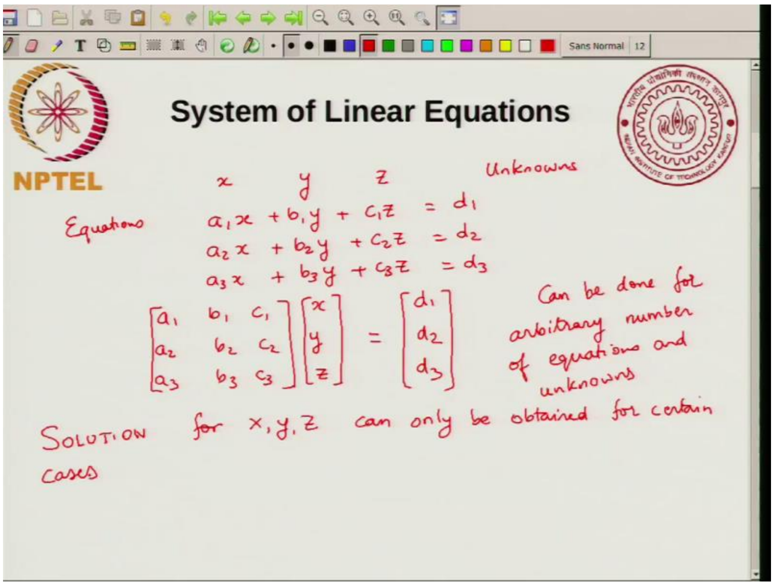
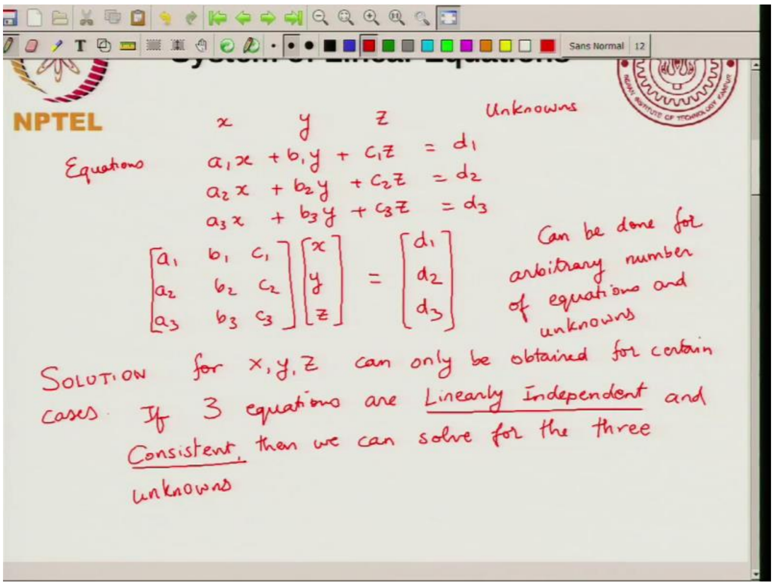
Mathematics for Chemistry
Prof. Madhav Ranganathan
Department of Chemistry
Indian Institute of Technology, Kanpur

Module - 03
Lecture - 04
Cramers Rule

(Refer Slide Time: 00:26)



(Refer Slide Time: 04:56)



