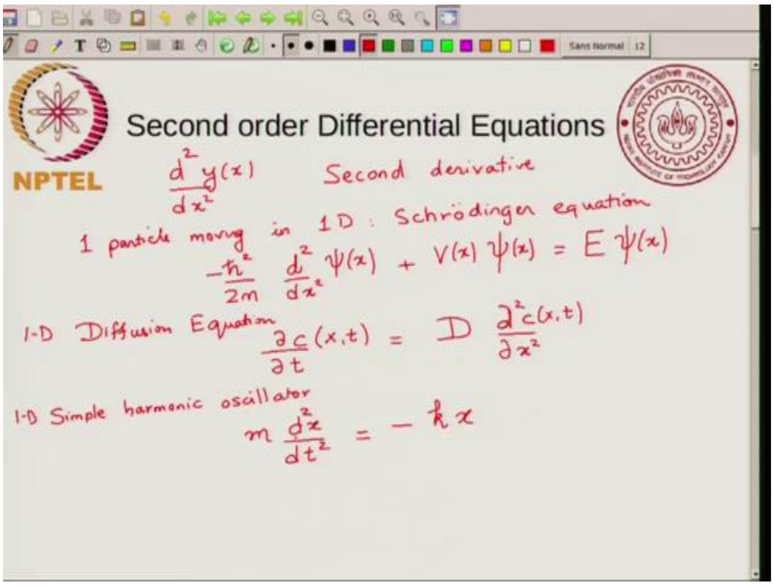

Let us start the lecture. Let us starts talking about second order ODEs. So, a second order differential equation; it is characterized by a second derivative. If you have y as a function of x then second order differential equation has d square y by dx square. Then it has a second derivative; second derivative is the highest derivative that appears. And second order differential equations are very common in all branches of science and engineering.

You have the spring equation, so that is a second order differential equation. You can have the quantum mechanics, the Schrodinger equation is a second order differential equation and so on. Just for example, if you take one particle moving in 1 D, then the Schrodinger equation of this particle has a form minus h bar square by 2 m d square by dx square of psi of x plus v of x psi of x is equal to some constant E times psi of x. So, this is a time independent Schrodinger equation, or this gives you the stationary states where energy is fixed.

So, E is constant. If you look this is a second order differential equation in the variable for the quantity psi. So, psi is your dependent variable and x is the independent variable. And v affects is some function of x which is known as a potential energy function. So, this is a second order differential equation, because the highest derivative of psi that appears is second derivative. In the other terms you do not have any derivatives of psi and you just have psi to a constant.

Another example is what is called the diffusion equation. So, here you have a concentration which depends on x and time. And what you write is that you write partial derivative with respect to time; this is equal to D times. Now x is just let us say- I am checking the 1 D diffusion equation one dimensional diffusion equation, then this is just d square c by or dou square c by dou x square. Again c is a function of x and t.

So, the highest derivative of c that appears is a second order differential equation. This is actually a second order partial differential equation. And the highest partial derivative with respect to t that appears is first order. So, this is one example. The wave equation will have a d square by dt square with a very similar looking right hand side. So, essentially these sort of equations are something that you see all the time, I will just add one more equation so the simple harmonic oscillator.

If you have let us say one dimensional again, so then you have d square x by dt square so times the mass. So, this is the force on the particle attached to the spring and this is equal to minus k times x, so where k is a spring constant. Now this is a second derivative of x. So, this is also of a second order differential equation. Now there are various types of second order differential equations. I will just mention a few types of second order differential equations.

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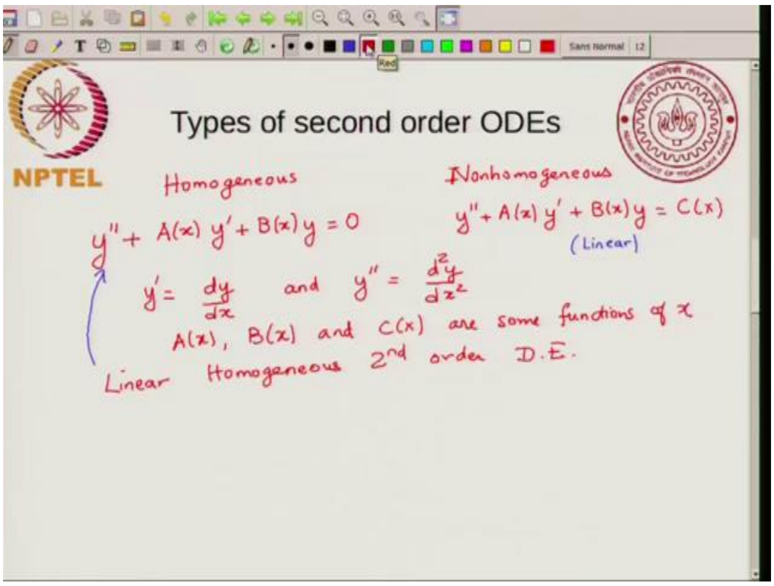

So, you can have second order differential equations that are let us say- homogeneous or inhomogeneous or non-homogeneous. So, what does a homogeneous second order differential equation look like? I will just write one form. Suppose, I write my second order differential equation as y double prime plus A of x y prime plus B of x y equal to 0. And similarly a non-homogeneous equation will look like y double prime plus A of x y prime plus B of x y equal to C of x.

