Mathematics for Chemistry
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Module - 04
Lecture – 16
Rank of Matrix, Inverse of a Matrix

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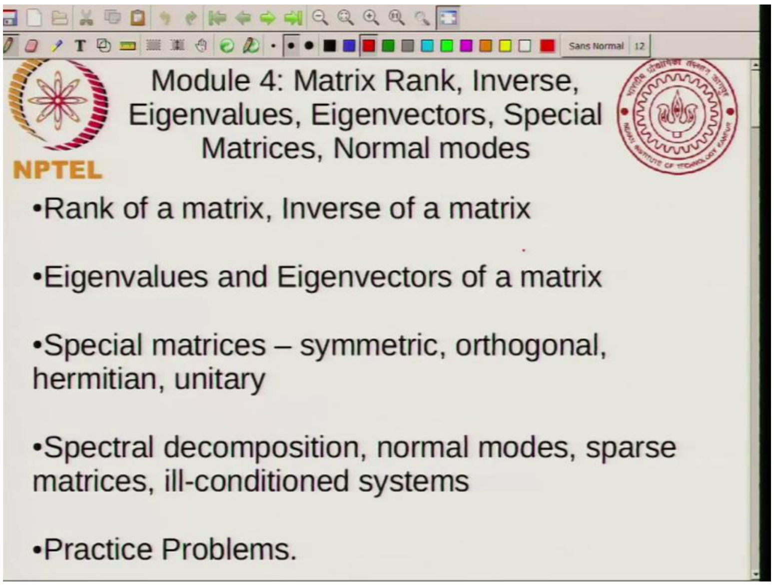

So, now we will start module 4 and in this module, we will complete the discussion on matrixes. We will talk about matrix rank, inverse of matrix, Eigen values, Eigen vectors and then we will talk about special mate matrices and normal modes. So, today I will be talking about the rank of A matrix and the inverse of a matrix.

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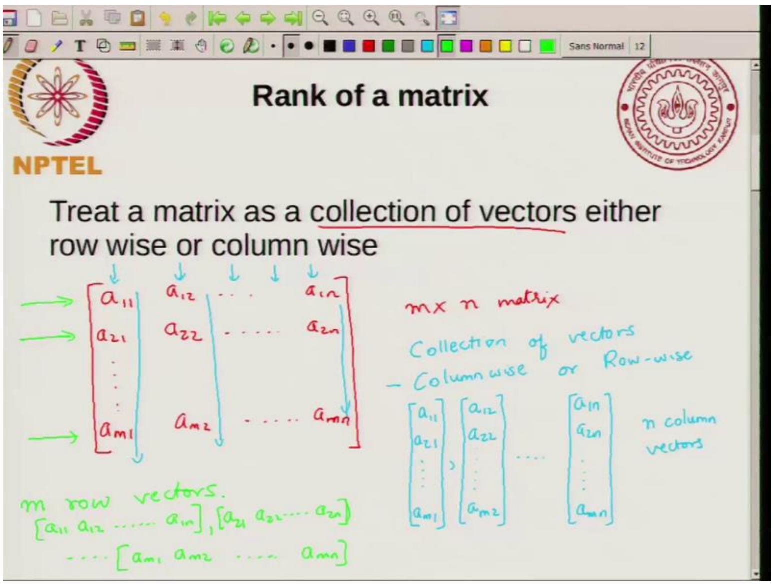

So, let us get to the rank of A matrix. Now, in order to define the rank of A matrix it, it helps to treat the matrix as a collection of vectors. So, your matrix is treated as a collection of vectors. So, how do you think about this and you can treat it either row wise or column wise. So, suppose you have a matrix a typical matrix. So, I will just take a typical over typical matrix a 11, a 12 etcetera up to a 1n, a 21, a 22 all the way up to a 2n and then similarly all the way let me take am1, a m 2 all the way up to a m n. So, this is an m by n matrix. So, this is an m by n matrix.

Now, I said that you can treat this as a collection of vectors. So, you can think of each row as a vector, I will just use a different color. So, you can treat each column as a vector in this way. So, you can think of this as a collection of vectors either as column wise. Column wise of row wise. So, how do you think of it? So, suppose I was to think of this as a collection of column wise vectors. So, column wise I would just say a 11 a 21, I will treat this as one vector, then I would think of another vector a 12 up to a m 2. So, you can say you can see all the way. So, what I am doing is, I am treating this column as a vector, this column as a vector each of these columns as a vector. So, all the way up to a 1 n, a 2 n all the way up to a m n.