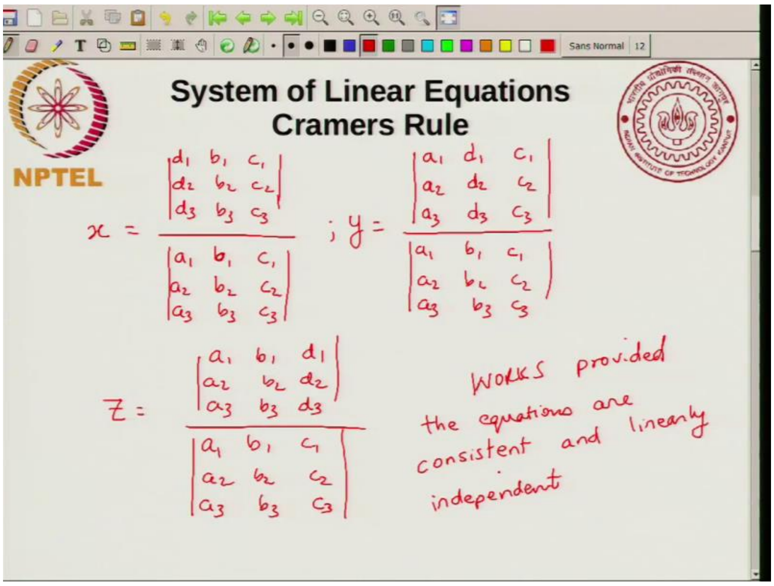
So, the next thing we want to discuss about matrices is how to use matrices to solve the system of linear equations. And in this, we will talk about Cramer’s rule. So, what is a system of linear equations, and where does matrix where do matrices play a role in system of linear equations. So, suppose you have a system of linear equations, let say you have unknowns are x, y and z; these are unknowns. And our equations; we have equations denoted by let say I will just call this a 1 x plus b 1 y plus c 1 z equal to d 1, a 2 x plus b 2 y plus c 2 z equal to d 2, a 3 x plus b 3 y plus c 3 z equal to d 3. Where a 1, b 1, c 1, d 1, a 2, b 2, c 2, d 2, a 3, b 3, c 3, d 3 are all scalars, they are all scalars that means, they can be real or they can be complex numbers.

You can write this in the following form this is exactly the same as writing a 1, b 1, c 1, a 2, b 2, c 2, a 3, b 3, c 3 of this matrix multiplied by another matrix that has just one column that that matrix is denoted by x, y, z. So, as I said the rules for matrix multiplication you multiply a 1 by x multiply add b 1 by y add c 1 with z which is exactly this, and this is equal to d 1. Similarly, if you do a 2 into x b 2 into y c 2 into z you will get d 2; if you do a 3 x plus b 3 y plus c 3 z, you will get d 3. And I can write the system of linear equations in matrix form and you can do this for any number of equations, you can do this when for any number of variables and any number of equations so for arbitrary number of equations and unknowns.

Now, I said you can do it for arbitrary number of equations and unknowns, but you know very well that there are conditions when this system of linear equations can be solved. So, it can be done for arbitrary number of equations and unknowns, but solution for x, y, z can only be obtained for certain cases. That means, if you have this arbitrary number of if you have this system of equations then it is essential that I mean you would intuitively imagine that the number of equations should be equal to the number of unknowns. So, if you have 3 unknowns and you have 3 equations then you can solve and you can get a unique value of the unknowns and that is what you expect.

There are more you know there can be more I would say complicated cases, this is in general you expect that, but we will see later that if these equations are linearly dependent then it turns out that you know 3 equations cannot solve for 3 unknowns. So, actually in order to see when this can be solved, you need to get the idea of something called a rank, which will talk about later, but the basic point right here is that if you have a system of equations then you can represented in matrix form. So, now, let say you have the case where, so if 3 equations are linearly independent and consistent then we can solve for the 3 unknowns. So, I will come to what is meant by linearly independent and consistent in a few minutes, but let us assume that you can solve for the 3 unknown. So, how would you imagine solving for the 3 unknown?

(Refer Slide Time: 05:46)



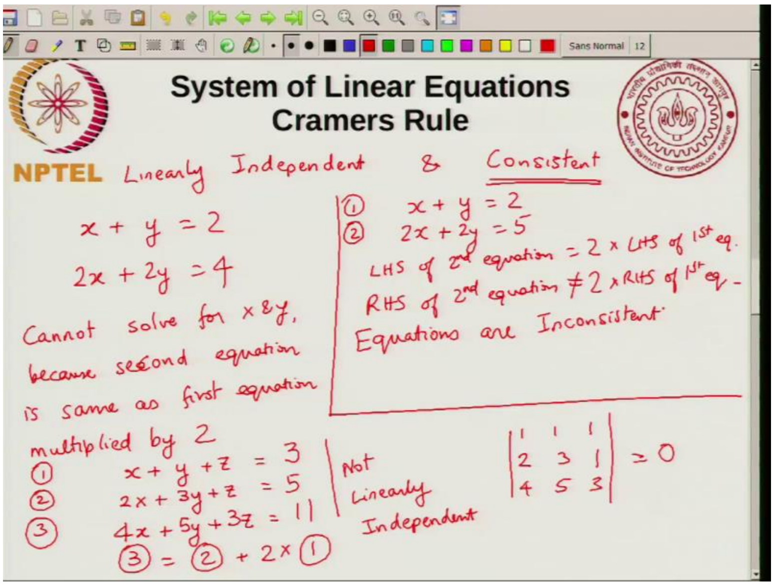
Now, there is one rule called the Cramer's rule which can be used to solve for the 3 unknowns. And I will just mention this right here. So, again this is only in the cases when these equations are linearly independent and consistent. So, if they are linearly independent and consistent then I can write x as a ratio of determinants and what is that ratio of determinants. So, I will write the denominator first. So, denominators a 1, b 1, c 1, a 2, b 2, c 2, a 3, b 3, c 3 which is nothing but the set of these coefficients, which is nothing but the determinant of this matrix. Now, obviously the value of x should depend on also on d 1, d 2, d 3. So, the numerator looks like d 1, b 1, c 1, d 2, b 2, c 2, d 3, b 3, c 3.

