Mathematics for Chemistry
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Module - 02
Lecture - 05
Practice Problems

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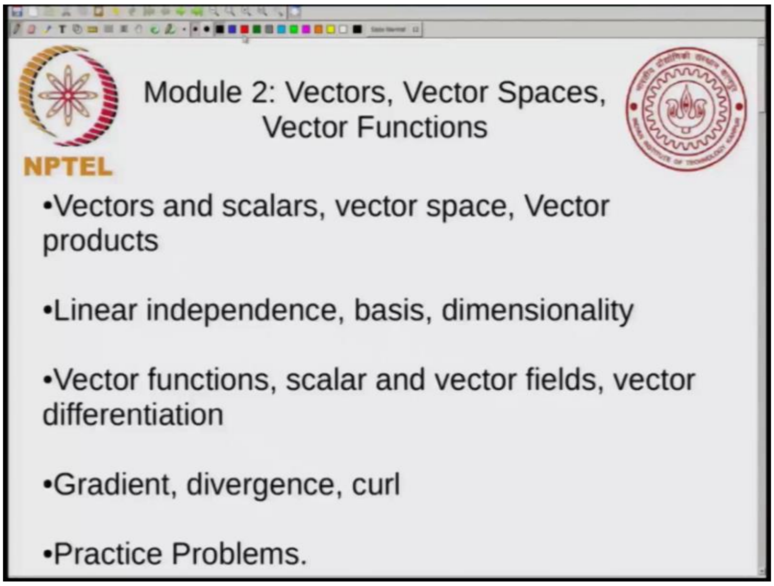

So today will be the final lecture of module 2 where I will be working on some practice problems. So, in this module, you have studied the basics of vectors; vector spaces and vector functions, and we studied some properties of vectors like linear independence and the dimensionality. And I will be focusing the practice problems today on these base concepts of vectors, and we will look at more problems related to scalar and vector field in the problem sets of the next module.

So, today I will show you some elementary problems that deal with vectors, vector spaces and vector products, the end the concepts of linear independence.

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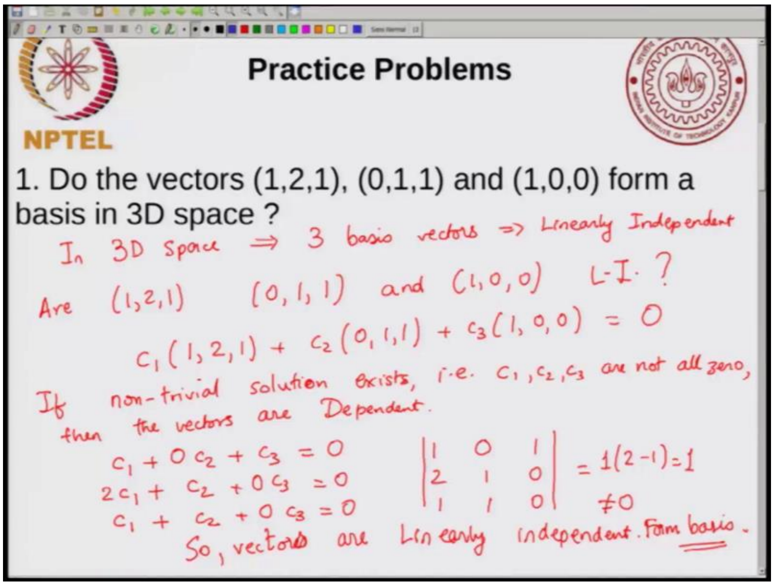

So, let us take the first problem. So, the first problem; ask, you do the following vectors form a basis in 3-D space. So, what do you need to check for a basis, so the condition; so, in 3-D space you can have only 3 basis vectors; now they can there are many choices for basis vectors, but the conditions for them to be basis vectors is that they should be linearly independent. So, if you have 3 linearly independent vectors you can choose them as a basis in 3-D space. So, what you have to check is whether these vectors are linearly independent. So, are 1, 2, 1 0, 1, 1 and 1, 0, 0 are these vectors linearly independent. So, that is the question that we have to answer.

