Mathematics for Chemistry
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Module - 04
Lecture - 20
Practice Problems

So, now we have almost finished with the discussion in matrix and just to conclude I will do some practice problems. In particular I will look at what are called as rotational matrices.

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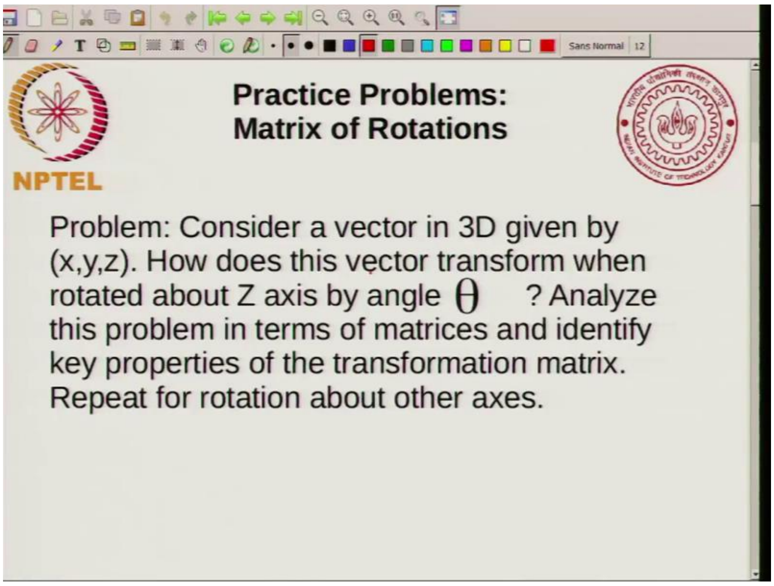

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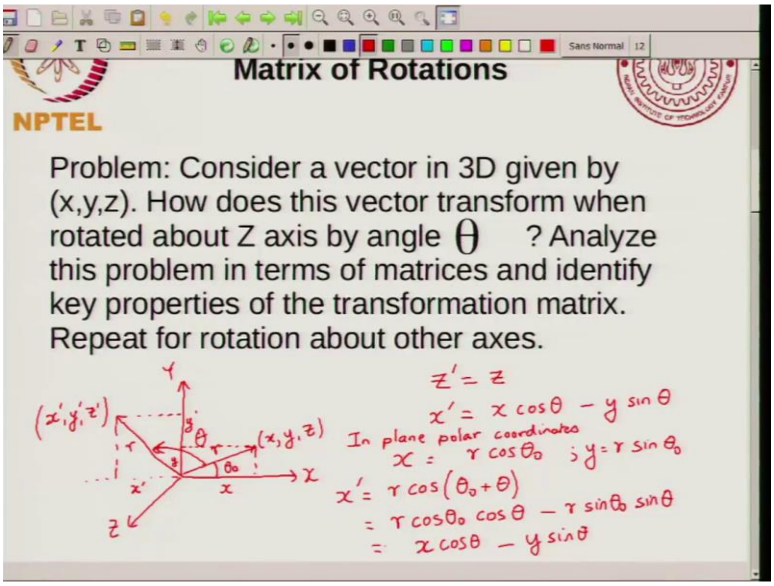

So, the problem is tough, I will just state the problem in the following way. So, suppose you consider a vector in 3 D and let us say that vector is given by x, y, z. How does this vector transform when it is rotated about Z axis by angle theta? And what I said is analyze this problem in terms of matrices and identify key properties of the transformation matrix and repeat for rotation about other axis.

So, this is an illustrative exercise on use of rotation matrices and. So, let us just look at what the problem says. So, you have some coordinate system x, y, z. I will show this as x this is y and let us say z is a plane that is coming outside the third coordinate system and now what you have is you have some vector. This is the point x, y, z, this is your vector and you are rotating it about the Z axis. So, you imagine that you are rotating it by angle theta and you get a new vector now; obviously, the length of this new vector should be the same as the original vector. I will call this x prime, y prime, z prime. This is rotated by angle theta about the Z axis.

