Mathematics for Chemistry
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Module - 05
Lecture - 23
Integrating Factors

Today I will be talking about the Integrating Factor method, which is a method for solving differential equations; first order differential equations which cannot be expressed as exact differentials.

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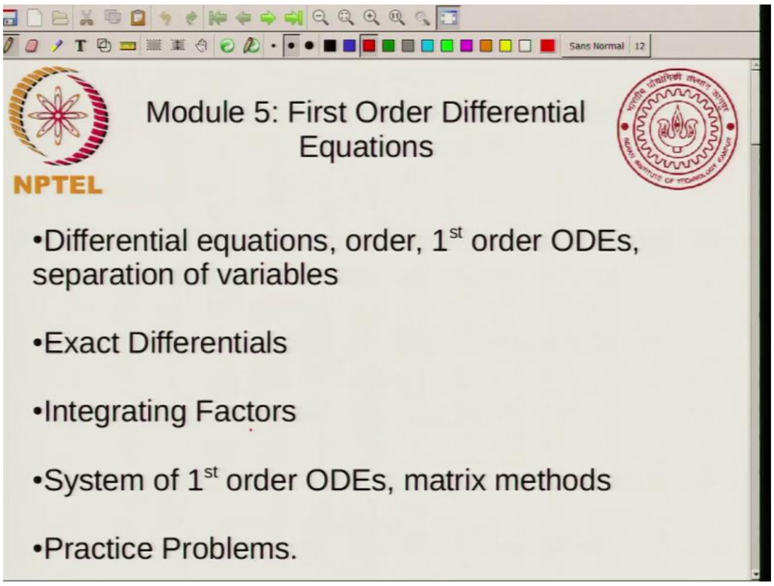

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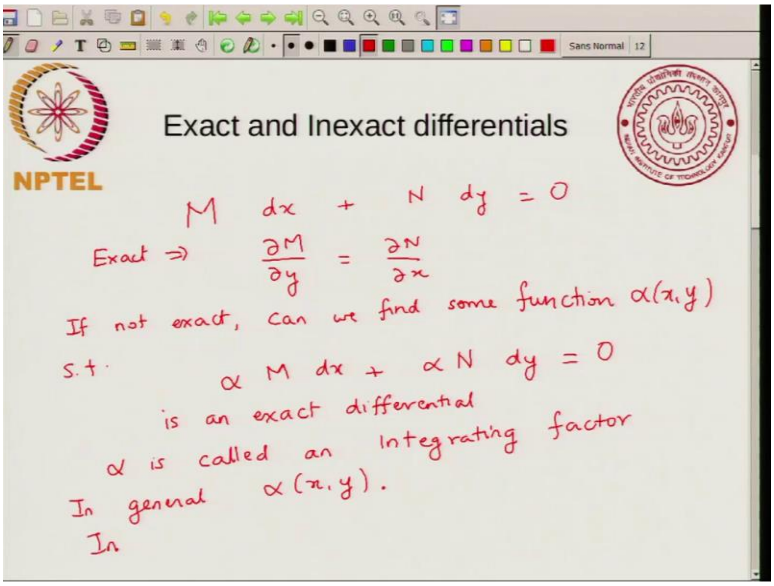

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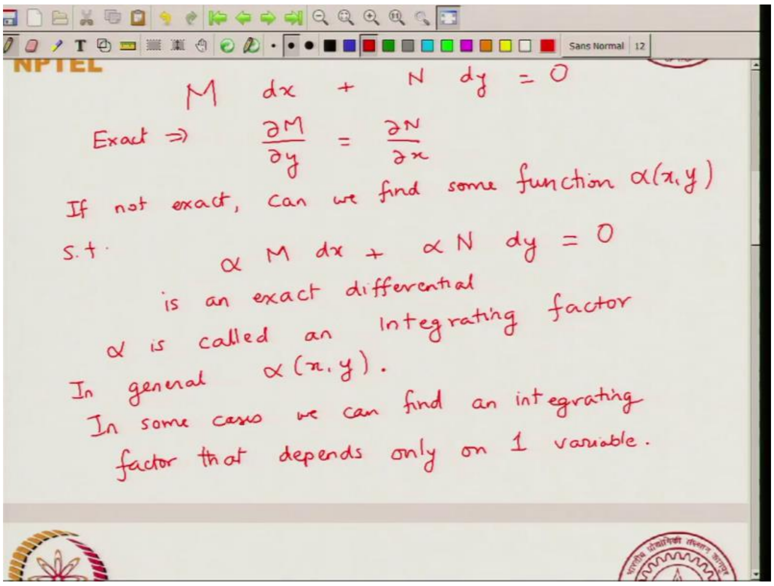