Similarly, I can write y as again you have the same denominator, same denominator a 1, b 1, c 1, a 2, b 2, c 2, a 3, b 3, c 3, but in the numerator what you have is a 1, d 1, c 1, a 2, d 2, c 2, a 3, d 3, c 3. So, notice what you did and similarly for z, I will just right what is z. So, z is equal to a 1, b 1, d 1, a 2, b 2, d 2, a 3, b 3, d 3, a 1, b 1, c 1, a 2, b 2, c 2, a 3, b 3, c 3. So, what we did is we have this determinant of the matrix so that is the same. So, the denominator for x, y, z are the same. In the numerator, in the first case you replace the first column with d 1 d 2 d 3; in the case of y, you replace the second column with d 1 d 2 d 3; in the case of z, you replace the third column with d 1, d 2, d 3. So, the

Cramer’s rule can be used to quickly solve a system of linear equations. So, works provided the equations are consistent and linearly independent. So, what that means, is that the number of equations should be equal to the number of unknowns and the equation should be internally consistent, so the; that is the condition for this.

Now I will just come back to this idea of what it means for the equations to be linearly independent and what it means for them to be consistent. So, again let me emphasize one more point that the Cramer’s rule that we have done, I have shown it for a 3 by 3 matrix. So, for 3 equations and 3 unknowns, you can do this for an n by n matrix, for n equations and n unknowns you can write exactly similar expressions and Cramer’s rule will be valid in that case also only thing in those cases your determinants will be much larger, but Cramer’s rule is still valid.

(Refer Slide Time: 10:04)



So, now let us look at some equations that are linearly dependent or inconsistent. So, what does, so we said linearly independent and consistent? So, we said that only if the equations are linearly independent and consistent can we apply Cramer’s rule. So, what would be an example of a case whether by the equations are not linearly independent, and what would be an example of a case where the equations are not consistent. So, let us take a simple example. So, suppose I have x plus y equal to 2 as my first equation, my second equation is 2 x plus 2 y equal to 4. So, I have 2 equations and 2 unknowns, 2 equations and 2 unknowns; however, can you solve for x and y, the answer is no you cannot solve for x and y. So, cannot solve for x and y, because second equation is same as first equation.

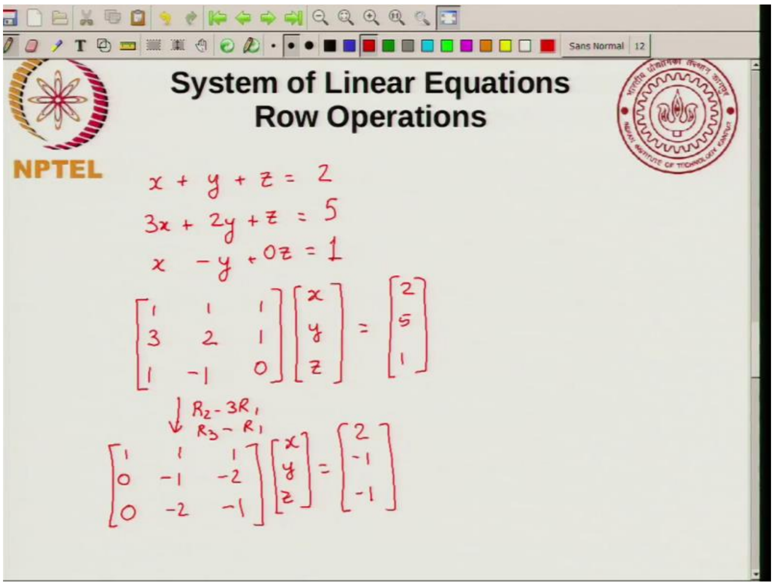
So, second equation is the same as first equation if you just divide by 2. So, as first equation multiplied by 2, this is a simple example I did for a 2 equations and 2 unknowns. If you have 3 equations and 3 unknowns then it is import you to look whether one equation whether one of the left hand in the left hand side one of the terms can be written as a linear combination of the other 2. For example, let us take x plus y plus z equal to 3, and let us take another equation that looks like 2 x plus 3 y plus z equal to make it 5. Now, the third equation I will make it as 4 x plus 5 y plus 3 z equal to 11.