So,. So, if I treat it column wise I have n vectors, n column vectors. Now alternatively and you can see this I do not, I do not really need to emphasize this this is row wise. So, if you think of it row wise, you can think of it as m row vectors and these row vectors will look like a 11, a 12, a 1 n then you will have similarly you can have a 21 a 22 all the way up to a 2 n and then all and then and then you can have various vectors all the way up to a m 1, a m 2, all the way up to a m n. So, sometimes it is very useful to think of this matrix as a collection of vectors and you can think of it either as a collection of m roll vector or you can think of it as a collection of n column vectors. So, this this basic idea of thinking about matrices as collection of vectors is what you need in order to define the ranks once you think about this matrix as a collection of vectors then the question you ask, who you can ask is are they linearly are these vectors linearly independent or dependent.

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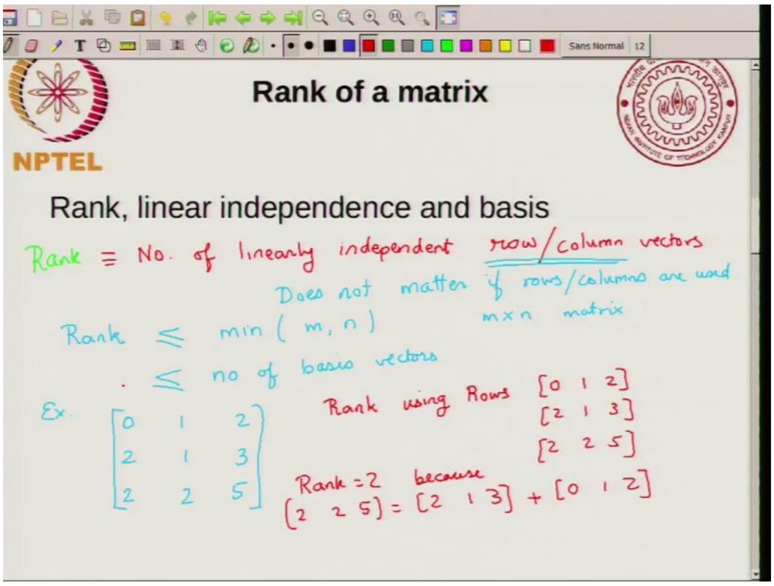

So, now, the rank is nothing but, is nothing, but number of linearly independent rows or column vectors. So, if you want to find the rank of A matrix you what to do, if you think of these you think of this matrix either is a collection of n column vector or you can think of it as a collection of m row vectors then of these. So, suppose you are thinking about it as n column vectors then you can also question how many of these column vectors are linearly independent. So, you will have some number which is less than n (Refer Time: 05:24) it is a rank of A matrix.

Alternatively you could think of row vectors and you will get a number, now you ask how many of these row vectors are linearly independent. You will get a number that is less than m and that is the rank of the matrix. Now what is important is that, it does not matter whether you take row or column vectors you will get the same rank and this is fairly easy to show. So, it is just underline this in a different color. So, does not matter if this rows or columns are used.

So, the idea is that you can use either row or column and you are fine to calculate the rank. So, what this implies is that this implies that your rank must be strictly less than or equal to the minimum of m and n. So, what that means is, if you have fewer rows and columns then the rank must be either equal to or smaller than that number of rows. Similarly, if you have fewer columns and rows then the ranks must be lower than the number of columns. So, rank has to be less than number of rows and it has to be less than or equal to number of rows and less than or equal to number of columns and. So, in so, we can write this as saying that the rank is less than the minimum of, of the number of rows and the number of columns and remember you have an m by n matrix this is for an n by m matrix. So, now, this is what it helps you the rank and you can immediately see that, since we already said that the maximum number of linearly independent vectors is equal to the basis of.

So, maximum number of linearly independent vectors in any spaces the is basically the number of vectors in the basis your ranks will give you some. If you look at a vector space and you consider various vectors in that space then the maximum number of linearly independent vectors is related to the basis. So, in that way you can also you can also relate rank of matrix rank of matrix has to be less than or equal to basis. So, it is less than or equal to number of basis vectors, vectors. So, it is less than or equal to number of basis vectors in that in in the appropriate space that you're considering.