So, just to remind ourselves about exact and in exact differentials. We said that, we can write our first order differential equation as M times dx plus N times dy equal to 0; where M and N are functions of x and y. And what we said is at an exact differential implies. So, exact implies dou N by dou y equal to dou N by dou x. So, then you say it like that. If it is not exact; so if not exact then can we find some function alpha of x y such that alpha times M dx plus alpha times N dy equal to 0 is an exact differential. So, what I mean is that the left hand side is an exact differential. So, what I did is I just took this equation multiplied it by alpha. So, the right hand side will be 0 into alpha which will still be 0.

Now, what I have is alpha M and alpha N instead of M and N. Now if this is an exact differential. So, can we find such an alpha? So, this is the main goal of this idea of integrating factor. So, alpha is called an integrating factor. So, if you find such an alpha that satisfies this then you have an exact differential and you can use exactly the methods that we use to solve for exact differentials

Now the question is how do you find an alpha that satisfies this. Now in general alpha is a function of x y, but maybe in some cases you can find an integrating factor that depends only on one of the variables. So, in some cases we can find an integrating factor that depends only on one variable. For example, you might find an integrating factor; for example, you might find alpha of x, integrating factor that depends only on x.

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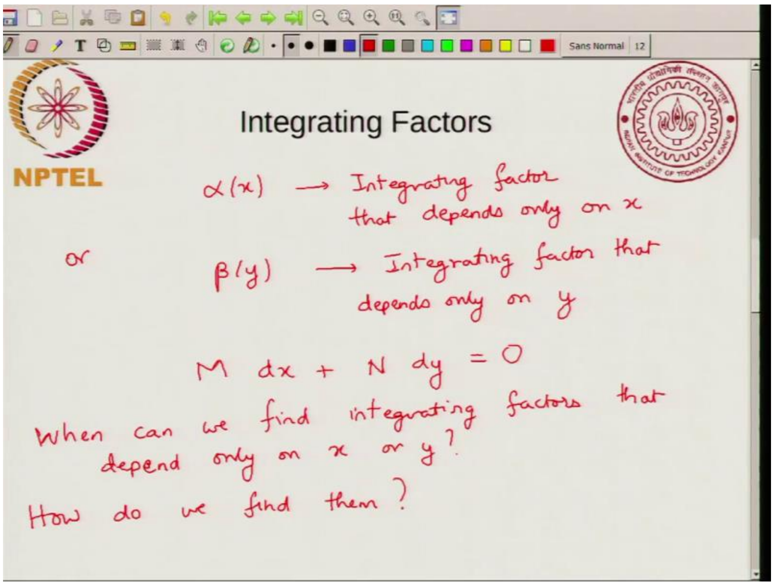

Now, notice I am using the term integrating factor, but this is actually a function of x; it is not a constant it is a function of x. In some cases you can find an integrating factor that depends only on x or you may find some other function. So, some other function I am just calling it beta of y; this might be integrating factor that depends only on y. So, differential equation was M dx plus N dy equal to 0, this is an in exact differential equation. And what we want to do is we want to find an integrating factor such that it becomes exact.

Now, it turns out that you know under certain conditions you can find an integrating factor that depends only on x or under other conditions you can find an integrating factor that depends only on y. So, what are these conditions? When can you find an integrating factor that depends only on x? When can you find an integrating factor that depends only on y? And how do you find these integrating factors?

So, I will just specify this when can we find integrating factors that depend only on x or y. And how do we find that? So, these are the questions that will address in the rest of today's class. And I should mention that integrating factor is you know very very widely used method. And i you will see that it is a very effective way of solving differential equations.

Now, if you have an integrating factor that depends only on x. Suppose, you had an integrating factor that depends only on x.