Now, here I have used y prime as a notation for dy by dx and y double prime as a notation for d square y by dx square. A of x B of x and C of x are some functions of x. So, any differential equation that has this form, where you can write y double prime plus some function of x times y prime plus some function of x times y equal to 0; this is called a homogeneous equation. The reason it is called homogeneous is because each term has y up to power 1, y or any of its derivative up to power 1. So, this has the second derivative up to power 1, this has the first derivative up to power 1, this has the 0 derivative or y up to power 1.

So, the power of y or any of its derivatives is the same for each of the terms. Therefore, it is called a homogeneous equation. And in particular this type of equation is called a linear homogeneous; linear homogeneous second order differential equation. So, I will just mark; so this is an example of a linear homogeneous second order differential equation. The reason we said it is linear is because the power of y or any of its derivatives in each term is 1. The only difference between a homogeneous and a non-homogeneous differential equation is that you have this extra term C of x which have written as C of x.

Now here this is again this is also linear, because the power of y in each term is 1 or 0. The maximum power of y is 1 so this is also linear, but this is non-homogeneous. It is non-homogeneous because right hand side contains; it is not equal to 0 and it does not contain y to power 1. So, the right hand side does not contain y to power 1. So, there is a term that does not contain y or any of its derivatives to power 1. So, this is an example of a non-homogeneous equation. It is still linear because each term has y up to a maximum of power 1.

So, we can think of second order differential equations, we can we can check whether they are linear or they are non-linear, and then you can check whether they are homogeneous or non-homogeneous. And there are some very nice properties of solutions of these equations that we will look at as we go along.

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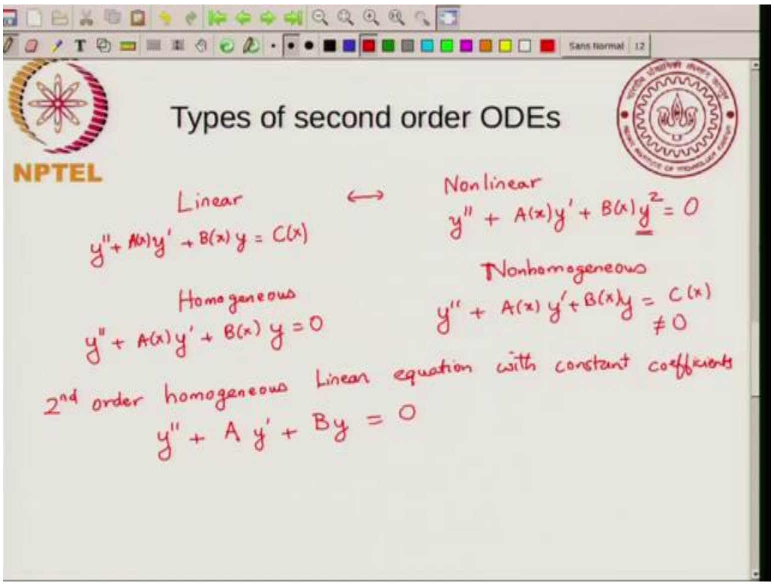

Let us once again mention that we have linear versus non-linear. In a linear equation will look like y double prime plus something times y prime plus A of x times y prime plus B of x times y this is equal to C of x. This is a real general linear second order differential equation it need not it is homogenous if C of x is 0. A non-linear equation will have could be many many ways an equation could be non-linear, but the simplest way I would say is if you have A of x y prime plus let us say B of x y square this is 0.

Now clearly this is a non-homogeneous equation because you have y square. So, the y square term is a it is a non-linear term. Then you could have homogeneous and non-homogeneous. The homogeneous one is given by y double prime plus A of x y prime plus B of x y equal to 0 or C equal to 0. And in this case we have y double prime plus A of x y prime plus p of x y equal to C of x not equal to 0. These are the typical types of second order differential equations that will encounter.

