Introductory Nonlinear Dynamics
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Lecture 09
Lyapunov exponents; Invariant measures.

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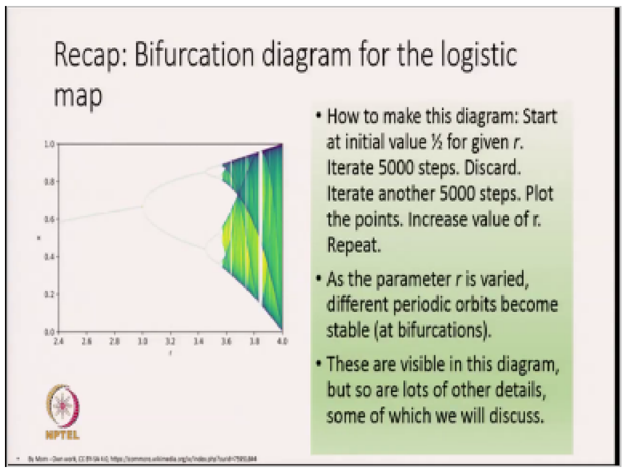
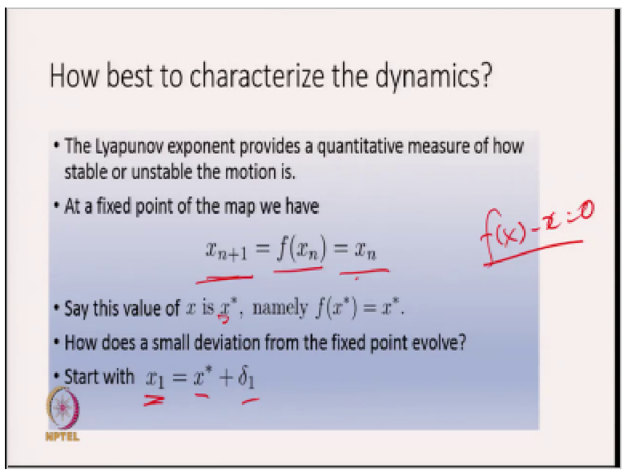


So, in the last two lectures we saw some of the phenomenology of nonlinear dynamics in the logistic map, how the or different orbits are born. We saw the bifurcation diagram, and we noticed that as you change the parameter r you have period 1, period 2, etcetera, etcetera. The period doubling bifurcation cascade ends at r infinity and this doubling bifurcations are characterized by this functional renormalization group and explained by these two numbers delta which tells you how the windows of stability keep narrowing and the number alpha which tell you how the different parts of the orbit come together. How does a make a diagram like this? We have seen Singers theorem which says that, if there is a stable periodic orbit the point half is going to be attracted to this particular periodic orbit. It is very important when doing these kinds of studies to look at the asymptotic behavior and so it is necessary to discard transients. And so, the practical way for those of you who would like to be able to generate such a picture is to do the following. You start with the maximum for technical reasons instead of starting at exactly the point half you can start at 0.50001 or something which is close to half, but not exactly half. You iterate the map for let us say 5000 steps, discard it as transient behavior, iterate it for a another 5000 steps and you plot these points. If the map has come to the stable fixed point over here all the 5000 points at let us say 2.6 will be a single point, but as you go for larger values of r these 5000 points will sweep out over the interval. So, what one does over here is you keep changing the value of r slowly, you start again with this initial value, iterate, discard, iterate, plot, increase the value of r continue etcetera. Now, as you keep varying this parameter r, the different stable periodic orbits become visible and you can see the bifurcations. But in addition to these stable periodic orbits we can see a lot of other behaviour, you can see these periodic windows over here, you can see some very evident kinds of lines and what we will try to discover in this particular series, in this lecture today is how does one understand a picture like this. (Refer Slide Time: 03:35)



Now, let us start by wanting to characterize the dynamics. The Lyapunov exponent is the quantity which provides a very quantitative measure of how stable or unstable the motion is. At a fixed point of the map we have the following condition, that

$$x_{n+1} = f(x_n) = x_n$$

. So, x_n gives you x_{n+1} this is a fixed point. We determine this by looking for this equation $f(x) - x = 0$, you find the 0s of this equation and that gives you a value x^*. Now, how does a small deviation from the fixed point evolve?

