Mathematics for Chemistry
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Module - 05
Lecture - 25
Practice Problems

In the last lecture of week 5 we will do some practice problems on first order differential equations.

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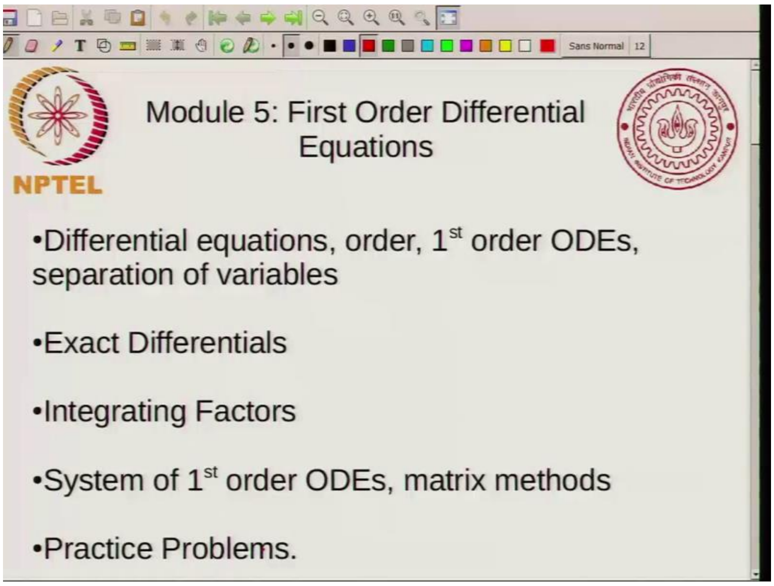

So, I will just do a couple of problems that illustrate the techniques that you use in first order differential equations.

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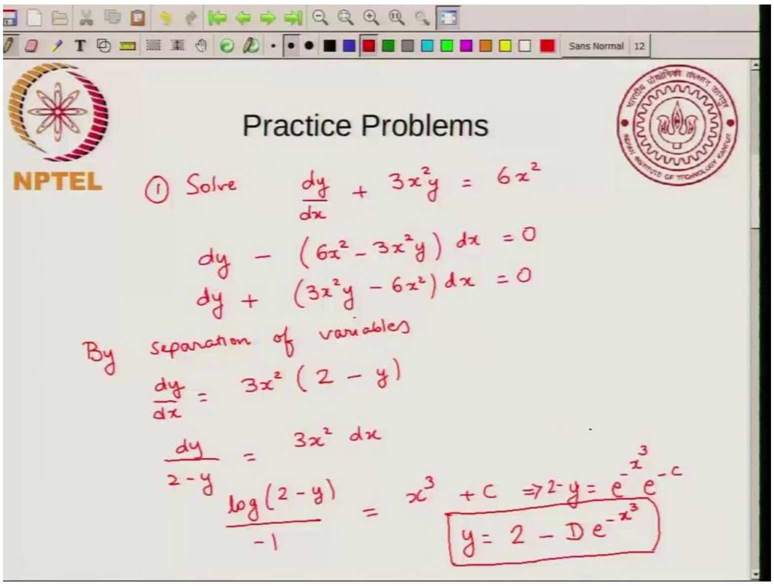

Let me take the first problem. So, the first problem is to solve this differential equation, and I will give you the differential equation. Solve the differential equation is dy by dx plus 3 x square y equal to 6 x square. And when we see this differential equation we will solve it in one way that that illustrates the use of variation of parameters.

So, let us write this in the form dy minus equal to 0 or rather I will write it as dy plus 3 x square y minus 6 x square dx equal to 0. I deliberately wrote it in this way; we will see why we wrote it this way, but let me mention right at the beginning that there might be other ways to solve this equation also. For example you could just see this, because you have an x square here and an x square here.

So, by separation of variables; so what do what do I mean by separation of variables? You would write dy by dx is equal to 3 x square times 2 minus y. And then therefore, you would write dy by 2 minus y is equal to 3 x square dx. And if you integrate both sides you will get log of 2 minus y divided by minus 1 this is equal to x cube plus constant.

So, if I go ahead and multiply it out what I will get is I will get 2 minus y is equal to get e raised to minus x cube into e raised to minus c; c is the arbitrary constant. And now I can write y is equal to 2 minus, e raised to minus c I can just call this some constant D e raised to minus x cube. So, this is the solution. You can solve this differential equation just by separation of variables.