Now, I have 3 equations and 3 unknowns, but notice that that if I call this 1, 2 and 3, we notice that that 3 equal to 2 plus 2 times 1. So, 4 x is 2 x plus 2 times x, 5 y is 3 y plus 2 times y, 3 z is z plus 2 times z, 11 is 5 plus 2 times 3. So, the third equation is nothing but the second equation plus 2 times the first equation. So, this implies that these equations are not linearly independent. So, you cannot solve using Cramer’s rule for this system. In fact, in fact what you will get is that the determinant will turn out to be 0, in fact, what you will get is that your determinant of this matrix, so if you take 1, 1, 1, 2, 3, 1, 4, 5, 3, this determinant equal to 0. You can verify this you can easily verify that this goes to 0. So, Cramer’s rule has not worked and you cannot solve this system of equation then it is obvious because you do not really have 3 independent equations you have only 2 independent equations and you have 3 unknowns. So, you cannot solve it.

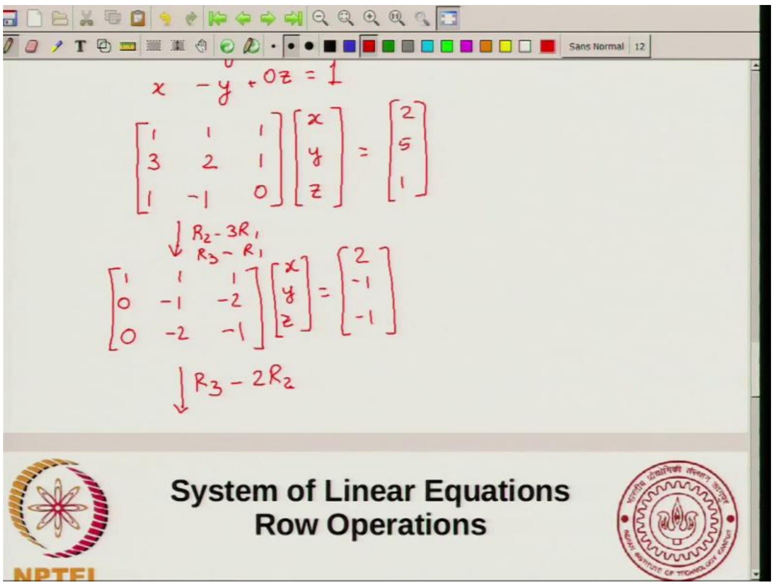
Now, what about the idea of consistency to see, and I will just do it in this part. So, now suppose you have an equation let say x plus y equal to 2 x plus 2 y equal to 5. You look at them, there are 2 equations, but you notice that the left hand side of the second equation is twice the left hand side of first equation, but the right hand side of the second equation is not twice the right hand side of the first equation. So, left hand side of second equation equal to twice into left hand side of first equation, but the right hand side of second equation is not equal to twice right hand side of first equation.

In this case the equations are said to be inconsistent that means, if I just take the first equation multiplied by 2, I will get 2 x plus 2 y equal to 4, but here I have 2 x plus 2 y equal to 5. So, there is no solution. So, these equations do not make sense or they are inconsistent. So, clearly you cannot apply Cramer’s rule you cannot solve this system of equations. And I show this for a 2 by 2 system, but you can you can extend for other systems also. So, it is important that before you try to solve the equations, you verify before we try to solve a system of equations, you should verify that they are both linearly independent and consistent only then you can use this Cramer's rule.