In order to find that we start the point x_1, which is $x_1 = x^* + \delta_1$ and iterate it.

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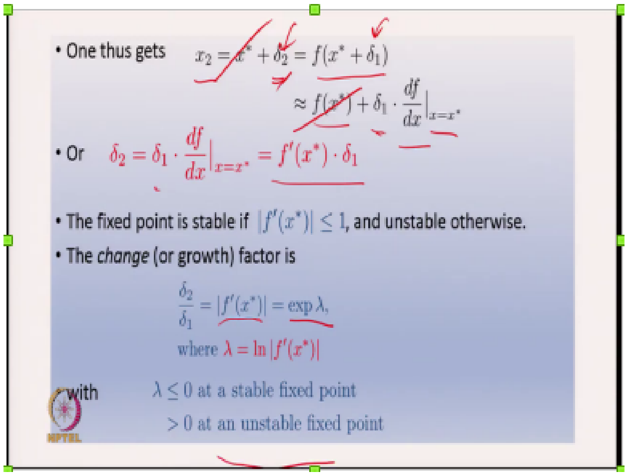
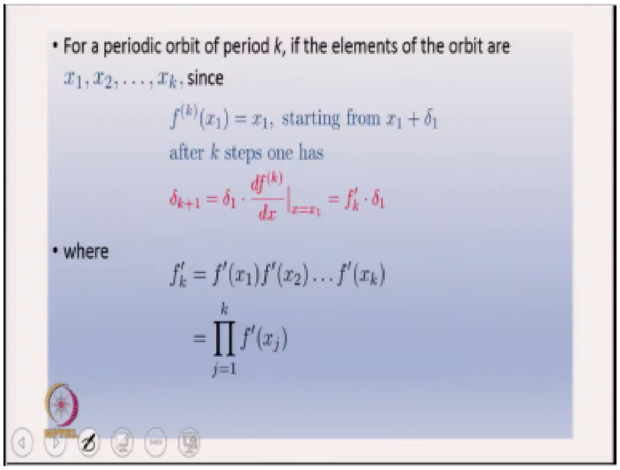


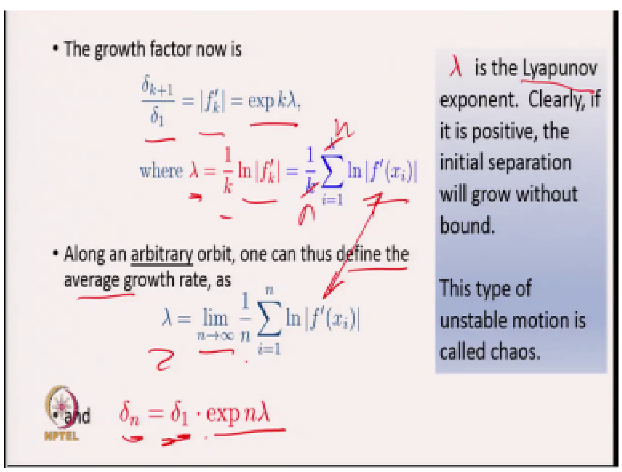
If you iterate it you get x2, which is $x_2 = x^* + \delta_2$. This is just $f(x^*) + \delta_1$ and doing a Taylor expansion and retaining just the first term, says that this is approximately $f(x^*) + \delta_1 . \frac{df}{dx}$ at the point x^*. Since, $f(x^*) = x^*$, we find that