Now, let us, let us go and see, see some examples where you can determine the rank of A matrix. So, suppose I take, suppose I take a matrix that looks like that looks I will just take some example. So, 0 1 2, 2 1 3 and I take and I take the, I take this as 2 2 5. Now, now if you want to determine the rank of A matrix you can use either the row for the columns. So, rank using rows. So, if you use rows then you have 3 vectors. So, you have this vector 0 1 2 you have another vector, 2 1 3 and you have vector 2 2 5 and if you want to ask how many of these vectors are linearly independent you can immediately just by inspection you can see that, you can see that rank equal to 2 because 2 2 5 is equal to 2 1 3 plus 0 1 2. So, in this case you can immediately see the rank you can see that the rank should be equal to 2.

Because, because I can write this third vector as a linear combination of first 2 vectors and not only can I do that, but addition to that you can see that that if you take, if you take the first and second vectors they are clearly linearly independent. So, the rank is not one rank is equal to 2. So, in general you can take any matrix you can do you can do the same using columns and you will get the same answer and basically the rank is independent whether you take the rows or columns, alright.

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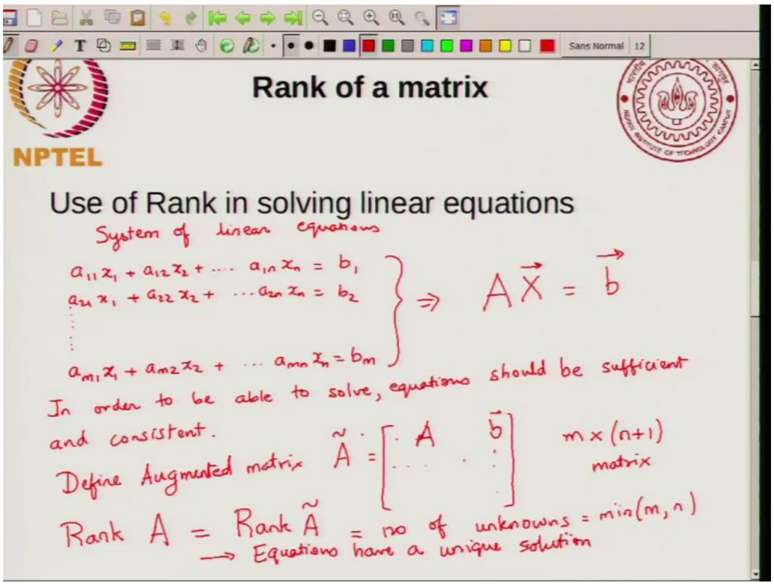

Now, now we can ask the next, next question as to where do you use the rank? So, now, rank of A matrix is actually extremely useful in solving linear equations. Now, now, you remember that we wrote, wrote your system of linear equations. So, your system of linear equations, we can write it in matrix form oh. So, if you remember we had at a 11 x 1 plus a 12 x 2 plus all the way up to let us say a 1 n x n is equal to b 1 and you had the system of equations. So, you had a 21 x 1 plus a 22 x 2 and all the way up to a 2 n x n equal to b 2 and we had this had this same thing all the way up to a m 1 x 1 plus a m 2 x 2 plus all the way up to a m n x n this was equal to b m.

And we wrote this whole system in a compact matrix on form a, which a is a matrix. X, x is a vector and this is equal to b. So, we had this simple notation where a was basically this matrix that we had before, yeah with, with coefficients a 11, a 12 and so on, oh now if you remember we had we, we when we were solving linear equations, we said that in order to be able to solve, in order to be able to solve these equations the equation should be both sufficient and consistent. So, should be sufficient and consistent. So, in general we said that the number of number of unknowns should be equal to the number of equations. So, in general we said that not only the number of unknowns should be equal to number of equations, but also that the equation should be internally consistent with each other. You cannot have equations that are inconsistent.

So, so, now, we can express this in a, in a very nice form using the rank of A matrix. So, this condition can be expressed in a compact way using rank of A matrix. So, we will define, define augmented matrix A tilde. So, we define an augmented matrix A tilde which what does it look like it basically looks like it looks like a matrix that has formed all these are just A, A is here and then and then you, you put the column of b right on the right side. So, you have all the elements of A and then you have 1 column 1 extra column containing elements of b. So, this is a m cross n plus 1 matrix. So, now, if you define this m cross n by 1 matrix A tilde which is called the augmented matrix. Now what is the condition for this system of equations to be both sufficient and consistent is that the rank of, of A should be equal to rank of A tilde. So, the rank of A should be equal to the rank of A tilde and this should be equal to number of unknowns. So, if this condition is satisfied then, then you know that the equations have unique solutions. So, then equations have unique solution, unique solution.