Now what I will do for illustration is to solve the same equation using a slightly different procedure and where we will use the integration factor; so the same equation where we know the solution. So, we know that the solution has the form y equal to 2 minus D e to the minus x cube; where D is some arbitrary constant and to determine D you need some boundary condition.

So, now let us try to solve this using a slightly different procedure. In order to do this I will write the differential equation in this form. So, dy plus 3 x square y minus 6 x squared dx equal to 0.

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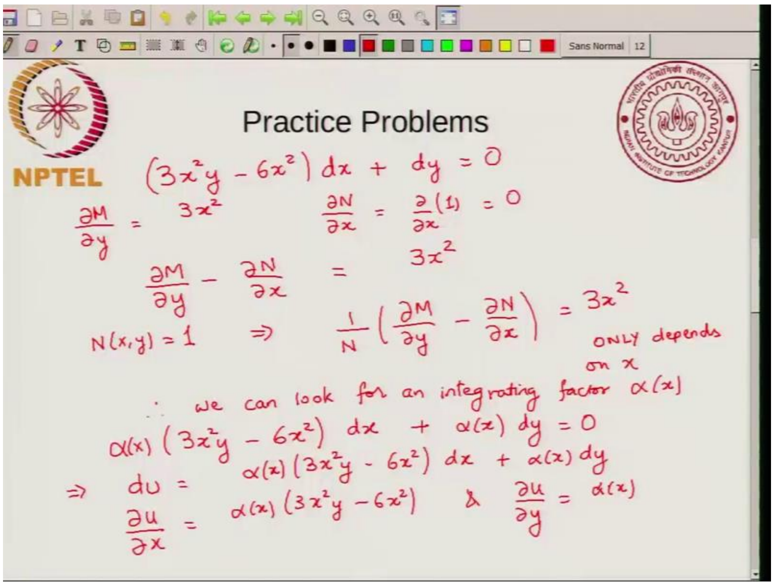

Let us rewrite this again. So, I will take the right hand side that is 3 x square y minus 6 x square dx plus dy; 3 x square y minus 6 x square dx plus dy equal to 0. So, now what you can do is you can take this equation, you can check for exact is this an exact differential. So, calculate dou M by dou y is equal to 3 x square and dou N by dou x; N is just a constant, N is just 1; equal to dou by dou x of 1 that is equal to 0. So, dou N by dou x is 0. So, clearly dou N by dou y is not equal to dou N by dou x.

Now if I take dou M by dou y minus dou N by dou x. So, if I take this difference then what I find that this is equal to 3 x square because dou N by dou x is 0. So, now 3 x square; so you immediately realize that n of x y equal to 1. So, this implies 1 by n dou M by dou y minus dou N by dou x is equal to 3 x square. This is only a function of x; only depends on x. Therefore, we can look for an integrating factor alpha of x; alpha that depends only on x.

So, we go ahead and we put this integrating factor, then our equation becomes; alpha of x times 3 x square y minus 6 x square dx plus alpha of x dy equal to 0. And now we know that this equation is exact that implies that this whole left hand side is some function u. So, d u equal to alpha of x 3 x square y minus 6 x square dx plus alpha of x dy. So, we immediately know that dou u by dou x is equal to alpha of x 3 x square y minus 6 x square and dou u by dou y is equal to alpha of x.

Now, we will come to this a little later, but let us go back to the condition for exact differential.

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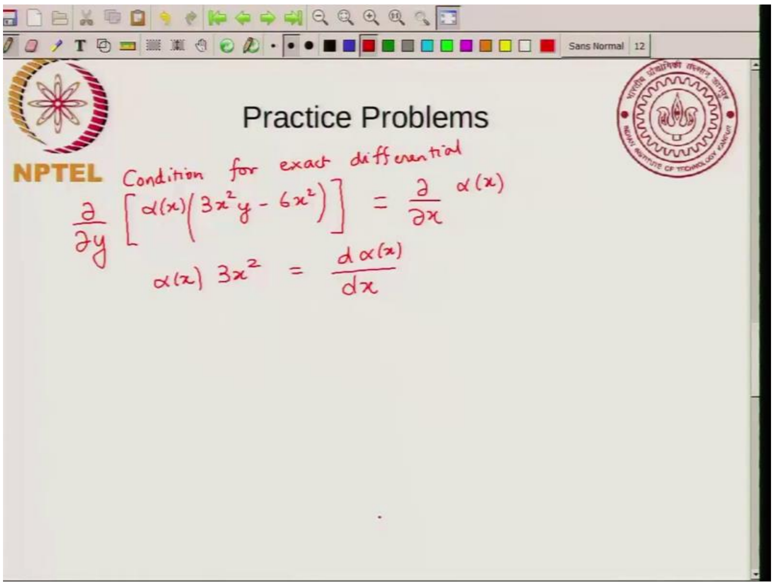