So, now what can you say about this x prime, y prime, z prime and how it is related to x, y, z. So, this is the basic problem that we are asking. Now in order to do this, you can immediately say one thing if you are rotating about the Z axis then the z coordinate will not change. So, you can immediately say that z prime has to be equal to Z. Now what about x prime and y prime? So, x prime and y prime you can actually easily find out you can just do simple cosine algebra and you can see that this is my x and this is my x prime and you can easily see that, I mean if you want you can use plane polar coordinates, but you know even that is not required. You can easily see that that x prime can be written as x times cos theta minus y times sine theta.

So, how do you show this? Now I just said it is not very difficult to show. So, suppose you had used spherical coordinates and let us say this angle was theta naught then, what you would have is you write x. So, in plane polar coordinates let us say the length of this vector is r, x equal to r cos theta naught and y equal to r sin theta naught. So, this is how we would define. So, x and remember this is your y coordinate, this is your y, this is your x and this is your y prime.

Similarly, what you will have is. So, this is what I have, what you had and what is x prime? x prime is nothing, but r cos of theta 0 plus theta and. So, what is this equal to? This is equal to r cos theta 0 cos theta minus r sin theta 0 sin theta and now r cos theta 0 is nothing, but x. So, I can write this as x cos theta minus r sin theta 0 is y. So, I can write y sin theta which is what we had here.

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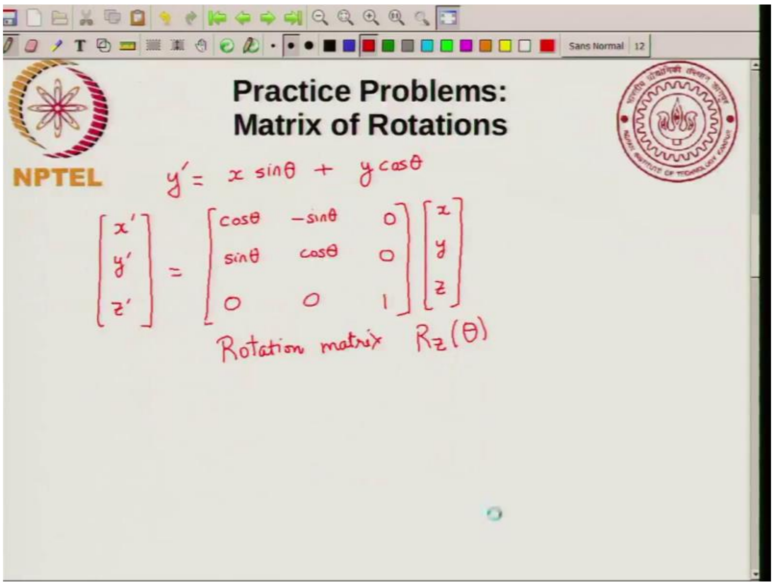

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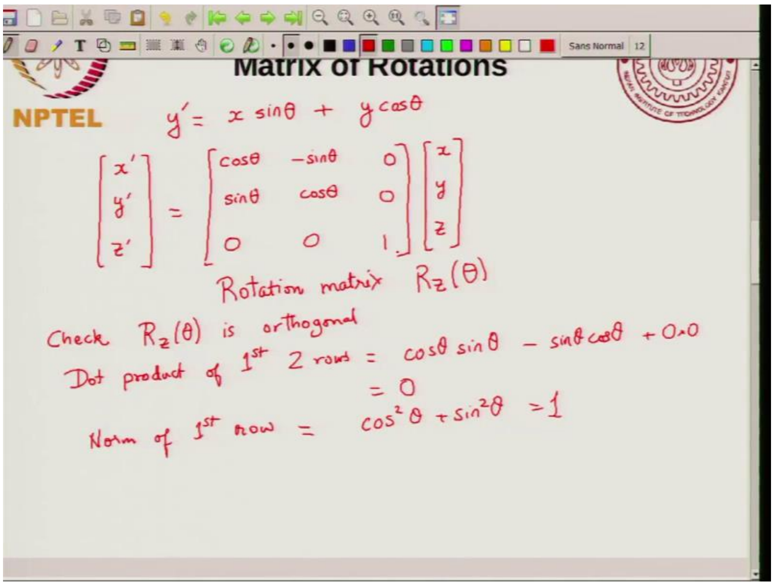