So, if the rank of A equal to rank of A tilde then you have guaranteed that the equations are consistent. Now further if the rank is equal to the number of unknowns then you know that the equations have a unique solution number of unknowns. So, in other words you can say rank is equal to minimum of m by n. So, it is whichever is less between m and n; so it should be equal to that.

So, then you can be sure that the equations have a unique solution. Now this is a very powerful result because, you can determine the rank of A and rank of A tilde fairly easily just by checking for linearly linear independence and you can use this to actually say whether your equations are consistent or not, whether they have one solution you can also extend it to other cases you can ask questions like what happens if rank of A is.

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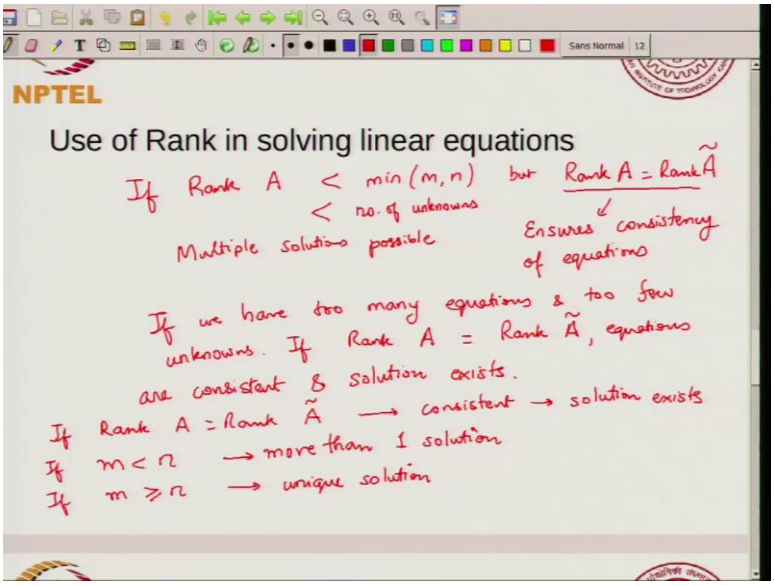

Let us say less than or equal to minimum of m cross n, but the rank of A equal to rank of A tilde. So, suppose you had a case where the rank of A is equal to the rank of A tilde. So, this guarantees consistency ensures consistency; that means, the equations will not be inconsistent and when rank is less than or equal to the minimum of m and n you can have more than one solution so; that means, you have some equations that are actually dependent on the other and what; that means, is that you can have multiple solutions.

So, rank of A less than number of unknowns. So, what I meant to say here is that it should be strictly. So, if it is less than the number of unknowns. So, if they, if your rank is less than the number of unknowns then, then you have multiple solutions possible and we will see examples of this over the over the course over, over the next few lectures, but basically this way of using ranks, you do not need to actually check each of the equations to see whether they are consistent or not you can directly use the idea of ranks to check for consistency. So, now, what happens is if we have too many equations any equations and too few unknowns. We have too many equations and too few unknowns, then you would say that some of the equations are redundant. If they are consistent then they have to be redundant. So, if rank of A equal to rank of A tilde then equations are consistent, consistent and solution exists. So, if you have too many equations and too few unknowns you can see, but if the equations are consistent then the solution will exist. So, even if you have too many equations. So, long as you have rank of A equal to rank of A tilde you are guaranteed the solution.

Now, if you want a unique solution then the number of unknowns should not be less than the number of equations. So, then you can ensure that the solutions are consistent. So, so, so, if you have too many unknowns and too few equations then multiple solutions. So, I will just, I will just write this explicitly. So, if rank A equal to rank A tilde then the consistent the equations are consistent and solution exists.

Now, if we have if m is less than n. So, if m less than n then multiple solutions or you know mo more than 1 solution then more than 1 solution exists. So, if m is greater than equal to n and then unique solution unique solution. So, this is the basic use of rank and solving linear equations.