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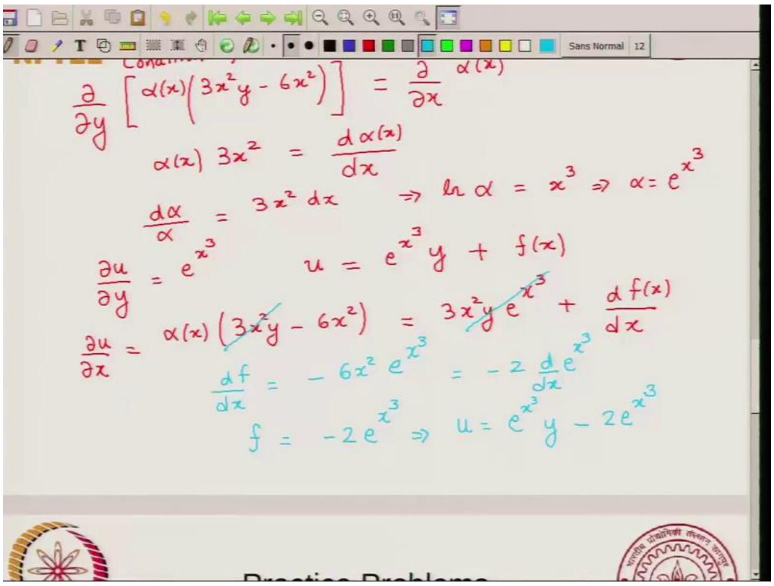

So, the condition for exact differential says that differential. So, this states that dou by dou y of alpha of x times 3 x square y minus 6 x square this should be equal to dou by dou x of alpha x. Now, this is just d by dx of alpha x, and on this side what I will get is alpha x I can take it outside; so I have alpha of x. Now derivative with respect to y of this quantity that will just give you 3 x square; equal to d alpha of x by dx. And I can rewrite this as d alpha by alpha is equal to 3 x square dx. And if you integrate both sides you will get alpha, the natural log of alpha is equal to x cube or alpha equal to e raised to x cube.

I am not writing the constant of integration that will not be necessary here. So, alpha equal to e raised to x cube. And you recall here we had this factor of e raised to minus x cube in the actual solution 2 minus D e raised to minus x cube. And we will quickly see how this will also give the same result.

Next what can you say; now you go back to these equations. So we know that dou u by dou y is alpha x, so therefore you can say that dou u by dou y is equal to e raised to x cube. Now if I integrate both sides with respect to y what I will get is u; u is a function of x and y. So, u can be; now if I integrate the right hand side with respect to y then e raised to x cube is like a constant. So, you just have e raised to x cube multiplied by y. And you will have a constant of integration, but since you are integrating with respect to y that constant can be any function of x. So, this is what you get.

Now, in order to determine this function of x we use dou u by dou x is equal to. So, we go back to our expression; so dou u by dou x plus alpha x 3 x square y minus 6 x square. So, this is alpha of x 3 x square y minus 6 x square. So, now if you substitute this form of u then you can see that dou u by dou x is nothing but derivative of e raised to x cube that is 3 x square y e raised to x cube, and you have plus df of x by dx. So, this is equal to this term here. Now alpha of x is nothing but e raised to x cube. So, what you will see is that these two terms will cancel and what you will get is df by dx is equal to minus 6 x square e to the x cube. And you can solve this and you can immediately see from here you can see that derivative of e raised to x cube will be e raised to x cube into 3 x square. Soso this is nothing but equal to minus 2 d by dx of e raised to x cube. And so what you get is that f is equal to minus 2 e raised to x cube.

And now you can substitute in here now u becomes e raised to x cube y x cube into y minus 2 e raised to x cube. And, remember this is when u is this exact differential, so d u is this. So, the differential equation is d u equal to 0, so d u equal to 0 is the differential equation. So, the solution is u is a constant ok solution is u is an arbitrary constant. So, then I can write my solution as this equal to constant; constant c and if I just multiply by e raised to minus x cube then what I will get is, this implies y is equal to c e raised to minus x cube.