$$\delta_2 = \delta_1 . \frac{df}{dx}$$

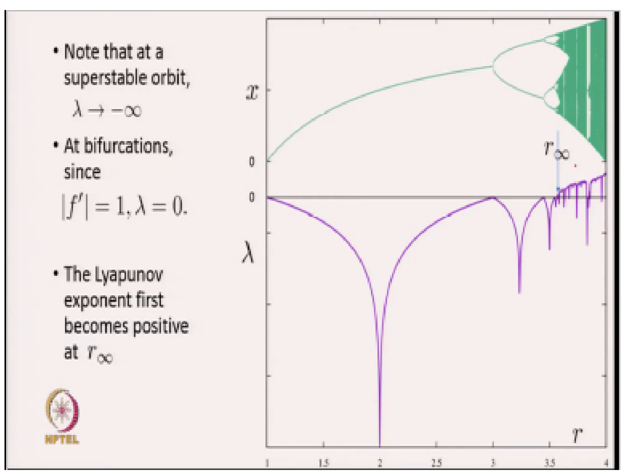
at x^* or just writing it in more compact notation as $f'(x^*).\delta_1$. Now, this fixed point is going to be stable if $f'(x^*) \leq 1$ in modulus and it is unstable otherwise. So, the initial separation which was delta 1 it has become delta 2, and the change factor or the growth factor it could be less it could be a shrinkage factor whatever, so the change factor is just delta 2 by delta 1 and this is simply the modulus of f prime of x star. If I were to write this as the exponential of a quantity lambda, where lambda is just the logarithm of the f of x star, then this factor delta 2 by delta 1 will be less than 1 if lambda is less than 0. And it will be greater than 1 then it will be less than 1, namely it will shrink if lambda is less than 0 it will grow if lambda is bigger than 0. And, so this quantity of lambda being bigger than 0 or less than 0 is more or less the same as f prime being less than 1 in modulus. But this can generalize. (Refer Slide Time: 06:45)



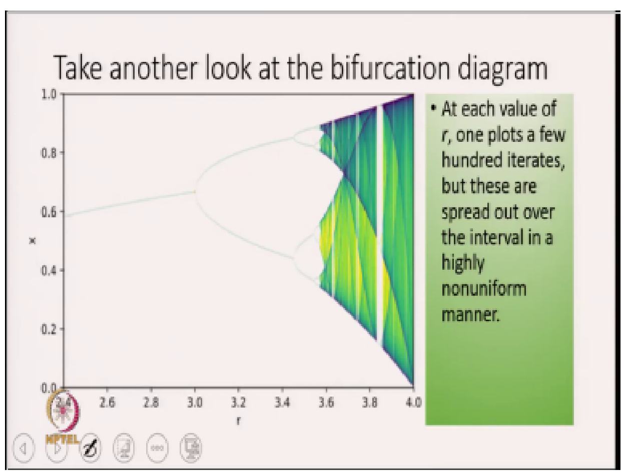
So, if we now look at a periodic orbit of period k, and if the elements of the orbit are x 1, x 2 etcetera all the way up till x of k. Now, f to the k of x 1 is equal to x 1 that is the statement of the periodic orbit. So, if we start with the point x 1 plus delta 1, after k steps one has delta of k plus 1 is delta 1 plus the derivative of f k with respect to x evaluated at this point which is just the same as f prime of k at delta 1, it times delta 1. f prime of k is f prime at x 1 evaluated f prime evaluated at x 2 all the way up to f prime evaluated at x of k. Namely, it is the product of the slope of the map at all the points of the orbit. (Refer Slide Time: 07:50)



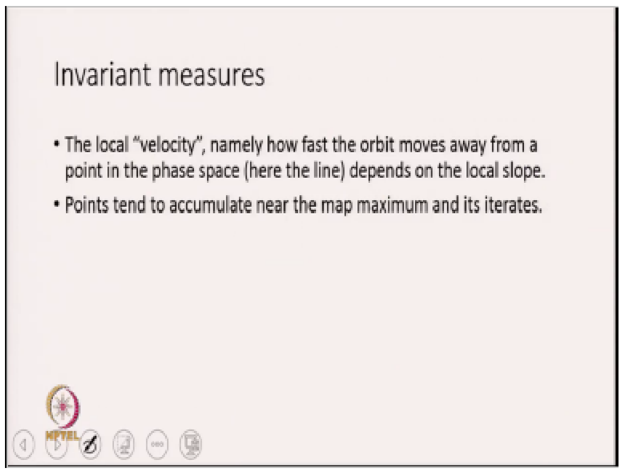
Now, the growth factor; after k steps is delta k plus 1 divided by delta k. And this is given by the modulus of f prime of k, and I write this as exponential of k times lambda, where lambda now is 1 by k times this quantity log of f prime k and I can notice that f prime k is product of f primes. So, the logarithm, sorry the logarithm of this product is the sum of the logarithms of the individual slopes, so my expression for lambda is 1 by k, sum over the various points of the orbit log of the modulus of this slope. Now, that I know how to characterize the stability of a fixed point or of a periodic orbit, I can extend this idea to any arbitrary orbit and define the average growth rate of perturbations in exactly the analogous way. Except now I write lambda as the limit of n going to infinity of 1 by n times the sum of the logarithms of the slope along this orbit, exactly like this particular statement, except that I have interchanged exchange k for n and I allow n to go to infinity. So, I have the average value of this particular growth factor. And after n steps delta n will just be delta 1 times the exponential of n times lambda. This quantity lambda is termed the Lyapunov exponent and it gives you the average rate of growth of small displacements from some reference or fiducial point. Clearly, if the this Lyapunov exponent is positive then the initial separation will grow without bound because it is just the exponential of n times lambda. This kind of unstable motion because these small separations growing uncontrollably this is what is called chaos; alternatively, if the Lyapunov exponent is positive then the motion is termed chaotic. (Refer Slide Time: 10:41)