Now, in general the way to check linear independence is to take a linear combination of these, and set it equal to 0 and see if you get a non trivial solution. So, what you will say is that you will say the you will say c 1 times 1, 2, 1 plus c 2 times 0, 1, 1 plus c 3 times 1, 0, 0 equal to 0. Now if non-trivial solutions, solution exists that is c 1, c 2, c 3 are not all 0 then the vectors are dependent or linearly dependent. So, what you have to do is you have to set this condition and you have to check whether there exists a non trivial solution. Now this condition is actually the way this is actually a set of 3 equations because this is the vector equation.

So, if I write out the 3 equations. So, the first equation is c 1 plus c 2 plus c 3; c 1 plus 0 c 2. So, if you just look at the x component you have c 1 here, you have 0 c 2 here, and c 3 here. So, this would be equal to 0 if you look at the y component then you have 2 C 1, and you have plus c 2, plus 0 c 3 equal to 0. If you look at the z component then you have c 1, plus c 2, plus 0 c 3 equal to 0. So, you have 3 equations and 3 unknowns and these are what are called homogeneous linear equations, so the condition for there to be a non trivial solution of this that is this equation. So, if I set c 1 equal to c 2 equal to c 3 equal to 0, I will automatically satisfy these equations, but is there a possibility of c 1, c 2, c 3 not all 0 that is what we have to ask and you might be knowing this, but you can easily show this that you evaluate this determinant, you look at the coefficients you put them in the form of a determinant and. So, you put 1, 0, 1, 2, 1, 0, 1, 1, 0 and you calculate this determinant and it is not hard to show this determinant is just. So, it will just be 1 times 2 minus 1. So, equal to 1 times 2 minus 1 equal to 1 not equal to 0, so since if this determinant is not equal to 0 that means, the only solution that exists is for c 1, c 2, c 3 to all be 0; so vectors are. So, that would mean the vectors are linearly independent and they form a basis.

So, just to recap what we did was we wanted to check whether these 3 vectors are linearly independent. So, what we did is you write the condition for linearly linear independence, and what you will find is that you will get the condition that this determinant if it is 0 then the vectors are linearly dependent, if it is not equal to 0 then the vectors are linearly independent. That means, you cannot write one as a linear combination of the others, and if you cannot write one as a linear combination of the others then these vectors will form a basis. And it is fairly straightforward to actually just a look at the 3 vectors and you can immediately see that they are linearly independent, because if you look at the second vector it has no x component whereas, the third vector has an x component and it has no y and z component, and the first vector has all 3 components.

So, clearly if you I mean, so it is clear that that you know you need what you are checking is whether this vector can be written as a combination of these 2, and just by inspection you can see that is not possible. So, this is the problem that has dealt with linear independence.

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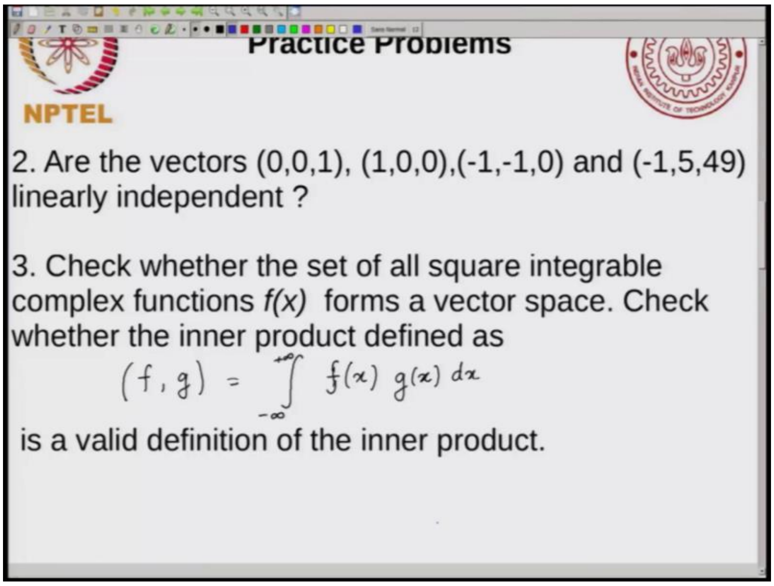

