Mathematics for Chemistry
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Module - 05
Lecture - 23
Exact Differentials

So, we saw what a first order differential equation, we saw different types of first order differential equation and we said that the general strategy of solving differential equations is to separate the variables. Now we look at cases where you cannot separate the variables then what can you do and today's topic will be exact differentials which will be the way of solving differential equations where you cannot easily separate the variables.

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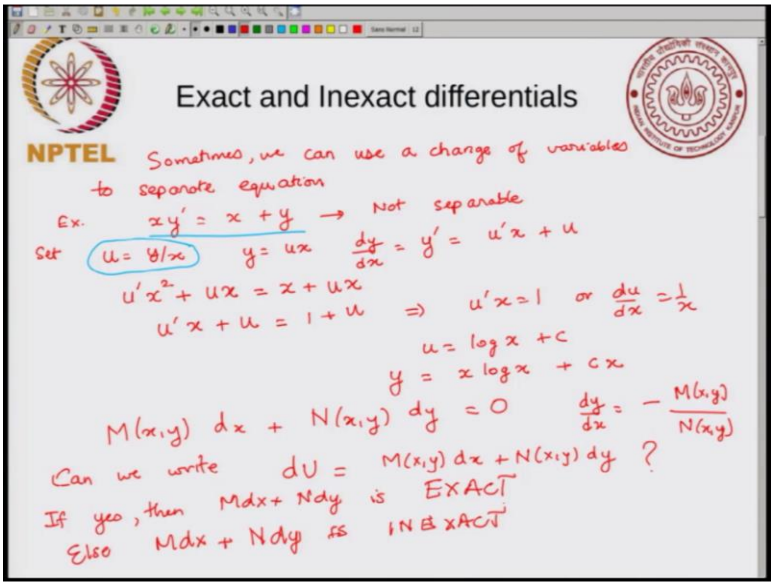

So, before talking about exact differentials I just want to mention one thing that sometimes you have, we can use a change of variables to separate equations equation. So, your differential equation might be converted to separable form by a change of variables.

So, an example of that is suppose you have a differential equation. So, example suppose you have x y prime equal to x plus y now you can see this you cannot try to separate, so you cannot separate this not separable; however, let us say u equal to y by x. So, suppose you set u equal to y by x set u equal to y by x. So, you are changing the variables from your dependent variable from y to u. So, suppose you set u equal to y by x then you can you can immediately write y equal to u x and you can write d y by d x is equal to which is also denoted by y prime this I can write it as u prime x plus u.

So, how does this help you? So, you just if you just replace y by u prime x plus u or if you replace y prime by u prime x plus u then what you will get is u prime x square plus u x equal to x plus y is u x and what you can see is that immediately 1 x will cancel. So, you can cancel one x. So, what you get is u prime x plus u is equal to 1 plus u and again also u will cancel. So, this you can write implies u prime x equal to 1 or d u by d x equal to 1 by x and you can you can solve this and you will get u is equal to log of x plus constant and you can go back and you can write y is equal to u x. So, that is y equal to x log x plus constant times x.

So, that this is the solution. So, you solve this equation which did not look separable initially by doing a change of variables. So, sometimes a change of variables can convert the differential equation into a separable form now often you cannot you cannot separate differential equations; however, there is still something to say. So, in this particular case we just looked at the differential equation and we and we chose this we chose this transformation of variables, but suppose you are asked how do you know whether a differential equation will have such a transformation of variables how do you know that a differential equation can be converted to separable form.

So, to answer that question, we will we will address the issue of exact and inexact differentials and here what we will do is we will rewrite the differential equation in a slightly different form. So, we will rewrite the differential equation in the form M of x y d x plus N of x y d y equal to 0. So, I just read out the differential equation not every differential equation can be written in this form, but let us assume that you know you can write the differential equations in this form.

So, notice that this basically implies d y by d x equal to minus M of x y divided by N of x y. So, it is an explicit differential equation whatever your d y by d x is you can, if you can write an expression for d y by d x then you can write it in this form. So, this is the differential equation and now you know that when you when we studied vectors and we studied gradient of a vector we studied a potential theory. So, when you studied potential theory we said that we said that force can be expressed as a gradient of a potential if the various components satisfy some relation and exactly the same idea can be used here.

So, can we write d u is equal to M of x y d x plus N of x y d y. So, is there is there some u some u which is a function of x y such that d u can be written in this form. So, can we write this and the answer is if you can if yes then you will say M d x plus N d y is exact else M d x plus N d y is inexact. So, this is the basic this is the basic idea of exact and in inexact differentials that if you if you have this differential equation and you and you want to write this whole differential on the left as d u.