Now, if I multiply by e raised to minus x cube this will cancel this and you will get plus 2, which is exactly the same as what we got earlier. You got 2 minus D e raised to minus x cube which is exactly the same as what we have here. So, you can think of c as minus d and you will get exactly that same equation. So, c is an arbitrary constant just like what we had d here and you get exactly the same equation

So, this problem illustrates how you can solve a simple differential equation using both integration factor or using separation of variables. And now in this case we could do the separation of variables, because are straightforward. Since our equation was straightforward we could also do separation of variables. But there are cases when you cannot do separation of variables when you have to go for hinting factors.

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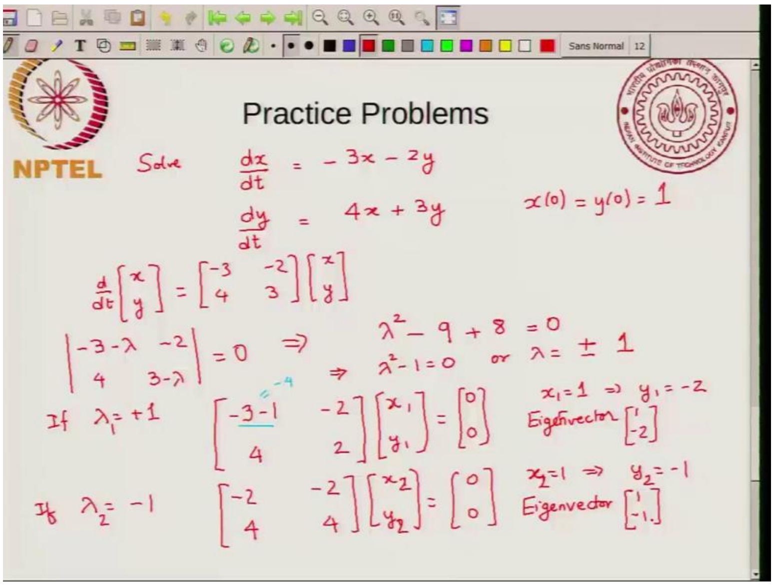

So, this is the first problem I wanted to do. Now the second problem that I will do we will illustrate the use of matrix methods. So, here what you will do is, you will solve these equations. So, solve the equations, so I will say dx by dt is equal to minus 3 x minus 2 y. The other equation is dy by dt is equal to 4 x plus 3 y. And the initial conditions are x of 0 equal to y of 0 equal to 1. Here, since I have two differential equations I need two initial conditions x 0 equal to 1 and y 0 equal to 1. So, these are the two initial conditions. When you finally solve this equation you will have no undetermined constants.

So, now what you could do is; I can immediately write this in matrix form. So, I right d by dt of this matrix x y is equal to minus 3 minus 2 4 3 times x y; so the set of linear equations with constant coefficients. So, then what I will do is solve for the eigenvalues and eigenvectors. So, what we want to do is to solve for the eigenvalues and eigenvectors of this matrix. So to solve for eigenvalues we use minus 3 minus lambda, so this determinant 3 minus lambda minus 2 4, this determine should be equal to 0. And that implies can this has so minus 3 into 3 is 9, and then I have minus 3 into minus lambda that is 3 lambda. And I have 3 into minus lambda, so that is again minus 3 lambda; so that will cancel and then I will have lambda square.

So, lambda square minus 9 is what I get from this. And then I will get plus 8 equal to 0. So, this implies lambda lambda square minus 1 equal to 0 or lambda equal to plus or minus 1. So, our two eigenvalues are plus 1 and minus 1. Now if lambda equal to plus 1; so if lambda equal to plus 1 then to solve for the eigenvectors we have to use the condition. So, minus 3 minus 1 minus 2 4, now 3 minus 1 is 2, this times x 1 y 1 equal to 0 0.

So, we have this equation and you can clearly see minus 4 minus 2 4 2 these are the same equation. Obviously, they should be linearly dependent, and you can clearly see this because this gives you minus 4. So, this term is equal to minus 4. So, you have minus 4 minus 2 in the first row and you have 4 2 in the second row. And clearly the first row is just minus 1 times the second row. So, clearly these two rows are linearly dependent. So, I choose x 1 equal to 1. So, if I choose x 1 equal to 1 then I will get. So, if I choose x 1 equal to 1, then what you will get is that you will get 4 plus 2 y 1 equal to 0, so y 1 equal to minus 2.