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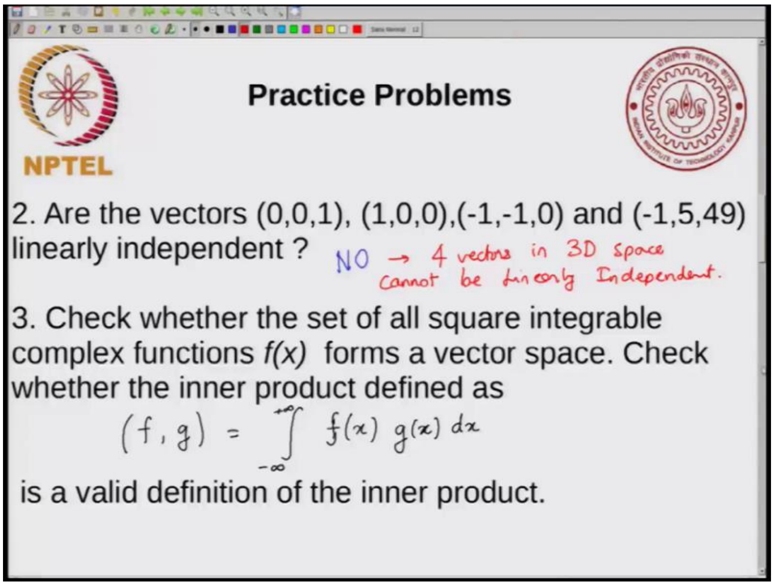

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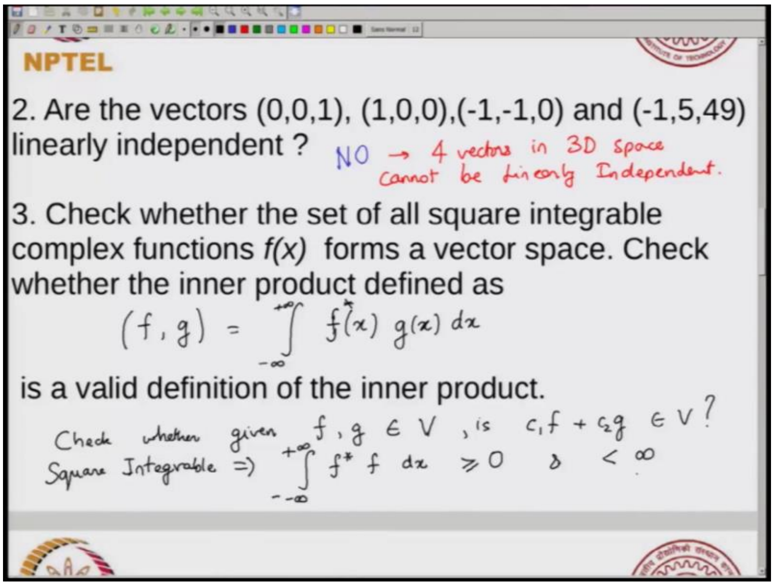

Now, I will give you another problem and this is so the second problem. So, you are asked whether the vectors 0, 0, 1, 1, 0, 0 minus 1 minus 1 0, and minus 1, 5,49 are linearly independent.

Now, the answer is no, and the reason is that 4 vectors in 3-D space cannot be linearly independent. So, if I give you a 3 dimensional space the maximum number of linearly independent vectors is 3 because the basis of that space is 3. So, you can never have 4 linearly independent vectors in a 3 dimensional space. So, without any inspection without checking anything you can immediately say that these vectors cannot be linearly independent.

Now, the next question has to do with vector spaces and inner product spaces. So, there are 2 parts. So, first is you have to check whether the set of all square integrable complex functions f of x forms a vector space, I will just put a star in this definition. So, what you have to see is whether the set of square integrable complex functions forms a vector space, now actually this is not a real vector space this is what is called a complex vector space, but we will just check whether these axioms are satisfied. So, if you want to check whether the set of certain functions forms a vector space, then what you have to do is to check whether given f and g belonging to the vector space V is c 1 f plus c 2 g contained in v. So, if you take any linear combination of this is also contained in the vector space. So, that is essentially the most important condition to check.

Now, a square integrable function in a complex space satisfies. So, it is a square integrable implies integral f star f d x, and we will just take a one dimensional space so from minus infinity to plus infinity, this should be greater than equal to 0 and less than infinity. Why this should this should be greater than 0 because f star f is positive and if it is less than infinity then you say that the function is square integrable.