So, that is what we want to do and we want to ask can we do it if you can do it then you say that it is an exact differential if you cannot do it you say it is an inexact differential. So, what is the condition for exact?

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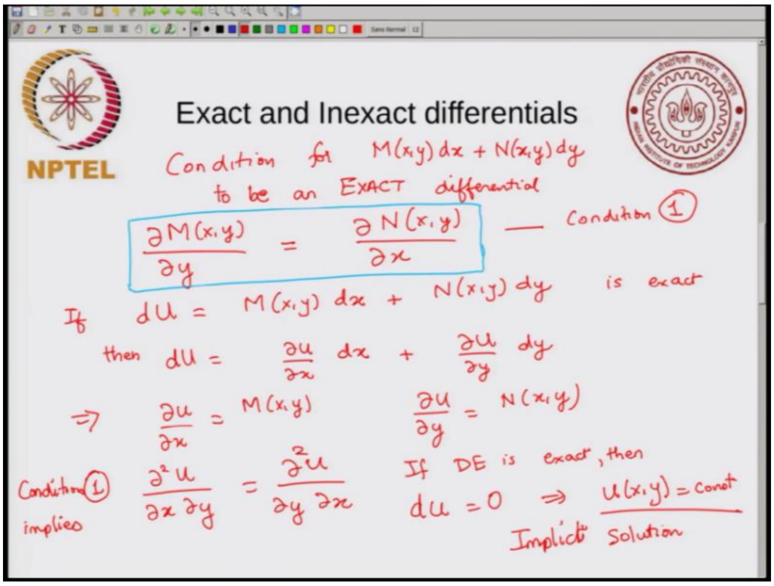

And inexact condition for M of x y d x plus N of x y d y to be an exact differential so what is the condition and I will write the condition and then it will become clear why that is the condition. So, the condition is partial derivative of M of x y with respect to y. So, the partial derivative of y means you keep x fixed. So, this should be equal to partial derivative of N of x y with respect to x again you keep y fixed and you have already seen such conditions when we did potential theory or when you do thermodynamics you see such conditions.

So, this is the condition for M d x plus N d y to be an exact differential and it is very easy to see, why this condition helps to make this an exact differential. So, to understand that what we want to do is if d u equal to M x y d x plus N of x y d y is exact then we have d u is equal to dou u by dou x d x plus dou u by dou y d y.

So, this implies dou u by dou x is equal to M of x y and dou u by dou y is equal to N of x y. So, if this were an exact differential then this is what it really implies about M and N. So, now, you can see the condition. So, therefore, the condition; your condition this condition basically implies the following. So, if you replace M by dou u by dou x then you have dou by dou y of dou u by dou x. So, that is dou square. So, I will call this condition one. So, condition one implies dou square u by dou x dou y is equal to dou square u by dou y dou x and if u is a smooth function of x y then this is satisfied.

So, therefore, you can see you can see the connection why we insisted that this should be the condition for exact differential. So, now, so what does this mean this means that you can take any differential equation you can check is it exact is it not. So, so and that is a very nice thing you know we are; you can easily check that it is exact or not.

Now, if it is exact if your differential equation is exact then what happens if d e is exact then have these 2 conditions you have dou u by dou x equal to m. So, u then d u equal to 0 implies u of x y equal to constant this is your implicit solution. So, you can solve the differential equation very easily if it is exact.

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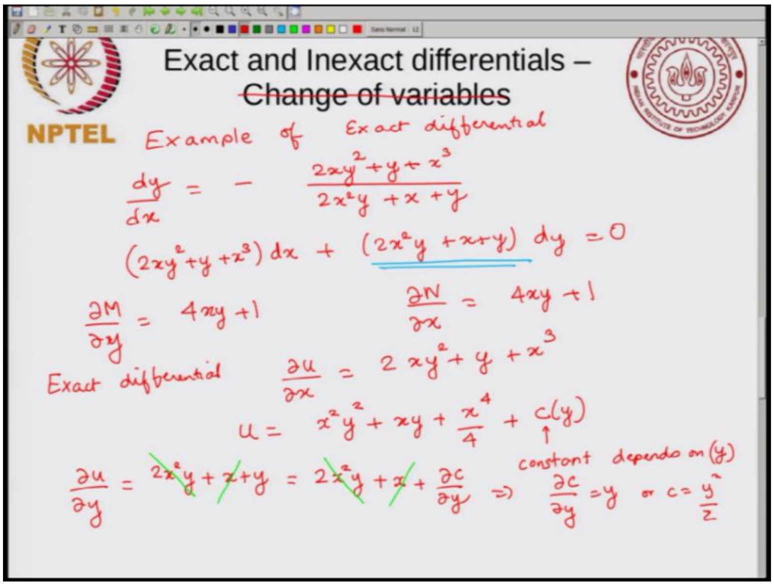