What else can you do? You can also write the y prime. So, I can also write y prime is equal to and by the same method what I will get is y prime equal to x sin theta plus y cos theta. So, our three transformations are basically z prime equal to z, x prime equal to x cos theta minus y sin theta and y prime equal to x sin theta plus y cos theta.

So, I can write this in the following way. So, I can write x prime, y prime, z prime is equal to some matrix times x, y, z; what is this matrix? So, we had, we just look at this x prime equal to x cos theta minus y sine theta; so x prime. So, this will be nothing, but cos theta, this will be minus sin theta and it is independent of z. So, this will be 0 and similarly this will be sin theta cos theta 0 and this will be 0, 0, 1. So, z prime equal to z it is independent of x and y. So, you get. So, you can write these three equations in this matrix form. This is called, this is a rotation matrix. I will use the notation R z theta. So, this is rotation about Z axis by angle theta.

So, I can express this transformation in this form. Now you can clearly see that R z theta is orthogonal. So, check R z theta is orthogonal matrix. So, how do you check it is orthogonal? You can just take any 2 rows. So, let us say you take the first row and multiply it by the second, take a dot product of the first 2 rows. So, so if I take dot product of first 2 rows is equal to. So, what I will get is cos theta into sin theta. I will get minus sin theta cos theta plus 0 into 0. So, this is equal to 0.

Similarly, you take dot product of second and third row you will get 0. You take dot product of first and third row you will get 0. You take dot product of first and second column you will get 0. Dot product of first and third column, dot product of second and third column all of them will be 0. So, dot product of any 2 rows or columns is 0. Now further you can see that norm of first row. So, this is equal to cos square theta plus sin square theta equal to 1, norm of first row is just row dotted just the sum of squares of the coefficient and that is cos square theta plus sin square theta. Similarly norm of second row, norm of third row they are all 1. So, we say that any 2 different rows are orthogonal and the norm of each row is 1. So, therefore, this R z theta is an orthogonal matrix. So, we have seen and it is orthogonal. So, it will preserve the length and; obviously, we will preserve the length because it is a rotation.

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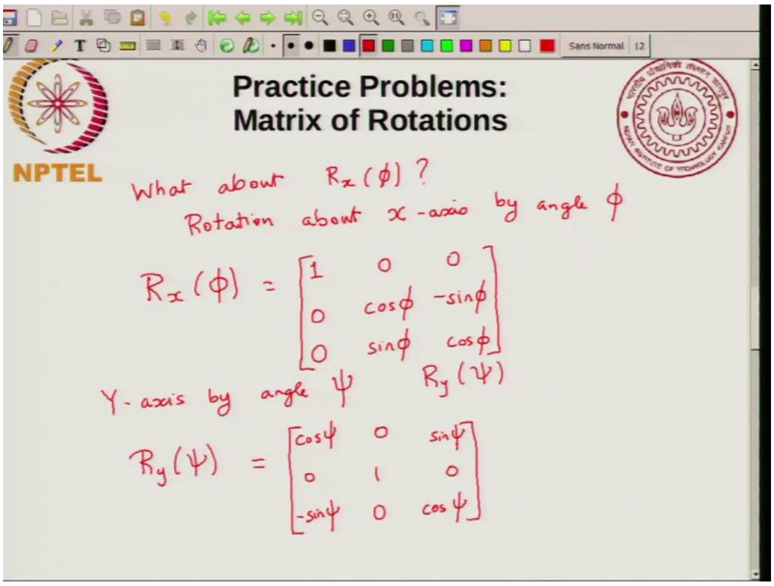