So, my eigenvector becomes is just 1 minus 2. Now first eigenvalue is plus 1 the eigenvector is 1 minus 2. Now if lambda equal to minus 1. Now, in this case what happens is you have minus 3 minus of minus 1 that is so minus 3 plus 1 that is minus 2 minus 2 you have a 4 here and 3 plus 1 is 4; x 1 y 1 equal to 0 0. So, the eigenvector in this case if I take x 1 x 1 equal to 1 then implies y or x 2, in this case I should write x 2. So, lambda 1 lambda 2; sorry this should be the second eigenvector. So, y 2 equal to minus 1, and your eigenvector is 1 minus 1.

So, we have the two eigenvalues and two eigenvectors, and now we can write the general solution.

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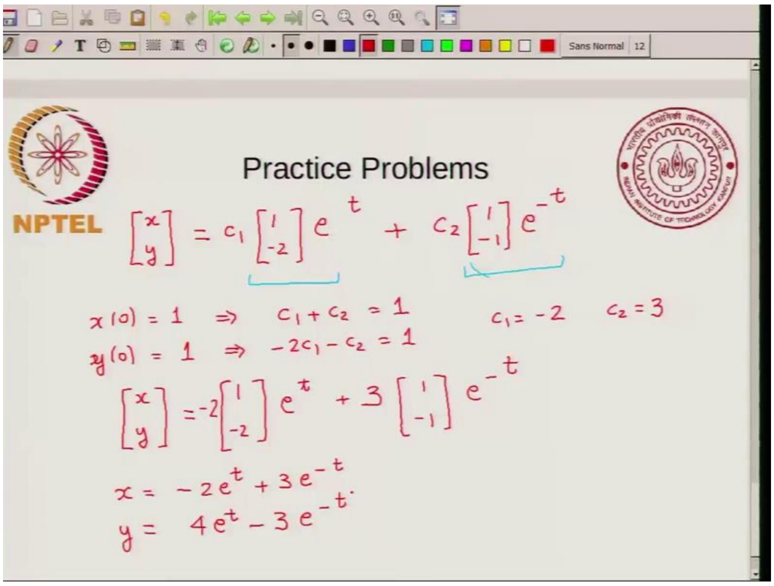

A general solution as in the following form: so general solution is this x y is equal to- so what I do is I take the first eigenvector which is 1 minus 2 multiplied by c 1 and then I put the e raised to this eigenvalue lambda 1 t; lambda 1 t is lambda 1 is just 1, so e raised to t. And the second term c 2 1 minus 1 e raised to minus t.

So, this is my general solution. And notice that the general solution has two undetermined constants. So, what do I do; I that each of these terms as a solution, I know that this is a solution, I know that this is a solution. So, once you know that these two are solutions, now we wrote the general solution as a linear combination of this of these two solutions. This is actually a special property whenever you have what is called a homogenous equation where each term contains the dependent variable to the same power. So, x to power 1 x to power 1 y 2 power 1 y 2 power 1 x to power 1 y 2 power 1 So, this is a linear homogeneous differential equation.

And whenever you have such an equation then if you have two solutions: you can write the general combination as a linear combination of these solutions. So, multiple of one solution will still be a solution. Similarly any multiple of the other solution will still be a solution. So, the general solution can be written as a linear combination of these two solutions. And the c 1 and c 2 are determined by the boundary conditions. So, we know that x of 0 equal to 1 implies c 1, so when t equal to 0 this you just have c 1 and you have plus c 2 equal to 1. And then you have y of 0 equal to 1 implies minus 2 c 1 minus c 2 equal to 1.

And if you add these two you will get the c 1 equal to minus 2 and c 2 equal to 3. So, if I just add these two I will get minus c 1 plus 0 equal to 2, so c 1 equal to minus 2. And then if I put c 1 equal to minus 2 then c 2 has to be equal to 3. And you can verify, so if this is 3 that this is plus 3, and if this is 2 is 3 so this is minus 3, and minus 2 c 1 is plus 4 so 4 minus 3 equal to 1. So, this is the solution. I can write my general solution in the following form: x y is equal to 1 minus 2 e raised to t plus or minus 2. So, plus 3 1 minus 1 e raised to minus t.

In other words I can write x is equal to minus 2 e raised to t plus 3 e raised to minus t and y is equal to 4 e raised to t minus 3 e raised to minus t. If I just equate the first row I will get this above equation, if I equate the second row I will get this equation.

In this way you can use the method of matrices to get the general solution of the differential equation. And just for practice you can verify that if you take dx by dt you get nothing but minus 3 x plus 2 y. And if we take dy by dt you will get 4 x plus 3 y. So, you can verify those for practice.

With this I will conclude the 5th week of classes. And next week we will start looking at second order differential equations.