(Refer Slide Time: 16:01)



(Refer Slide Time: 19:29)

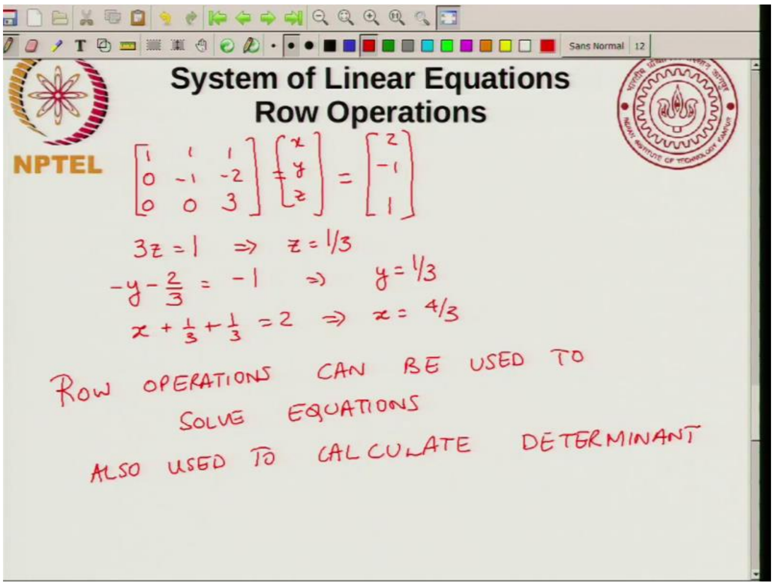


Now, you can solve a system of equations using row operations, you do not need to use Cramer's rule, you can actually use row operations. So, how would you do this? So, I will show this with an example and then you can do it for specific cases. So, let us just take one of the examples that we had suppose I take equal to 2, and I take 3 x plus 2 y plus z equal to let us say 5, and let us say I take x minus y plus 0 times z x minus y, I want to does one. So, suppose I take this system of equations.

Now, now you can solve for x, y, z. So, what you can do is if you if you solve for x, y, z what you will get is let us just calculate the determinant. So, I will write this in the following form 1, 1, 1, 3, 2, 1, 1 minus 1, 0, this multiplied by x, y, z is equal to 2, 5, 1. I will do a row operation that converge this 3 to 0. So, what I will do is I will do and I will do an row operation that that does R 2 minus 3 R 1. So, what that will do is wherever I have I, do 3 minus 3 times 1 that is 0 I will 2 minus 3 times 1, I will get minus 1 and I will do 1 minus 3 times 1 I will get minus 2. So, in addition, what I will do is R 3 minus r 1. So, I will do these 2 operations and it will be instantly clear why I am doing this.

So, what I will get is 1, 1, 1 R 1 is not change R 2 you subtract 3 times R 1. So, you will get 0 2 minus 3 is minus 1 one minus 3 is minus 2 and R 3 minus R 1. So, I will get 0 minus 1 minus 1 minus 2 0 minus 1 is minus 1. So, I did this on the left hand side, I have to do the same thing on the right hand side. So, what I will do is I will leave the 2 as it is, now 5 minus 3 times 2 is minus 1, 1 minus 2 is minus 1. So, I get this equation. So, what I did is I just did some subtraction of the equations, and I got this. Now, notice that I have 2 0s here. What I will do is I will try to get a 0 here by subtracting this row.

(Refer Slide Time: 20:36)