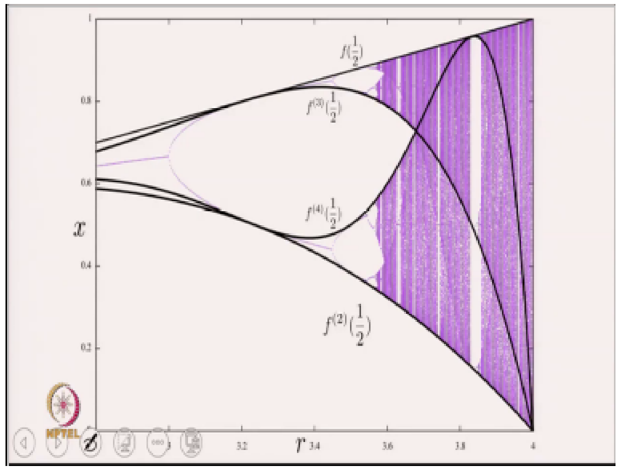
The Lyapunov exponent can be calculated for the logistic map using exactly this formula that we have over here. So, at any value of r, you iterate the map, you start with some arbitrary initial point, you keep iterating the map calculating the slope as you are going along take the logarithm, average it out over the entire map and we plot it as a function of r. Notice that if you have a super stable orbit the value of the slope becomes 1, the value of the logarithm of 0, sorry the value of the slope becomes 0, the value of the logarithm goes to minus infinity and that dominates the sum. So, at every super stable orbit you find that the Lyapunov exponent is actually negative infinity. Numerically, you do not always reach negative infinity, but you can see the indications of that at this super stable orbit, at this super stable orbit and many other super stable orbits all over the place. At a bifurcation on the other hand the slope of the map is equal to 1 and therefore, the Lyapunov exponent must be exactly 0. So, at every bifurcation you find that the Lyapunov exponent takes the value 0 that was at the trans-critical bifurcation. Here it is at the period doubling bifurcation and here tucked away inside that the tangent bifurcation, you find that the Lyapunov exponent is actually 0. The Lyapunov exponent actually first becomes positive only at r infinity, because before r infinity all the orbits are periodic. And for periodic orbits the Lyapunov exponents are negative because that is the way the whole thing is organized and this curve is entirely on then below the below the 0 line. So, you have chaotic motions so to speak only after the period doubling accumulation which is also one of the reasons why it is called the period doubling route to chaos. (Refer Slide Time: 13:07)



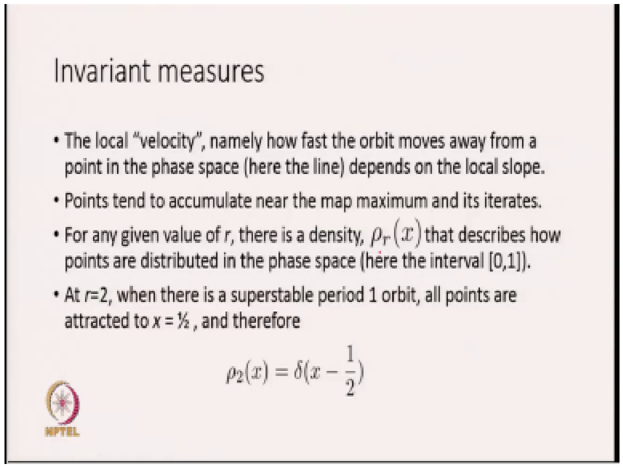
Let us take another look at the bifurcation diagram. As I mentioned right at the beginning at each value of r you discard a few hundred iterates and you plot a few hundred or a thousand iterates depending on your taste. But these are not you know even away from the periodic orbits. Over here you can see that these are not spread out uniformly in the region where they are present, but they seem to be concentrated along certain lines there are more of them at certain regions and fewer of them at certain other regions. This is these are spread out over the interval in a highly non uniform manner. (Refer Slide Time: 13:55)