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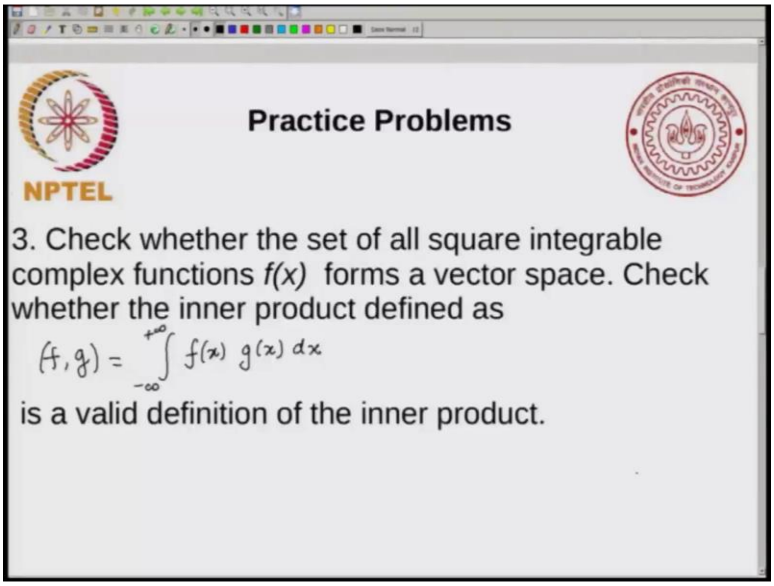

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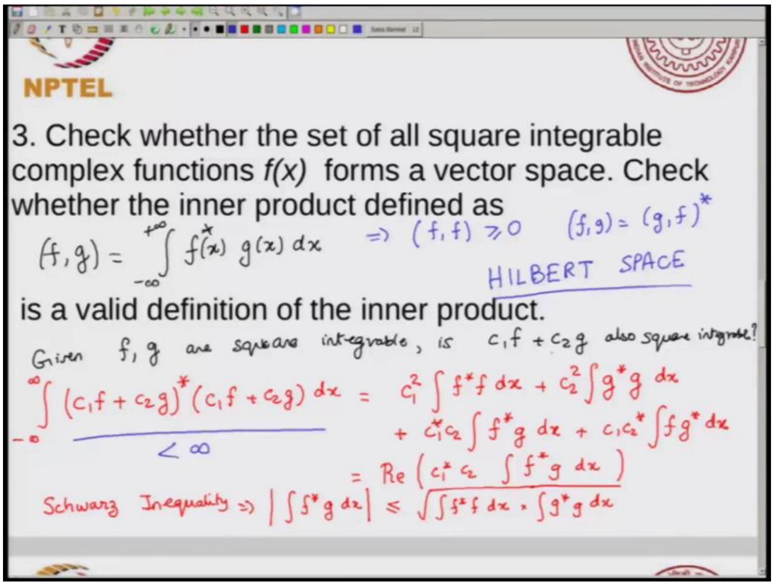

So, now, let us a check whether that forms a vector space. So, what you have to do if you have 2 functions that are square integrable, then is a linear combination of them also square integrable. So, what you have to check is whether given f g square are square integrable is c 1 f plus c 2 g also square integrable. So, this is also square integrable and this is what you have to check also. So, how do you check this? So, you calculate c 1 f plus c 2 g take a complex conjugate then you take c 1 f plus c 2 g and you do this integral d x from minus infinity to infinity, and you have to check whether this is less than infinity.

So, what you will get is that this will be c 1 square integral f 1 star f, d x plus c 2 square integral f 2 star or g star (Refer Time: 12:35) g star (Refer Time: 12:39) f f star f, this should be g star g g star g d x now you remember c 1 square is c 1 star times c 1. So, these are the 2 things I am not bothering putting the limits here and then you will have cross terms that look like c 1 c 2 I will write it as c 1 star c 2 integral f star g d x plus c 1 c 2 star integral f g star d x.

Now, the first 2 terms are clearly less than infinity since the square integrable f star f will be less than infinity, g star g integral of g star g and integral of f star f will both be less than infinity. So, the first 2 terms are clearly less than infinity what about the third term? So, the third term satisfied. So, what you notice is that the third and fourth terms are just complex conjugates of each other, and if we take a number and add its complex conjugate then what you will get is that you will just get the real part of that number. So, I can write this as the real part of c 1 star c 2 integral f star g d x, and now actually we will see this in a minute there is a condition that there is a there is a inequality called the Schwarz inequality, which basically says that integral f star g d x is less than or equal to. So, the absolute value of this is less than or equal to the square root of integral f star f, d x times integral g star g, d x.