Now what else can you do? Now let us talk about rotating about, what about rotating about let me call it R x phi, this is rotation about x axis by angle phi. Now in this case I do not need to do this in detail, but you can immediately write R x of phi as this matrix. Now since you are rotating about the x axis, the x coordinate will remain unchanged. So, you have a 1, 0, 0, here and then you will have this will look exactly like your other matrix, but instead of theta you have phi cos say phi minus sin phi, sin phi, cos phi. So, this is a matrix for rotation about x axis by angle phi. And similarly you can do for y axis also. So, y axis by angle I will just use another Greek letter psi, I will write it as R y of psi. So, this will look like now the y coordinate is left unchanged. So, here you will have 0, 1, 0, 0, 1, 0. Now what you will have here is cos psi minus sin psi and you have cos psi, sin psi.

So, these are your three rotational matrices. So, you have the R z of theta which is rotation about Z axis and now you have 0, 1, 1 along the in the z direction. Now here when you have R x of phi then, you have 1, 0, 0 at this is. So, the x coordinate is unchanged and here the y coordinate is unchanged. So, these are the three matrices of rotations. Now you can ask, what happens when you take let us say x, y, z and first you do R z of theta?

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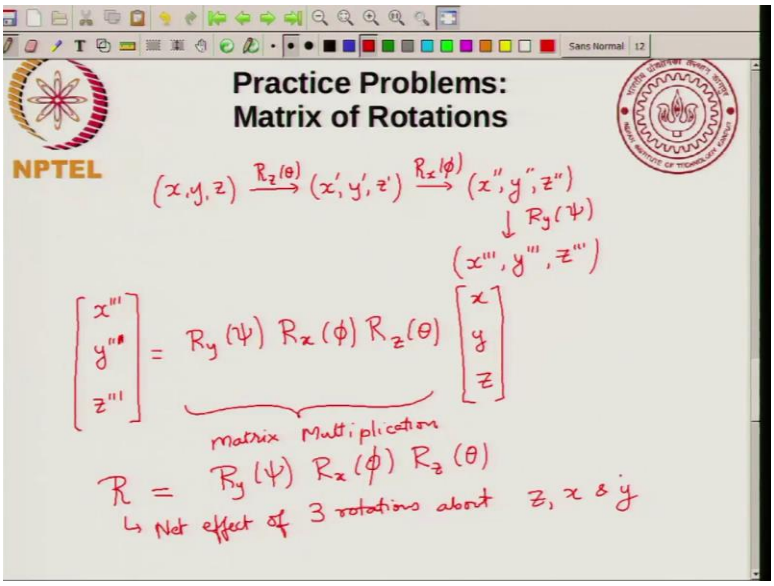

Then I get something that is x prime, y prime, z prime then, I again do by R x of phi then I will get something like x double prime, y double prime, z double prime and then I do by R y of psi I will get x triple prime, y triple prime, z triple prime. So, I imagine doing successively doing three different rotation operations.

Now, you can easily show again you do not you can show that x triple prime, y triple prime, z triple prime. So, this is after three rotations. So, I can write this as R y of psi times x double prime, y double prime, z double prime, but x double prime, y double prime, z double prime I can write as R x of phi times x prime, y prime, z prime. So, I can write it as R x of phi times x prime, y prime, z prime. Now x prime, y prime, z prime I can write as R z of theta times x, y, z times. Now this is nothing, but this is matrix multiplication. So, what you have is you can find the net effect of multiple rotations just by multiplying the individual rotations. So, the net effect of all these three rotations is essentially a matrix multiplication of individual rotations and. So, if I call this R, this is R y of psi, R x of phi, R z of theta. This is the net effect of three rotations about z, x and y.

So, I can write this matrix corresponding to this total rotation. I can write it in this way and what you can do is you can actually calculate this, you can calculate, what is R? Again it is straightforward. You just have to multiply the, these three matrices; you have multiply this matrix, this matrix and this matrix in the particular order. So, if you do that you have to preserve the order, you know matrix multiplication is not something that is commutative. So, you are first rotate about z, then by about x and then about y.