Further we might encounter some second order differential equations where the coefficients, where these functions A of x and B of x might have a specific form. So, suppose A of x B of x may be a constant, so we could have a second order homogeneous linear equation with constant coefficients. So, this would look like y double prime plus some constant I will just write it A primes y prime plus B primes y equal to 0. So, this is a second order homogeneous linear equation with constant coefficients.

It always helps to identify what type of differential equation you are dealing with, because for each type of differential equation there are very efficient ways to solve them. Now what about the solutions?

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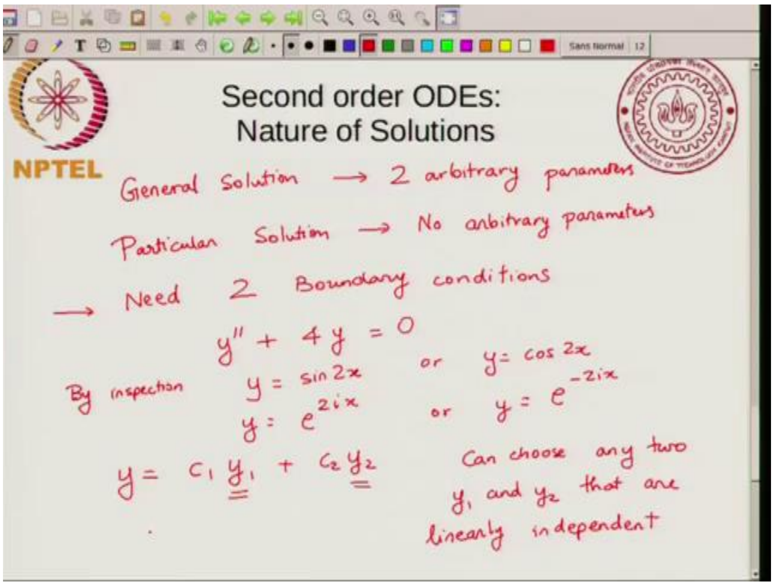

Suppose you solve the second order differential equation then in general solution; so there could be a general solution will have two arbitrary parameters. And of course, you could have a particular solution. This will have no arbitrary parameters, arbitrary constant you can just say constants parameters or constants. So, what the message is that: if you want to get a particular solution you need 2 boundary conditions. So, not just one boundary condition, but you need 2 boundary conditions in order to get the particular solution.

So, let us it take one example just to illustrate this point. If I take y double prime plus 4 y equal to 0, because this is actually a simple harmonic oscillator we have only thing instead of d by dt I have d by dx. Then I can immediately see that just by inspection y equal to sin 2 x or y equal to cosine 2 x or y equal to e raise to 2 i x or y equal to e raise to minus 2 i x are all solutions of this equation. So, each of these is a valid solution. You can you can verify. Suppose I take the first derivative of this I will get 2 cosine 2 x if I take a second derivative then I will get 2 into minus 2, so that is minus 4 sin 2 x. Then you have minus 4 as to sin to x then you have plus 4 sin 2 x, so it is 0.

Similarly you can verify that the cosine and the e to the 2 i x and e to the minus 2 i x also satisfy. So, how do you write the general solution? In order to write the general solution, what you write is y is equal to I will write it as c 1 y 1 plus c 2 y 2. In this particular case if sin 2 x is a solution even some constant multiplied by sin 2 x is also a solution for this differential equation. And we will see in the next class that there is a certain property of this differential equation that allows you to write the solution this way.

How do you choose y 1 and y 2? We have many choices for y 1 and y 2. So, the idea is that you can choose any two y 1 and y 2 that are linearly independent. So, you can take any two functions that are linearly independent, so it should not be 1 should not be a constant times the other. That is the only restriction you can write the solution. So, one way to write the solution would be to take let us say you can take y equal to A sine 2 x plus B cosine 2 x.

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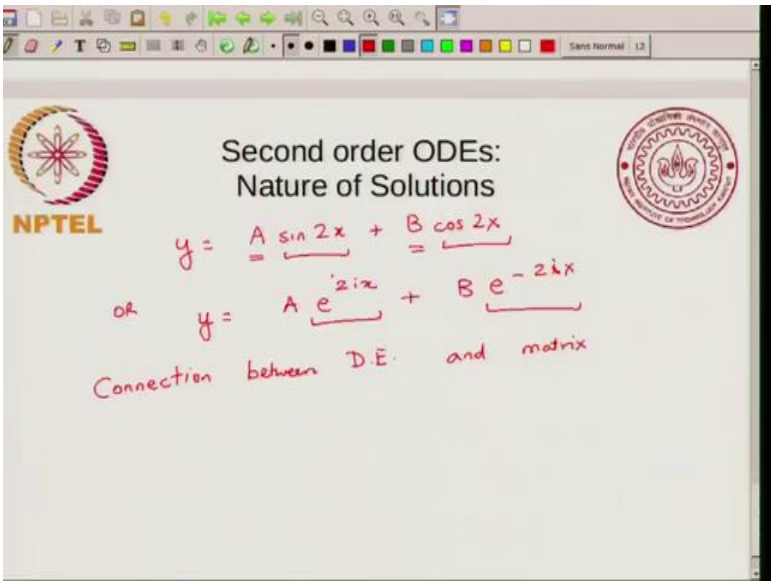