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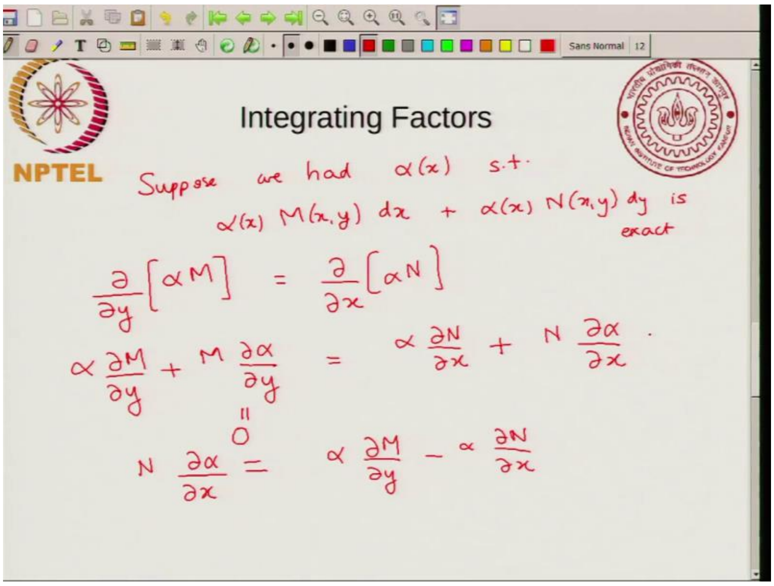

Suppose, we had alpha of x such that alpha of x times M of x y; I am writing explicitly the dependent on x and y here, plus alpha of x N of x y dy is exact. If this is exact, then you can immediately say that we use the condition for exact differentials we say that dou by dou y of alpha times M is equal to dou by dou x of alpha times N.

And if you work this out further what you can see is that this is; so this alpha times dou M by dou y plus N times dou alpha by dou y this is equal to alpha times dou N by dou x plus N times dou alpha by dou x. So, I just use the product rule and I wrote it in this form. However, we know that this is equal to 0; dou alpha by dou y is equal to 0 because alpha depends only on x. So, that is what we assumed. We assumed that alpha depends only on x so this term will go to 0.

So, we get this equation. I can rewrite my equation in the following form; I will just bring the dou alpha by dou x to the left. So, I will write dou x and I can write this as alpha times dou M by dou y minus alpha times dou N by dou x and I have an N here. So, N times dou alpha by dou x that is this term is equal to and I just took everything else on the right hand side. And I can rearrange this further and I can write 1 over alpha dou alpha by dou x is equal to 1 over N dou M by dou y minus dou N by dou x

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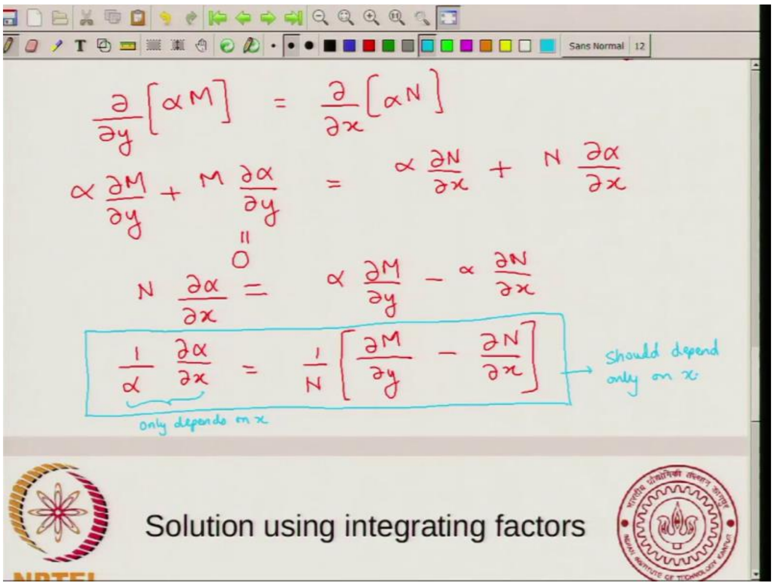

Now this equation that we have here, this equation let me emphasize this equation a bit. This equation is quite a remarkable equation. And the reason I said it is remarkable is that we started with the assumption that alpha is a function only of x. So, if alpha is only a function of x then the left hand side this only depends on x. So, left hand side only depends on x, therefore the right hand side; now the right hand side M and N and functions of x and y. So, their derivatives will also be functions of x and y.