We will show this in the in the next problem, but the basic idea is that this integral this where you have f star and g is basically less than the product of f star f, and g star g and under the square root sign. Now each of f star f and g star g is less than infinity. So, therefore, f star g into integral f star g d x has to be less than infinity. So, what you get is that this whole thing is less than infinity, because each of these terms are individually less than infinity. So, basically if f and g are square integrable this is also square integrable. So, with this you can basically show that this is a valid this forms a vector space.

Now, the next problem is to check whether the inner product and define this way is a valid definition of the inner product. So, whether the inner product define this way is a valid definition of the inner product and again you can show this easily, you can show that if the 2 conditions for validity of inner product is that f s f times f and inner product of f with itself. So, this implies inner product of f with itself should be greater than or equal to 0, and again you can easily see that the other condition is that the inner product of f with g should be equal to inner product of g with f.

Now, this is true for a real inner product space, for a complex inner product space actually this is modified with a star. So, these 2 conditions you can easily see again from this definition of inner product. So, this is a valid inner product, now I just want to mention one thing why I chose this complex vector space, the reason I chose this complex vector space is that this thing is something called a Hilbert space. So, this space is called a Hilbert space and this is the vector space in which the entire mathematics of quantum mechanics is performed. So, the your entire mathematical aspect of quantum mechanics works on this Hilbert space which is not quite a real inner product space, but it is a complex inner product space.

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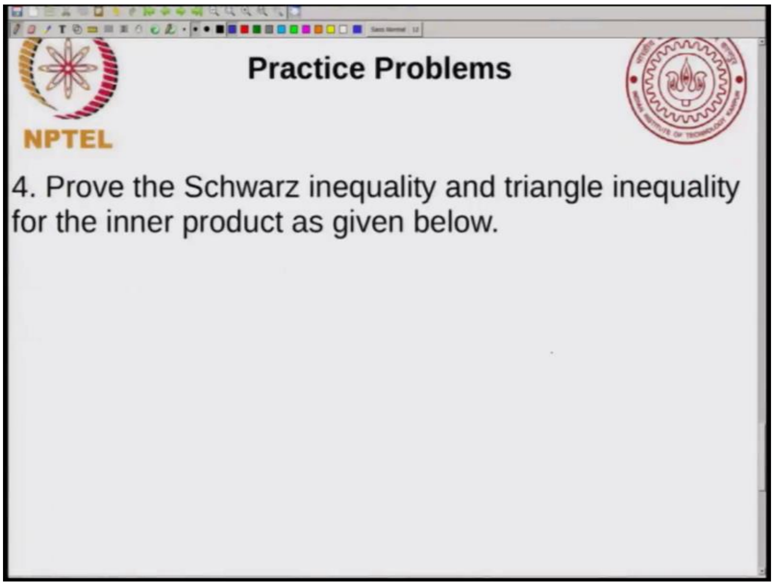

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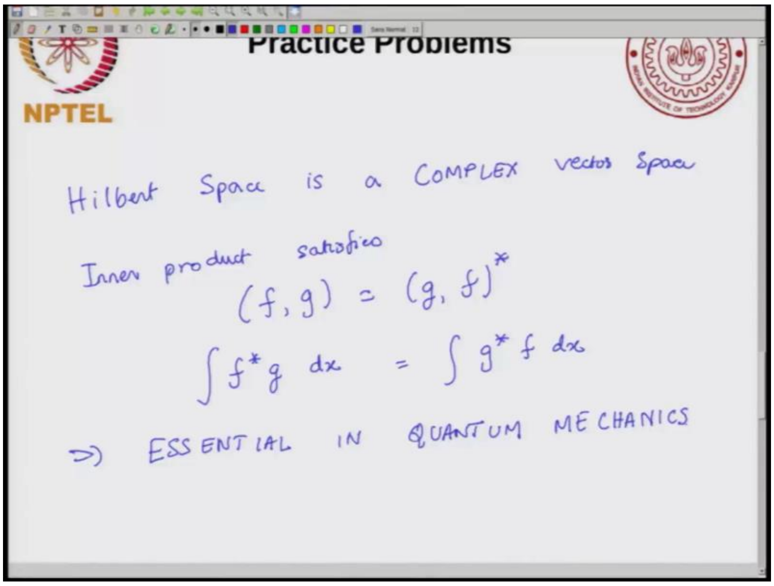

Now, the next question; so, just to emphasize I will just emphasize this here that Hilbert space is a complex vector space and inner product satisfies f g equal to g f star, and f g is given by integral f star g and d x, and this is given by integral g star f d x. So, the Hilbert space is actually this complex vector space where the inner product is defined in this way. So, and this is essential in quantum mechanics.