What is the probability of finding a point at a finding a group of points or orbits near any point in the phase space? This depends on this something like a velocity because, you have got you know we have got points moving around in phase space. So, the local velocity at a point tells you how fast the orbit is moving at that point. The velocity over here is just of course, the derivative or the local slope. So, that tells you whether stuff is moving fast or slow. Now, points are going to accumulate where the velocity is slow and the lowest value that this velocity this slope can take is 0. So, points tend to accumulate near the maximum of the map, and not just the near the maximum of the map also the points to which the maximum will iterate and iterate consecutively. So, the points will move very fast where the slope is high, they move slowly where the slope is low or it has a slow precursor. (Refer Slide Time: 15:15)



So, coming back to the bifurcation diagram this is at much lower resolution than the earlier one that I have shown. But you can still see that there are these very strong and prominent curves, and I just like to demonstrate that these really are the images of the map maximum and its iterates. So, let us look at that. There you can see very nicely that this, ok. So, the upper envelope is just where the map maximum iterates. The second curve over here this dark one that is moving around like so, this is where the map will iterate the maximum will iterate on the second term, then the third and the fourth and you can see over here are the particular curves. So, there on top is the maximum, here is the iterate of the maximum again, here is the third iterate, here is the fourth iterate and all the and there are even higher order iterates that are sort of visible, but all these different points tell you that along the line or in the phase space points are going to be moving around slower and faster they are going to tend to accumulate in certain regions and not in certain others. This gives us an idea that there could be along the line in the interval there is a density, ok; this density which we denote by the letter rho. For any given value of r there is this density rho r of x that describes how points are distributed in the phase space which is of course, here the interval 0 to 1. (Refer Slide Time: 16:56)



A simple example is useful to keep this concept in mind. So, when r is equal to 2, we know that there is a super stable period 1 orbit we have just gone through you know discussing this several times around. Now, if you have got a super stable period 1 orbit all points or almost all points let us say are attracted to this point x is equal to a half. What is the point that is not attracted to x is equal to a half? It is the point 0, because 0 is a fixed point an unstable one, but it is fixed. The point 1 goes to 0, so that is also not attracted to a half, but barring these two points everything else in the interval goes to the point half. Therefore, the density of points is actually just a delta function and this density of points is just delta of x minus half. I put the subscript two to indicate the value of r over here. So,

$$\rho_2(x) = \delta(x - \frac{1}{2})$$

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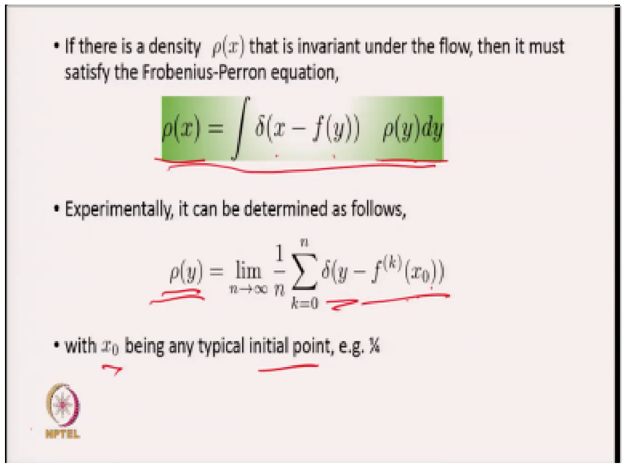
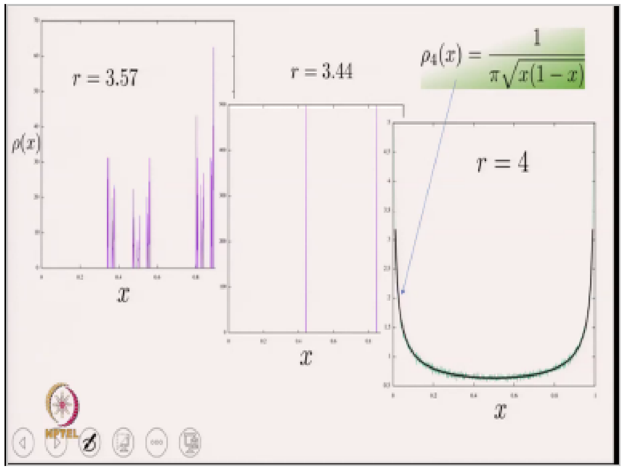