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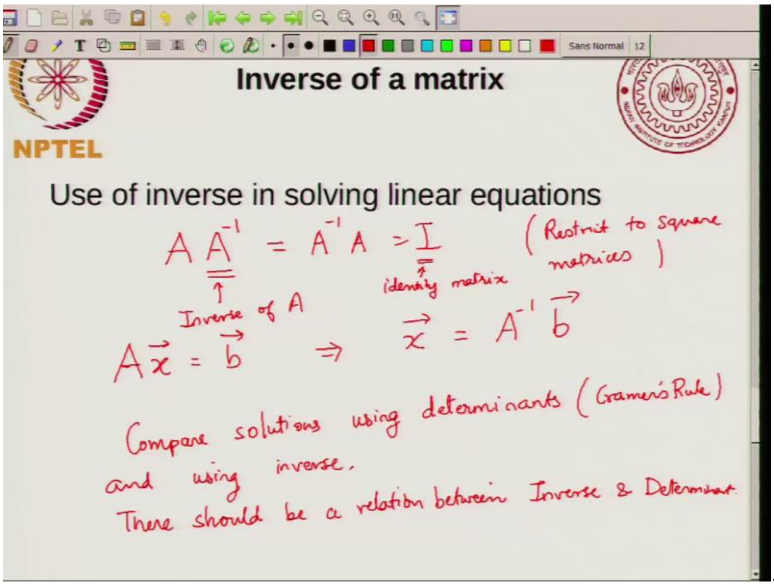

Now, next, next thing I want to talk about is the inverse of a matrix and how we use the inverse of a matrix in solving linear equations. So, we had already seen earlier that that if you have a matrix a then A inverse. So, this is called the inverse of A, this is called the inverse of A, this is equal to A inverse A equal to I so again, will restrict to square matrices. So, now, the then you have a square matrix you know that the matrix can have an inverse and if the inverse exists it is unique and the inverse satisfies this equation. So, now, suppose I had an equation A x equal to b. So, suppose I had my matrix my system of linear equations that we had earlier, then you can immediately see that this imply x is equal to A inverse b a inverse is a matrix. So, suppose I multiply both sides by A inverse then you can clearly see that A inverse A is nothing, but the identity matrix. So, this is what we call the identity matrix and what you have is that I can write x as A inverse b. So, basically if I know the inverse of this matrix a then I know the solution, then I can immediately solve for b. So, suppose I have a set of equations and I write it in matrix form and I calculate the inverse of that matrix I immediately know the solution of those set of equations.

Now, the; I mean you have you, you have already seen solving a system of equations using Cramer's rule. So, now, we can see; what is the connection? So, when you solved equations using Cramer's rule then you we saw determinants coming in. So, now, here this gives another solution using inverses. So, so, compare solutions using determinants, that is what we called as Cramer's rule and using inverse. So, what I mean is that, what I mean is that you have the same vector that can be expressed as two different ways one is using this inverse of a matrix and the other is using determinants. So, what that implies is that there should be a relation be relation between inverse and determinants. So, just based on this you would conclude that there should be a relation between the inverse of a matrix and the determinant of a matrix and you know various determinants involving matrices.

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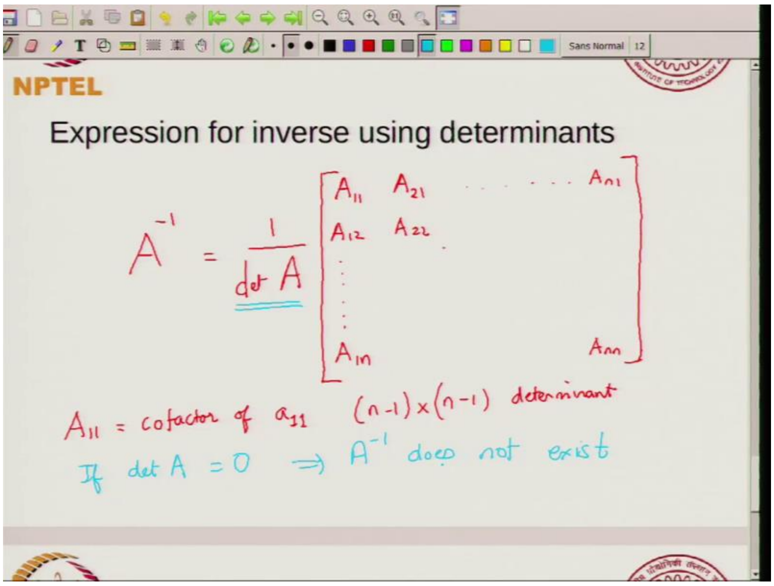

So, let us write an expression for inverse using determinants. I will just write the expression it is fairly easy to verify. So, suppose I have a matrix a I can write a inverse in the following form, I can write it as one divided by determinant of a and I will just I will just say determinant of a this is the determinant of the matrix and I will just and a inverses a matrix. So, it should have various terms here what are the various elements of this matrix. So, the first element I will call it as a 11, I am using the notation of capital A 11, this is a knot instead of 1 2 I will write a 21 here and I will write a 1 2 hear all the way up to a 1 n and here I go all the way up to a n 1, a 22 and all the way up to a n n.