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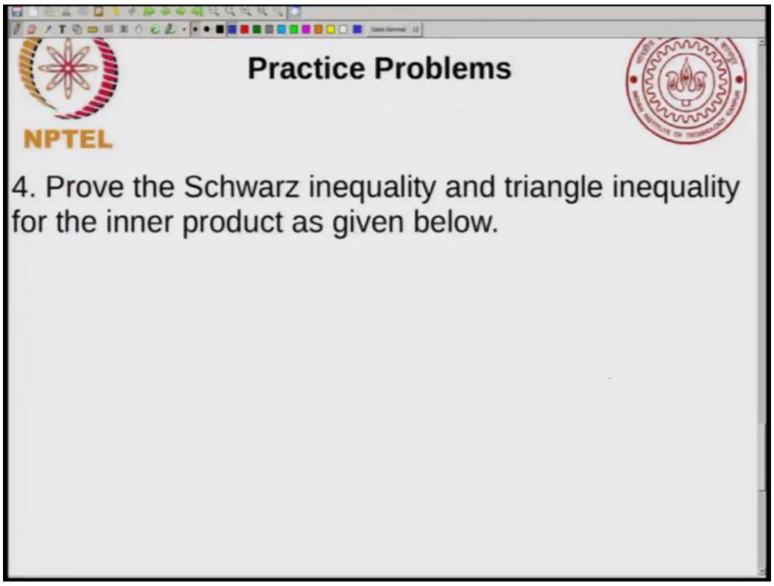

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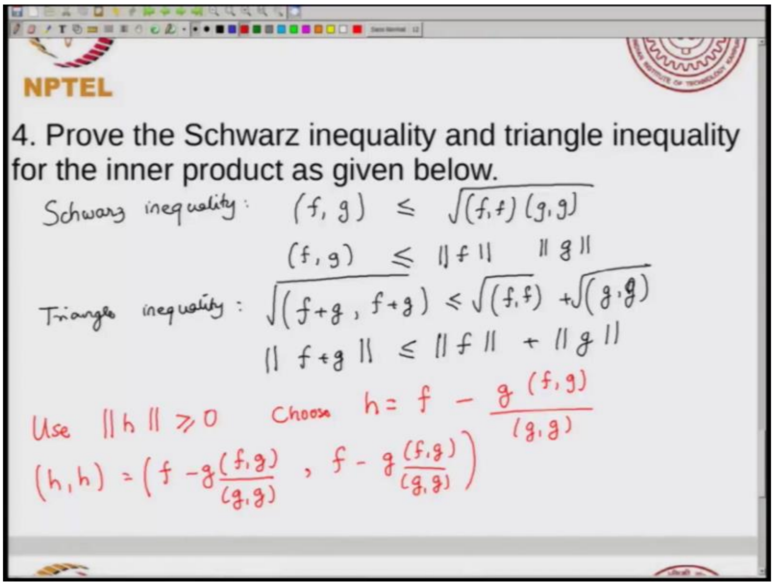

So, next practice problem is to prove the Schwarz and triangle inequality for the inner product as given below. So, what is the Schwarz inequality and what is the triangle inequality, and we will prove this in general. So, I will just tell you what the Schwarz inequality is. So, the Schwarz inequality; so for any inner product for any inner product space and for any inner product the Schwarz inequality basically says that if I take an inner product of f g, that should be less than equal to square root of f f and g g.

So, across in a product is always less than the product of the individual inner products the norms product of the norms. So, that is the. So, in other words I can write it as is less than or equal to norm f into norm g. So, this is the Schwarz inequality; the triangle inequality this is like saying that the sum of lengths of 2 sides of a triangle should be greater than or equal to the length of the third side. So, in terms of vector spaces, so what it says is that f plus g in a product with f plus g, this should be strictly less than or equal to f inner product with f plus g inner product with g and all this should be under the square root (Refer Time: 21:18).