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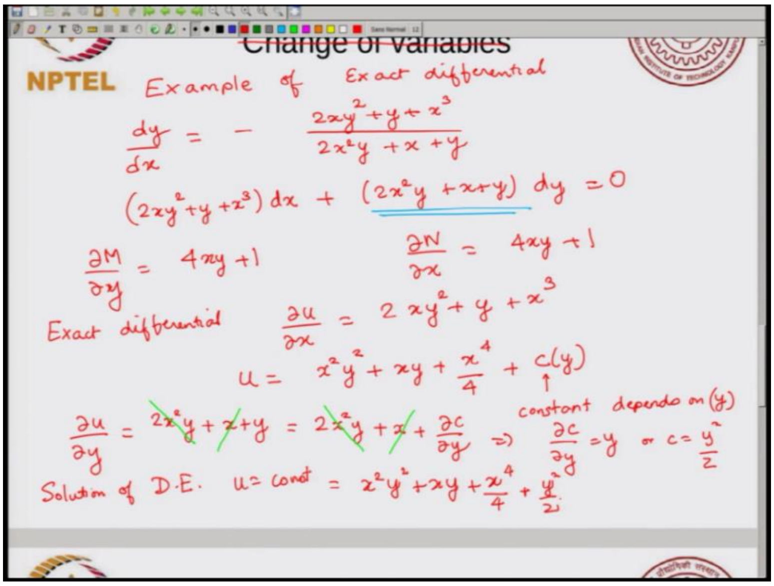

So, let us take an example I would not discuss the change of variables yet. So, let us take an example. So, I would not be discussing this let me take just take an example of exact differential. So, suppose you have d y by d x is equal to minus 2 x y square plus y plus x x plus x cube divided by 2 x square y plus x plus y. So, what I will do is I will rewrite this; I will rewrite this as, so this times d x, I will bring it to the left. So, I have 2 x y square plus y plus x cube d x plus 2 x square y plus x plus y d y equal to 0. So, this is my and you can verify. So, you are the term dou M by dou y is equal to 4 x y plus 1 dou N by dou x is equal to 4 x y plus 1. So, clearly it is an exact differential. So, this is an exact differential.

Now, how do you solve this exact differential equation? So, you know that this quantity is nothing, but dou u by dou x. So, so you say that dou u by dou x is equal to 2 x y square plus y plus x cube you can use either of them you can use either dou u by dou x or you can use dou u by dou y and now if I integrate the right hand side with respect to x remember when you do dou u by dou x you are keeping y fixed. So, you are in. So, now, this integration is very straightforward. So, you will immediately say that u is equal to x square y square plus x y plus x rise to 4 divide by 4 plus constant.

And remember the solution of the differential equation, now you have to be a little careful. So, this constant of integration, constant depends on y. So, it is, I should write c of y because I had a partial derivative with respect to x. So, any function of y if I take the derivative with respect to x it will go to 0. So, if I have this you can verify that a derivative with respect to x will give me exactly the above equation.

So, this is the form of u now further you know that dou u by dou y is equal to 2 x square y plus x plus y and this is from this is from the differential equation from here. So, what can you say so. So, therefore, what you will say if I take dou u by dou y here then I will get 2 x square y plus x plus dou c by dou y and what you immediately get is that dou c by now these 2 will cancel each other. So, I will just cancel this, similarly x x will cancel. So, what you will be left with this implies dou c by dou y is equal to y or c is equal to y square by 2.

So, by doing this we got this c of y and. So, I can write the total u. So, my solution of differential equation is u equal to constant equal to x square y square plus x y plus x 4 by 4 plus y square by 2. So, this is a constant. So, we solve this differential equation by this method of exact differentials.

So, I think I think this use of exact differentials to solve differential equations is actually a very powerful methods and I will just write; I will just emphasize the steps again, in the solution by exact differentials.