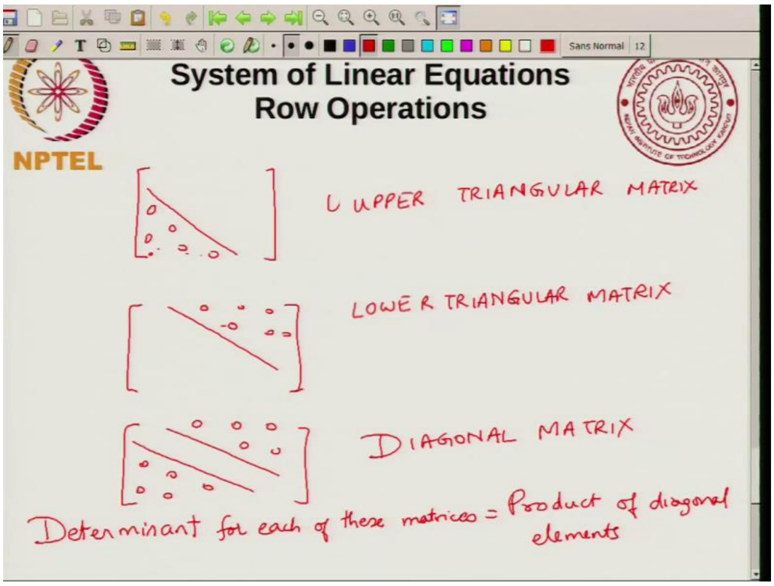


So, next what you do is, I will do the following. So, I will do R 3 minus 2 R 2. So, if I do R 3 minus 2 R 2, you can see what happens. So, I will just show it right here. So, what I will get is 1, 1, 1, R 1 is unchanged, now R 2 is unchanged from here, so 0 minus 1 minus 2. Now, R 3 minus 2 R 2, so this 0 minus 2 times 0 is just still 0, so that is important that I converted this to 0 first. So, this is not affected now minus 2 minus 2 times minus 1. So, minus 2 plus 2 that is 0, minus 1 plus 4 that is 3 this times x, y, z and this is equal to, now what I will have is 2 minus 1 now minus 1 minus 2 times minus 1 so plus 2 that is 1. So, you converted the equations to something like this. So, the same 3 equation you sort of did some linear combinations and you got this set of equations.

Now, you can immediately see just look at the third equation. So, the third equation says 3 z equal to 1 implies z equal to 1 by 3, immediately you do not have to calculate any determinants. The second equation says minus y minus 2 times z, so 2 by 3 is equal to minus 1. So, this implies y equal to 1 by 3. And now you have z equal to 1 by 3, y equal to 1 by 3 and so you immediately you look at the first equation, so x plus 1 by 3 plus 1 by 3 equal to 2 implies x equal to 4 by 3. So, using these row operations, we solve this equation this system of equations and without using Cramer's rule or determinants we managed to solve these equations.

So, row operations can be used to solve equations in a way we did without using determinants. If you have really large matrices then calculating determinants each times becomes with very tedious, whereas if you do row operations you can quickly solve the equations. In fact, even to solve, even to calculate determinants you know once you do row operations on a matrix row operations do not change the determinant. So, now once you get a matrix into this form where you have 0s then you can see that it is very easy to calculate the determinant. So, row operations can also used to calculate determinants. So, what you do is you convert this matrix into a form where you have 0s on one side.

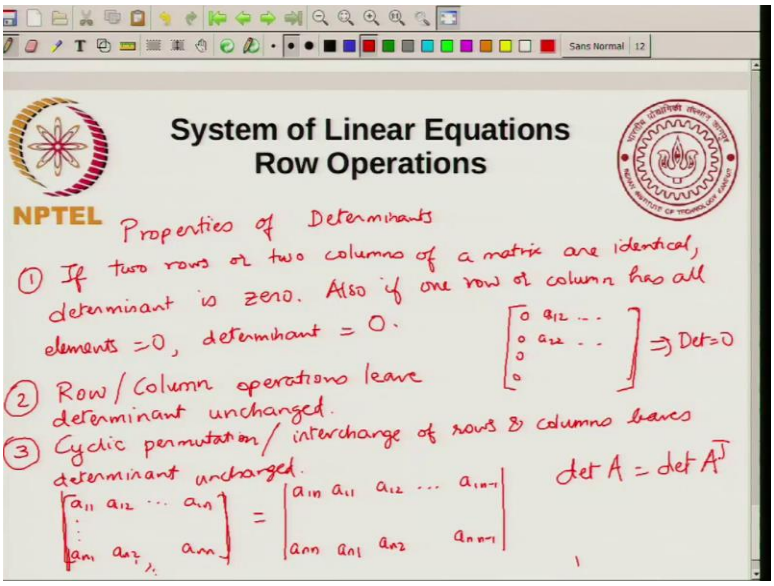
(Refer Slide Time: 23:07)