So, this should depend; in general it depends on both x and y but if you have this integrating factor alpha, then this should depend only on x. So, what this means is that we have a condition for existence of integration factor. If this right hand side depends only on x is not a function of y then you have an integrating factor alpha that depends only on x; that satisfies this equation.

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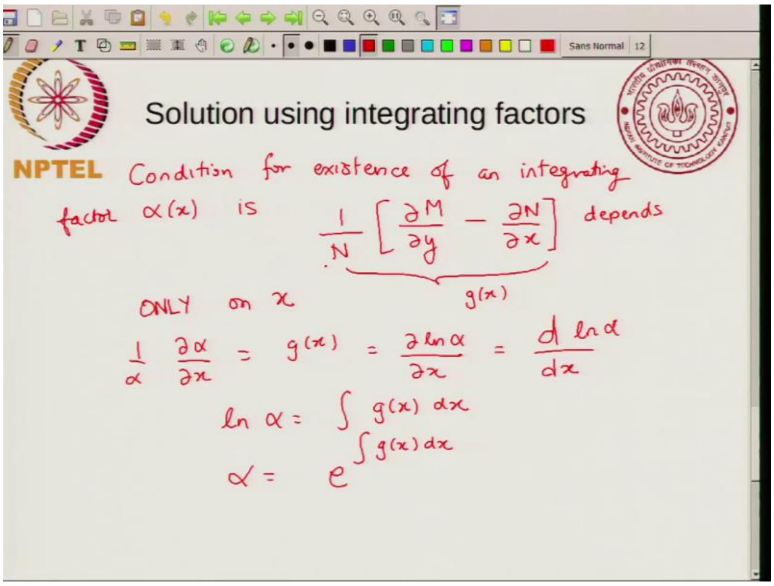

So, the condition for existence of an integrating factor alpha of x is 1 over N dou M by dou y minus dou N by dou x depends only on x. So, this should be independent of y. If this term is independent of y then you can find an integrating factor that depends only on x. So, what do you do? You just calculate this quantity, so you know M and N, you calculate this right hand side, you calculate dou M by dou y minus dou N by dou x divide by N, and you see it has it depend only on x. If it depends on x then you know that whatever you get here you can equate it to 1 over alpha dou alpha by dou x.

So, then I will just call as if this is g of x then you immediately know that dou alpha by dou x 1 over alpha dou alpha by dou x equal to g of x. Or I can write this is left hand side is dou ln alpha by dou x. So, this is dou ln alpha by dou x derivative of natural log of alpha will give me 1 by alpha dou alpha by dou x. I should write this as a regular differential, because alpha depends only on x ln of alpha. This whole equation has only one variable x, there is no y in this equation. And I can integrate this, so I can write ln alpha is equal to integral g of x dx and I can write alpha is equal to e raise to integral g of x dx. So, this is my integrating factor that depends only on x.

I can find this integrating factor that depends only on x. So, the condition for existence of an integrating factor that depends only on x is that this quantity dou M by dou y minus dou N by dou x into 1 by N should depend only on x. Now notice that if dou N by dou y equal to dou N by dou x, then of course you have an exact differential and you do not need to go through any of this. So, dou M by dou y is equal to dou N by dou x then you have an exact differential and you do not even need to find an alpha.

So, we are looking at how this is deviating from the exact differential and then dividing it by this N and seeing whether we can find an integrating factor that depends only on alpha.