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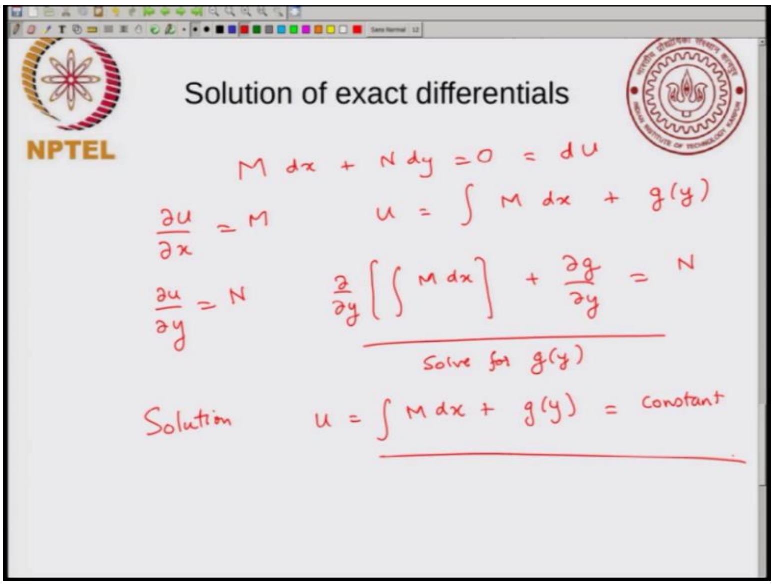

So, first we had M d x plus N d y equal to 0. So, so we took advantage of the fact that m. So, you; we set this equal to d u and what we wanted to do is to find this u is a function of x y just like M and N.

So, what you said is that dou u by dou x equal to M and. So, I can write u as integral M d x plus some function I will just call it g of y then you had dou u by dou y equal to n. So, therefore, I can write dou u by dou y I can write as dou by dou y of integral M d x plus dou g by dou y equal to N and this allowed us to solve for this gives a differential equation in g. So, you can solve for g for g of y and once you have solved for g of y then you have your solution of the form u equal to u equal to integral M d x plus g of y equal to constant. So, this is my solution of this differential equation.

So, we see how you can use the idea of exact differentials to actually solve a differential equation that otherwise if you did not use this idea then this differential equation actually looks very complicated. So, if you just take this differential equation then it does not look like you can separate x and y it looks like x and y are very; it looks like a very complicated equation where you cannot separate x and y. But by going through this by going through this idea of exact and inexact differentials we actually related at the numerator and the denominator to derivatives of some function which is the solution and the only reason we could do this was because you had this equation of this exact differential that dou M by dou y was equal to dou N by dou x. So, this is a very powerful method to solve first order differential equation and what it says is that even if you cannot separate the variables you can still go ahead and solve first order differential equations.

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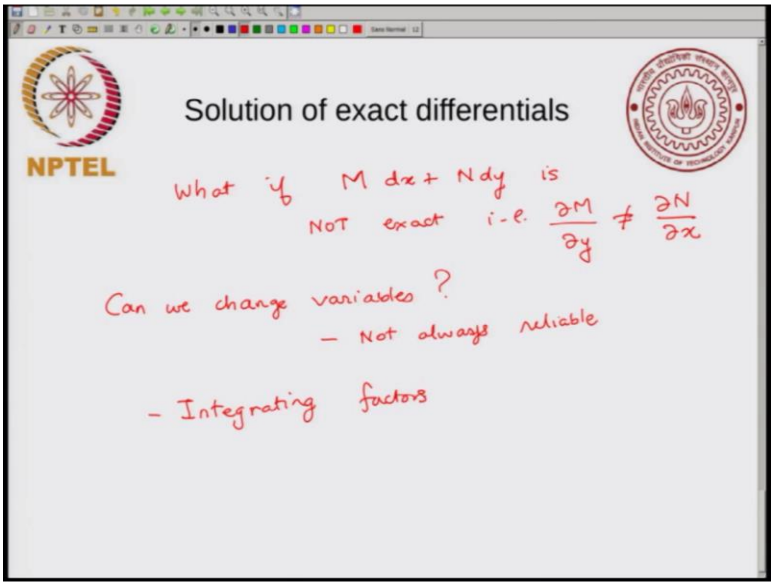

Now, what happens if M is not exact? So, what happens; what if M d x plus N d y is not exact and that is dou M by dou y is not equal to dou N by dou x what can you do in such a case? So, now, there are 2 things to. So, one is can we change variables. So, sometimes you are able to find some tricks where you where a where you suitably change variables and suddenly you get to the solution. So, this is not always reliable. So, it might not be possible.

So, you just have to see whether you can change variables and you can get to this. So, the other method that we will be discussing in the next lecture will be that of integrating factors and this is the other trick that will be using when you want to solve differential equations first order differential equations. So, in the next class I will be talking about integrating factors and how you can use the idea of integrating factors to solve differential equations where you cannot use the ideas of exact differential. So, I will stop here for today now.