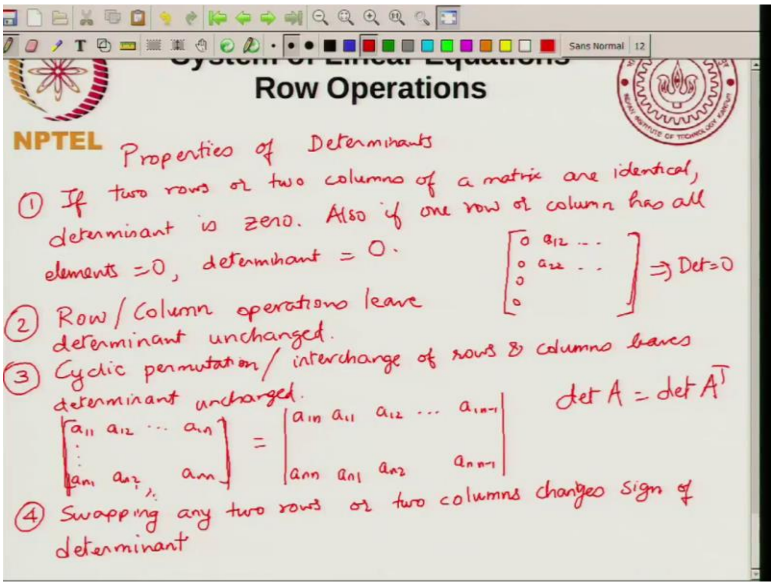
So, I will just mention a few more things.

So, this is related to the row operations. So, suppose you have a matrix that looks like this. So, it has 0s on one side. So, it can be an arbitrary large matrix, but it has 0s on one side of the diagonal. So, this is called the diagonal of the matrix. So, on one side of the diagonal, it has 0s then it is called an upper triangular matrix and similarly, if this matrix has 0s on this side of this diagonal. So, this is called a lower triangular matrix. And if the matrix has 0s on both sides of the diagonal, so it has 0s here, 0s here, then it is called the diagonal matrix. And you can see that whether you have a lower triangular upper triangular or diagonal matrix, the determinant for each of these matrices equal to product of diagonal elements. So, whether you have an upper triangular, lower triangular or a diagonal matrix, so whether you have an upper triangular, lower triangular or diagonal matrix, the determinant of each of these matrix is nothing but the product of the diagonal elements.

(Refer Slide Time: 25:16)



(Refer Slide Time: 29:43)



Now a few more a few more things about determinants and then will stop this discussion. So, suppose you have a matrix. So, there are properties of determinants, I will just list them and you can show them easily. So, first one is that if 2 rows or 2 columns of a matrix are identical, determinant is 0, also if one row or column has all elements equal to 0, determinant equal to 0. So, basically if you have a matrix, where on an entire row you have a 0. So, for example, if you have a matrix where you have 0, 0, 0, 0 and then you have some nonzero element say a 1 2 and so on, but you have one entire row being 0 then determinant equal to 0. So, this has determinant equal to 0.

Similarly, if you have a matrix where 2 rows or 2 columns are the same then the determinant is equal to 0. So, this is one property. The second property is row or column operations leave determinant unchanged. So, suppose I do these row or column operations that we talked about then the determinant does not change. The third one cyclic permutation or interchange of rows and columns leaves determinant unchanged so that means, what is the cyclic permutation I just take any one row or one column and then I sort of shift all the rows and columns by 1.

So, for example, what is the cyclic permutation? So, if you have I will just do one cyclic permutation of row. So, if I have a 1 1, a 1 2 up to a 1 n and then all the way up to a n 1 a n 2 up to a n n. Now, if I do a cyclic permutation. So, if I take the determinant of this matrix and i imagine doing a cyclic permutation. So, this determinant is the same as now, I take I shift a n 1 here a 1 1 and a n n and then I have a 1 1, a 1 2 up to a 1 n minus 1 and all the way up to a n 1, a n 2 up to a n n minus 1. So, what I did was I just took this last column and I moved it all the way to all the way to the first and I shifted all the columns to the right then the determinant is unchanged. Now, similarly if you interchange all the rows and columns if you swap the rows and columns; that means, in other words determinant of A is equal to determinant of A transpose. So, if you swap the rows and column then the determinant is unchanged.

There is one more rule I will just mention this briefly that if you change if you swap any 2 rows or columns. So, if you just swap 2 rows or columns then the determinant changes sign. So, if you swap just 2 rows or columns then the determinant changes sign. So, this is the last property, I will just write this here, so swapping any 2 rows or 2 columns changes sign of determinant. So, these are some of the properties of matrices and determinants. So, you can use these very do to solve equations very quickly. So, you whenever convenient your swap rows or columns or you add rows and columns or you take linear combinations, so that your calculations become easy. So, in the next class, I will try to do some practice problems, I will do some practice problems on these topics.

Thank you.