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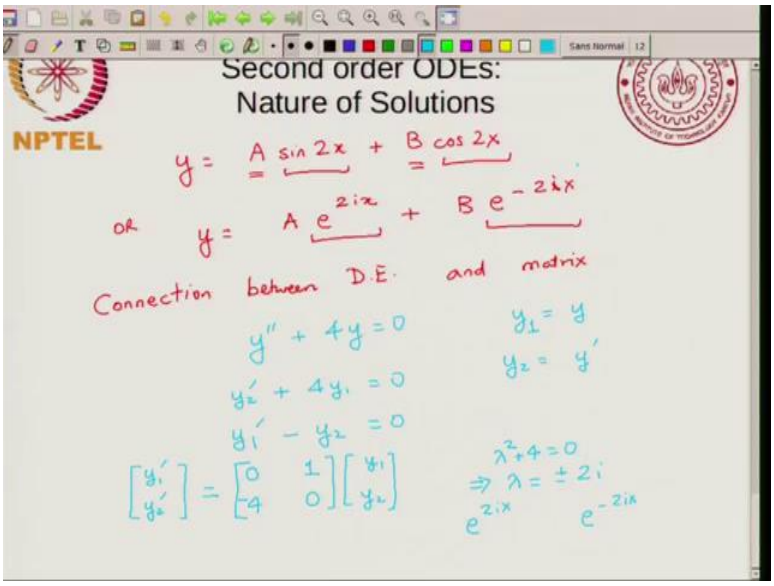

So, this might be one way where A and B are constants which are which are determined by the boundary conditions. Or y is equal to A e raise to 2 i x plus B e raise to minus 2 i x. So, you could also write it in this way. And in fact, there are also other ways to write the solutions. Now, what we mentioned is that these two solutions sin 2 x plus cosine 2 x have to be linearly independent. So similarly e raise to 2 or 2 i x and 2 raise to 2 minus 2 i x are also linearly independent.

So, there is a connection between differential equation and matrix. And we actually saw that in the classes during last week where we actually use matrix methods to find out solutions of differential equation. And if you remember we used it for a system of differential equations with constant coefficients. And notice that this is a second order differential equation with a constant coefficient. So, I can actually write this in a slightly different way.

So, I can write this as: so we can rewrite y double prime plus 4 y equal to 0. So, you define y prime equal to y 1 and or actually still let me do it this way so slightly differently I will write y 1 equal to y and y 2 equal to y prime. And you recall from last week that we said that this second order differential equation can be written as a set of first order differential equation with constant coefficients. So, if I take this equation then I can write this as; so y double prime is nothing but y 2 prime, so I can write this as y 2 prime plus 4 y 1 equal to 0, and the other equation I could write this as y 1 prime. So, this y 2 equal to y prime y y is same as y 1, so I can write y 1 prime equal to y 2; so minus y 2 equal to 0.

In other words I can write this equation in the following form; so y 1 prime y 2 prime. If I take these to the right I can write this in matrix form. So, y 1 prime does not have any term that depends on y 1, but it has a plus 1 that depends on y 2. Y 2 prime has a plus 4 or we will have a minus 4 on y 1 and it does not have any term that depends on y 2. So, this is my end to have y 1 y 2. What we show this at you can take the second order differential equation write it as a pair of coupled equations.

And you can immediately see that since this is it has by matrix methods you know that the solution looks like e to the lambda x, where lambda if lambda from this equation so you will get lambda square plus 4 equal to 0 or you will get lambda is equal to plus minus 2 i. So, the linearly independent solutions will look like e to the 2 i x and e to the minus 2 i x. Will have e to the lambda x, therefore it will be e to the 2 i x in e to the minus 2 i x.

And you can find the corresponding eigenvectors and you can solve this equation. I just wanted to illustrate this point that because this was a linear homogeneous different second order differential equation with constant coefficients you could actually have just used matrix methods that you learnt from first order differential equations and solve this system of equation. And notice that this solution is the same as what we had here; A e to the 2 i x and B e to the minus 2 i x.

So, I will stop here for today. In the next class, we will talk more about homogeneous second order differential equations.