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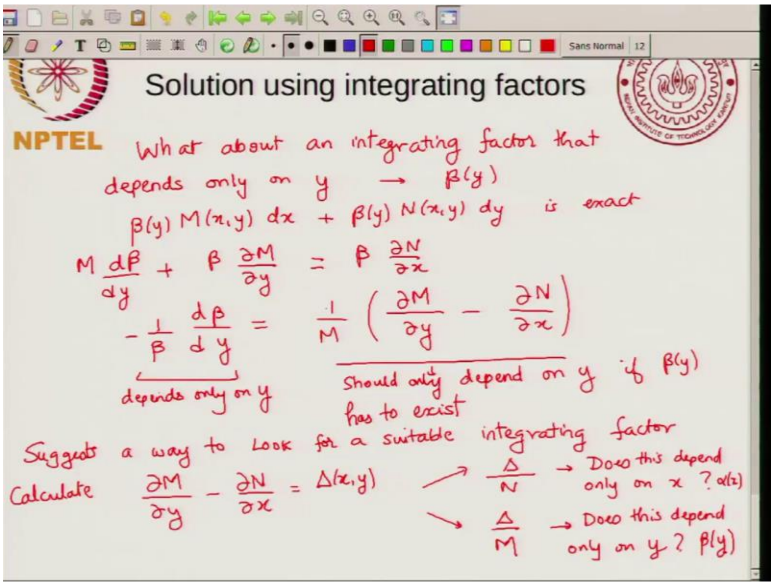

So, now we can do the same thing. We can ask, can we find an integrating factor that depends only on y. So, what about an integrating factor that depends only on y? Let me call this beta of y. So, beta of y is an integrating factor that depends only on y that is independent of x. So, we can do exactly the same thing. What you have is beta of y M of x y dx plus beta of y N of x y dy is exact. And if you go through the same exercise that we went through earlier what you would say is that you would put the condition if this is exact that means, derivative of this quantity with respect to y. Now that will have two terms: it will have M times d beta by dy plus beta times dou M by dou y this should be equal to; now we are taking the derivative with respect to x now beta is independent of x. So, I can just write beta times dou N by dou x.

Now again I can do the same sort of rearrangement, I can write minus 1 over beta d beta by dy is equal to 1 power M dou M by dou y minus dou N by dou x. Again you have this depends only on y; and again let me make the correction here I will call this d this is again since it depends only on y this is d beta by dy. So, the condition again we have is that this in order to find an exact differential that depends only on y, this should depend only on y. So, this should only depend on y, if beta of y has to exist.

Notice; let us let us just go back to what we did in the case of x. In the case of x we had this quantity 1 over N dou M by dou y minus dou N by dou x should depend only on x. And in this case we have almost the same factors, but instead of N in the denominator we have M in the denominator. So, this should depend only on y. So, the rest of the factor dou M by dou y minus dou N by dou x is the same. So, what this suggest is that this tells you how you can go about looking for integrating factors. So, this suggests a way to look for a suitable integrating factor. So, what is this suitable way in which you look for an integrating factor?

So, what you do is: so first calculate dou M by dou y minus dou N by dou x. So you will get something. And then let me call this delta of x y. First you calculate this delta of x y, it is in general a function of x y. Then you calculate two things: you calculate delta divided by N and delta divided by M. So, you take this delta you divide it alternatively buy M or by N. Now you check does this depend only on x and you check does this depend only on y.

So, these are the things that you check. You check whether this delta by N and delta by M depend only on one variable. If this depends only on x then you get your integrating factor alpha of x. If this depends only on y then you get an integrating factor beta of y. So, what this tells you is that you can use this method of integrating factors to actually convert an inexact differential to an exact differential. Finally, if you get one of these conditions satisfied then you multiply either by alpha or beta you multiply your differential equation either by alpha or beta. And now the differential equation becomes an exact differential equation.

So, this basically shows that if you are given a general differential equation then first thing you try to do is to separate the variables, if that does not work out you try to check whether it is an exact on an inexact differential; if it is an exact differential then you can solve the differential equation; if it is not an exact differential then you check whether you can use an integrating factor to convert it to an exact differential. And you can check whether you have you can get an integrating factor that depends only on x or on the other y. If you are not able to find an integrating factor that depends only on x or y then you cannot solve the differential equation using these methods. You might find some tricks some other tricks to solve it, but there is no standard method to solve a first order differential equation in that case.

So, in the next class I will discuss about the system of differential equations. So, I will look at a system of a first order differential equations and I will see how you can solve them. And how you can use techniques that we learnt from linear algebra and matrices to solve first order differential equations.