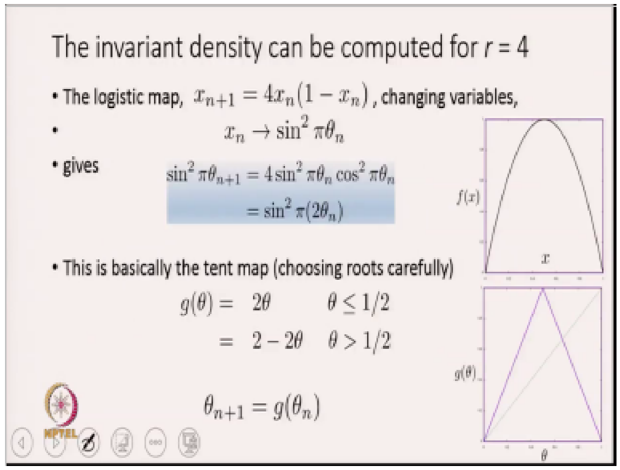
At different points a different values of r you can find different this density keeps changing, but if there is an invariant density then it must satisfy an equation that was first described by Frobenius and Perron, and it is just a statement of stationarity. Namely, if you have a density which is invariant ρ, then if you have got a function f that is the mapping then

$$\rho(x) = \int \delta(x - f(y))\rho(y)dy$$

all the points y when they map to the point x that must give you back $\rho(x)$ which is the same. So, an invariant density has to satisfy this equation of stationarity known as the Frobenius-Perron equation. To determine this density experimentally is not very difficult. Again, starting with some typical initial point x_0, if you just keep iterating this initial x initial point discarding transients and then looking at where they land up on the interval that gives you rho of y. One can anticipate a few things; for example, sorry. Over here, if I were to try to iterate at a point over here everything will go to these two fixed points, the period two orbit, so eventually everything must land up on two delta functions. If I iterate at r infinity there must be two to the infinity so to speak points. So, it must be so many delta functions this density must start looking somewhat complicated. At r is equal to 4 you see that the entire interval is covered and it will be interesting to see how this interval is covered by the various points. So, let us do this experimentally by starting with some initial condition, iterating this map and collecting all these values into this density. (Refer Slide Time: 20:54)



So, it r is equal to 3.4, as we had discussed we have these two delta functions. At r is equal to 3.57, you can see that this invariant density is quite a bit more complicated, although there is an interesting structure over here which we may come back to in a subsequent lecture. And it r is equal to 4 which is the most interesting point over here, you find that the density has a very nice shape which is given by an analytical formula 1 over the square root of x into 1 minus x, times the normalization factor. So, rho 4 has a very nice analytic form and what you see superimposed over there are the you know just a numerical experiment where we took a few maybe a million points or so, and calculated this particular density. Given the fact that you have a nice formula describing it is worthwhile trying to see how you can derive it. And this invariant density can actually be computed for the value of r equals 4. (Refer Slide Time: 22:00)



To do that it is useful to change variables. From the logistic map you note that x is a positive number that lies between [0,1], so the transformation over here is to a variable $\theta(x)$ goes to $sin^2(\pi\theta_n)$. And you can see that θ now lies between [0,1], and that will give you a very nice mapping between x and θ. In this equation just becomes $sin^2(\pi\theta_{n_1})$ and on this side it becomes $sin^2(\pi 2\theta_n)$ So, just analyzing this particular equation, looking at the roots of these various sins and so on you note that this is basically the tent map. Namely,