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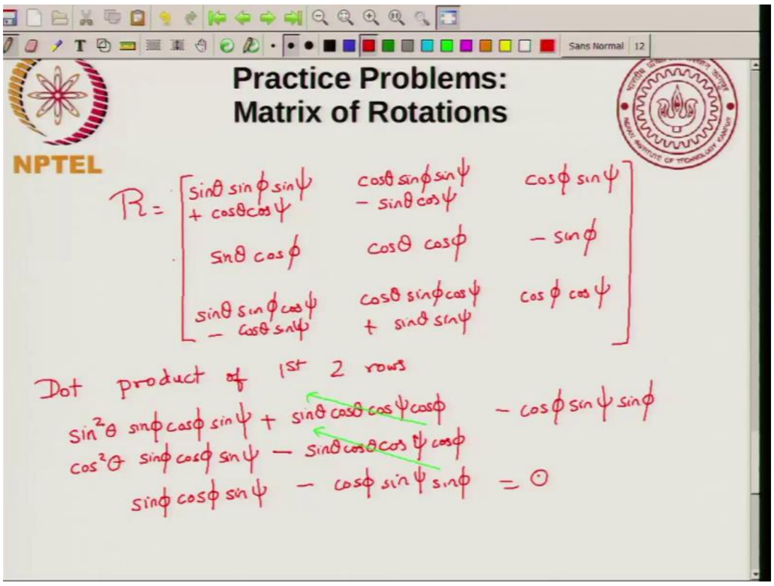

So, if you do that then what you will get is that this R is equal to I will just write the final result. Now it will looks a little complicated, but it is simply a matrix multiplication sin theta, sin phi, sin psi plus cos theta, cos psi and then you have cos theta, sin phi, sin psi minus sin theta, sin psi and you have cos phi, sin psi, sin theta, cos phi, cos theta, cos phi minus sin phi and then you have sin theta, sin phi, cos psi minus cos theta, sin psi and finally, you have cos theta, sin phi, cos psi plus sin theta, sin phi and the last term is cos phi, cos psi. So, this is the matrix corresponding to three successive rotations; first, about the z axis by theta then, about the x axis by phi and then, about the y axis by psi.

So, now you can easily verify that this is also orthogonal. So, let us just take the first and second rows and take a dot product of the first or let us take a dot product of the first 2 rows and here you will get lot of terms. So, you will get terms having you will have I just write the terms. So, you have the first term that looks like sin square theta you have and then you have sin phi, cos phi, sin psi and you have another term that looks like sin theta, cos theta, cos psi; cos phi. So, this is from the first element of these 2 rows.

Then from the second element you will get a term that looks like cos square theta. So, I will just write it below. So, it will become obvious; cos square theta, sin phi, cos phi, sin psi and you can clearly see that one has a sin square theta other has a cos square theta.

So, when you add these 2 this will go away and you have a minus, you have a sin theta, cos theta, sin psi, cos phi and the last term looks like minus. So, this should be cosine of psi. So, there is a small typo in that and. So, this will become cosine of psi. So, and then you have minus the last term you will have cosine of phi, sin of psi, sin of phi.

Now, this is the dot product of the first 2 rows. Now you can immediately see that these 2 terms actually cancel each other. Now if I add up these 2 terms then, what I will get is I will get sin phi, cos phi, sin psi, if we just add up sin square plus cos square is one and then I have minus the same thing; cos phi, sin psi, sin phi. So, this is equal to 0. So, what we showed is that, these 2 rows are orthogonal to each other and you can similarly show that all other rows are orthogonal to each other. So, and; obviously, it should be because this is a rotation and rotation preserves the length.

So, we have shown one type of rotation matrices, there are actually many other types of rotational matrices that are used in various branches of science and engineering, but this illustrates the power of matrix methods in for solving problems.

So, with that I will conclude this module on matrices and I end with this we have concluded the whole discussion in linear algebra. So, in the subsequent weeks we will start talking about differential equations.