So, what is a a 11 is called the cofactor of a 11. So, we saw earlier that the cofactor of any element is the value of the determinant you get when you remove that row and that column. It is the value of the determinant of the matrix obtained by removing that row and that column. So, each of these each of these is determinant. So, a 11 is an n minus 1 cross n minus 1 determinant. So, this is the n minus 1 it is A, it is the determinate. What I mean it is the determinant of a matrix of size n minus 1 times n minus 1.

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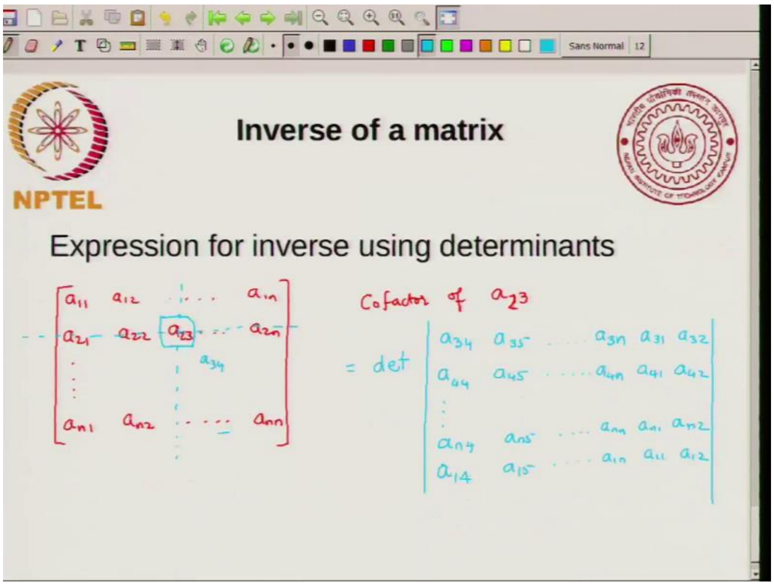

And once again let me remind you what the cofactor is. So, the cofactor if you, if you, if you had a matrix a 11, a 12, a 1 n, a 21, a 22, a 2 n, a n 1, a n 2 up to a n n. So, if you had a matrix like this, let us say if I want to calculate the cofactor of I will just take some element let me take a 13 or let me take a 23 a 23 that will make it that will make it more interesting. So, suppose I have a 23 here, now if I want to calculate the cofactor of a 23 then this is the second row and the third column. So, what I do is I want to calculate the cofactor of this this element and what, what I should do is I imagine that I imagine I consider a matrix where this row and this column are removed. So, if I remove this row in this column and then and then I and then what does my matrix look like. Now, now, I have to start from the a 23.

So, the matrix now will look the following way. So, my cofactor is the determinant of this matrix and that matrix is given by the following way. So, if I remove this row and this column then what I have is a matrix that is starting with this will be a 34. So, this this matrix starts with a 34, a 35 all the way up to a 3 n. So, then you go all the way up to a 3 n then you are still left with a a 31 and a 32.

So, it comes all the way up to a 21 a 32. So, that is how you write a cofactor then in the next row you will have straw will start with a 44, a 45 and you go all the way up to a 4 n, a 41, a 42. So, you go all the way up to a n 4, a n 5, a n n, a n 1, a n 2. So, that is that is what you will end up here with this last with this last row then you have to come back to the first row. So, then you still have 1 more row which is a 14 a 15 all the way up to a 1n and then a 11 a 12. So, this is an n minus 1 by n minus 1 matrix where you remove the second row and the third column and you take the determinant of this this is what is the cofactor. So, the cofactor of cofactor of any elements of a matrix is obtained it this way and the inverse of the matrix can be expressed in terms of cofactors.

Now, notice that there is a dominant that appears in the expression of the inverse now and it is a 1 over the determinant. So, what this implies is that if of a is equal to 0 then that implies matrix inverse does not exist a inverse does not exist again a very very powerful results if the determinant is 0 then the inverse does not exist and this is actually a very useful property of matrices that you can immediately tell whether you can invert a matrix or not by looking at the determinants.