$$g(\theta) = 2\theta, \theta \leq 2$$

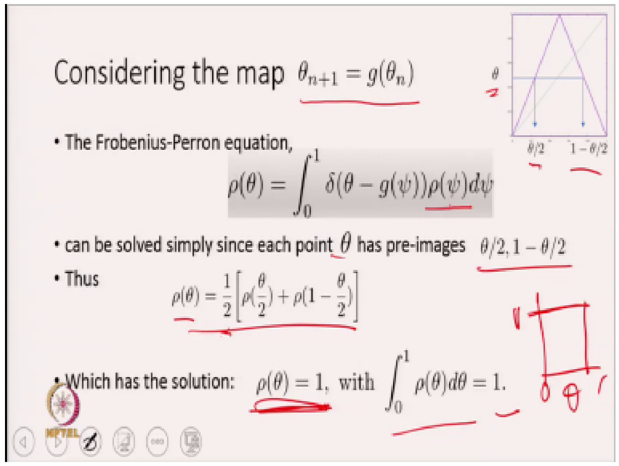
and

$$g(\theta) = 2 - 2\theta, \theta > 2$$

. So, you see this is the correspondence between these two maps, the logistic on the one hand, and the tent on the other hand. Through this transformation x goes to sin squared by theta. So, the tent map is

$$\theta_{n+1} = g(\theta_n)$$

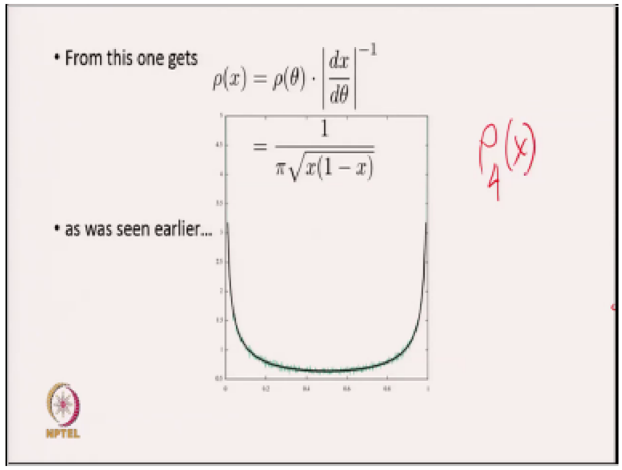
, where $g(\theta)$ is that function over there. (Refer Slide Time: 23:40)



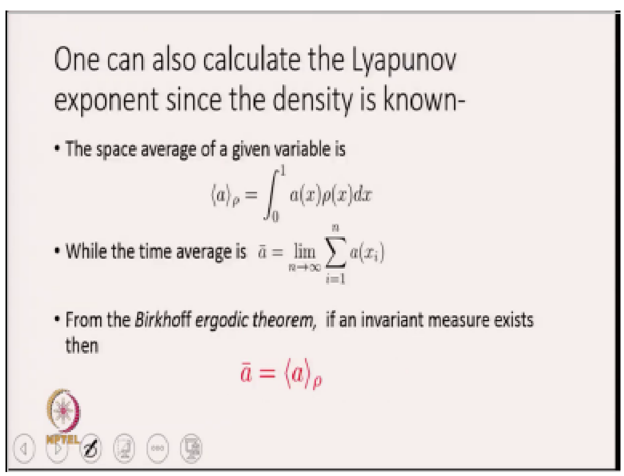
And when you look at g at this map $\Theta_{n+1} = g(\Theta_n)$, the corresponding Frobenius-Perron equation is actually quite simple to solve. And we note that this is the form that the Frobenius-Perron equation takes. You start with the density rho of psi of, rho of psi deep psi and then you map the point psi to theta by a you know to g psi which is theta and that gives you this function. And you notice that any point theta has contributions from only two points in the pre image. So, only two points from this equation mapped to a given Θ, $\Theta/2$ and $1 - \Theta/2$. So, the point theta has precisely two pre images, $\Theta/2$ and $1-\Theta/2$ and therefore, you can rewrite this particular equation as