In other words norm of f plus g should be less than equal to norm of f plus norm of g. So, this is the triangle inequality and what we are going to do is we are going to go ahead and show both these inequalities. Now there are many ways to prove the Schwarz inequality, the one common method that is used is that is to actually choose is to use the idea use norm of any function norm of any function should be greater than equal to zero. So, the. So, in the definition of the inner product one of the conditions is that norm of any function should be greater than 0, then what I will do is I will choose h is equal to f minus g times f comma g divided by g comma g.

So, I just took I just took a particular form of form of h and now and now if you calculate h comma h you will get it as if you write it out. So, you will get this inner product with itself, and what you will get is f minus g, f g divided by g g, comma f minus g, f g by g g.

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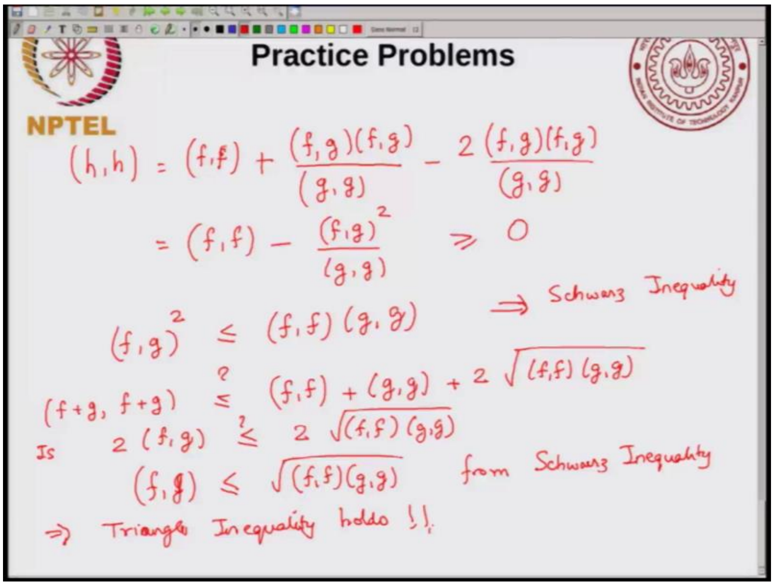

You can unfold these brackets and you can multiply them term by term what you will get is h comma h is equal to f comma f minus. So, you will get one term that takes inner product of this with it with these 2 terms. So, if I take the inner product of these 2 terms I will have g g and then f g f g twice.

So, now, one of the g gs will cancel. So, I will have f times g into f times g divided by g times g and then I will have 2 cross terms. So, at this multiplied by this. So, f times g and then and then I will have the same thing here. So, I will have (Refer Time: 24:03) this is the plus sign and I have minus 2 f times g f times g divided by g times g. Notice that this particular choice of h ensured that these 2 terms the second and third terms are basically the same. So, what you get is f comma f? So, this is these 2 are the same. So, what I get is minus f comma g whole square divided by g comma g and this because it is a norm of h. So, this should be greater than or equal to 0, and that immediately implies f comma g whole square should be less than equal to f comma f, times g comma g and this is Schwarz inequality.

So, we have proved the Schwarz inequality, next we will prove the triangle inequality. So, to prove the triangle inequality you take our let us look at our expression for the triangle inequality that we had here. So, this is the expression for the triangle inequality square root of f plus g plus g is less than this. Now if I square both sides then what I get is the following. So, f plus g f plus g; so, this on the left hand side, and on the right hand side we have to check is this less than or equal to f comma f. So, I have the norm of I have. So, if I square the right hand side I will get this term square plus this term square plus the cross term.

So, f star f. So, f comma f plus g comma g plus twice square root of f f g g and. So, the question is this less than this now again you can open this. So, if you expand the left hand side you will get f comma f, g comma g, and twice g comma f. So, what you have to prove is that is twice f comma g, less than equal to twice square root of f f, g g; if this is true then you satisfy the triangle inequality.

Now, clearly f comma g is less than equal to square root of f comma f times g comma g from Schwarz inequality (Refer Time: 27:24) is the stop here. So, we just showed this here. So, if you just take square root on both sides you will get exactly this expression. So, this implies that that this equality holds and it implies that this inequality also holds.

So, therefore, this implies triangle inequality holds. So, what we are showed is that the Schwarz inequality and the triangle inequality they hold for any arbitrary inner product space, and we have restricted to real inner product, but you can do the corresponding for the Hilbert space also. So, I will stop here.