$$\rho(\theta) = \frac{1}{2}[\rho(\theta/2) + \rho(1 - \theta/2)]$$

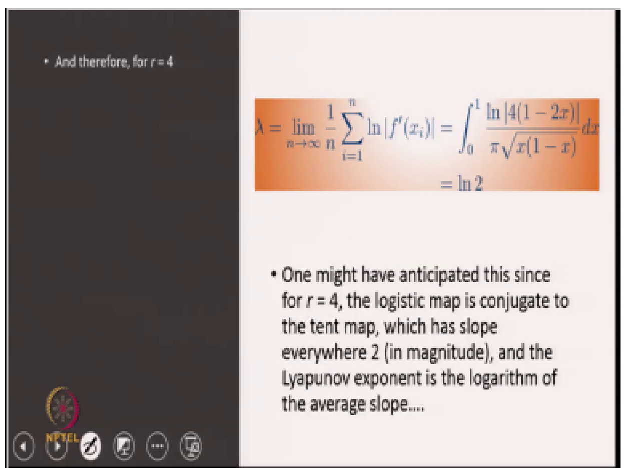
. And this half is not the average half, but it is the slope of two because of the delta function. And solving this particular equation says that rho of theta is equal to 1, because this is true for all theta the only solution is that rho of theta is 1, and this is a good invariant density because the integral over the interval is equal to 1. And this basically says that if you take the tent map and iterate an arbitrary point then points are uniformly distributed over the entire interval [0,1]. Uniform because $\rho(\theta) = 1$,with $\int_0^1 \rho(\theta)d\theta = 1$. So, it is just a uniform distribution. (Refer Slide Time: 25:56)



Now, these two maps the tent map and the logistic map are conjugate to one another. And therefore, you get that rho of x because we know what the transformation variable is x is just sin square theta. So, rho of x is given by this particular formula, it is rho of theta divided by dx by d theta and doing the little algebra you get that this is just equal to pi of under root x into 1 minus x. And we have already seen that in the in you know in the in the map earlier we saw that the numerical exploration also gave you this very nice verification of the same formula. (Refer Slide Time: 26:35)



You can do a lot once you have this invariant density. If you know the invariant density for a map you can actually do a lot. And we will use this invariant density and the analytical calculation of rho sub 4, namely the value of the invariant density for the value of the parameter r is equal to 4, we will use this to compute the Lyapunov exponent. Now, from the Birkhoff ergodic theorem, basic statement of the ergodic theorem is that if there is an invariant measure, then time averages and space averages are the same. So, given a particular variable a, its space average is the average over the entire face space given this invariant measure rho is just the integral from over the space in this particular case is just 0 to 1. It is a of x times rho of x dx. The time average on the other hand is indicated with the over bar over here, is the limit of n going to infinity, maybe there should be a factor of 1 over n over here, but it is the limit of this is this particular average a evaluated at the various points x sub i, all the way from 1 to n and I will add over here 1 by n over there. So, let n go to infinity you get the time average. (Refer Slide Time: 28:28)



The Birkhoff ergodic theorem says that if the invariant measure exists, then this time average and the space average are equal. And we notice that the Lyapunov exponent was defined as a time average. Namely, lambda was given as the limit of n going to infinity, 1 over n sum of the logarithm of f prime of x i, this particular average over here. Now, this has to be exactly equal to the space average. So, the same quantity f' which is the $ln(4(1-2x)$ times the invariant measure which is 1 upon $\pi\sqrt{x(1-x}$, dx. The integral is fairly simple, can solve that quite nicely to find the value of log 2. So, the Lyapunov exponent for the logistic map at the value $r = 4$ is known analytically to be equal to the $ln2$. We might have anticipated this because at r=4 the logistic map is conjugate to the tent map and the tent map has slope 2 in magnitude everywhere. And the Lyapunov exponent is just the logarithm of the average slope and over here the slope everywhere is true then the logarithm the average is also 2, in the logarithm would be log 2 and so log 2 is the numerical value of the Lyapunov exponent for the logistic map for the value of the parameter r is equal to 4. Now, this brings us to the end of this lecture, but there is lots of lots of other details that we would like to fill in. What more can one do with invariant densities, would be something which is interesting to know. But I would like to just leave with the following kinds of issues that in hand. The invariant density we can see at 4 was simple. For all the periodic orbits it was quite simple. But at the point of accumulation of all the period doubling bifurcation this is just a little above r infinity, it has a very interesting and nice structure. So, in the subsequent lectures we will turn a little to trying to describe, this kind of geometry which is known by the term of fractal geometry. So, we will come back to this issue in the next lecture of the series.