Introductory Nonlinear Dynamics
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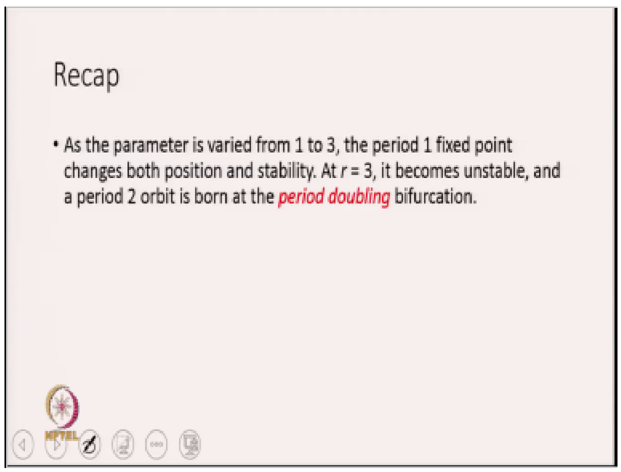
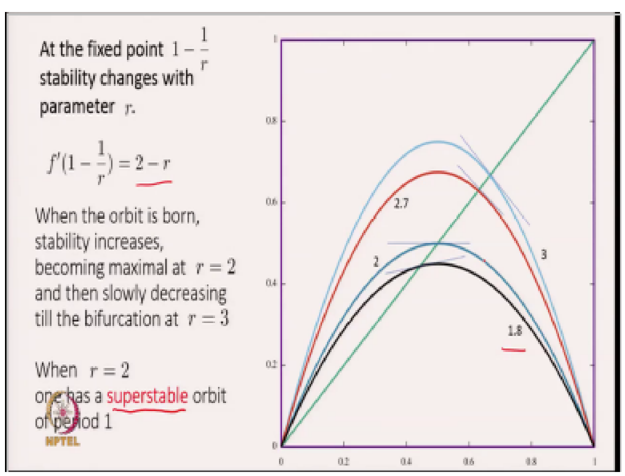
Lecture 08
Characterizing the period-doubling route to chaos.

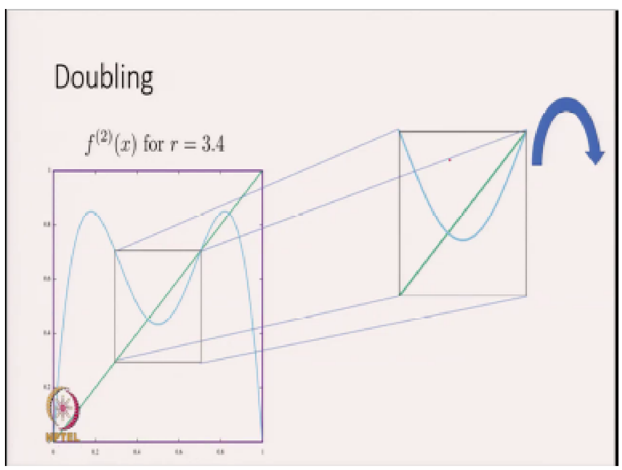
In the last lecture, we saw that the a map such as the logistic map when you vary the parameter, the period one fixed point it changes its location as well as its stability. (Refer Slide Time: 00:39)



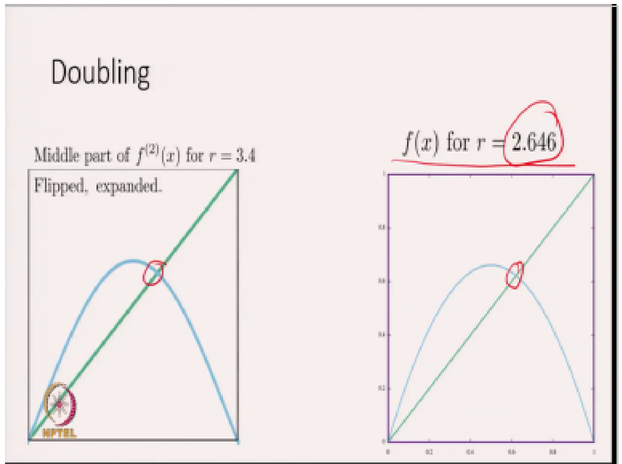
And, at the value $r = 3$, the period 1 point becomes unstable a period 2 orbit is born at the so called period doubling bifurcation. (Refer Slide Time: 00:45)



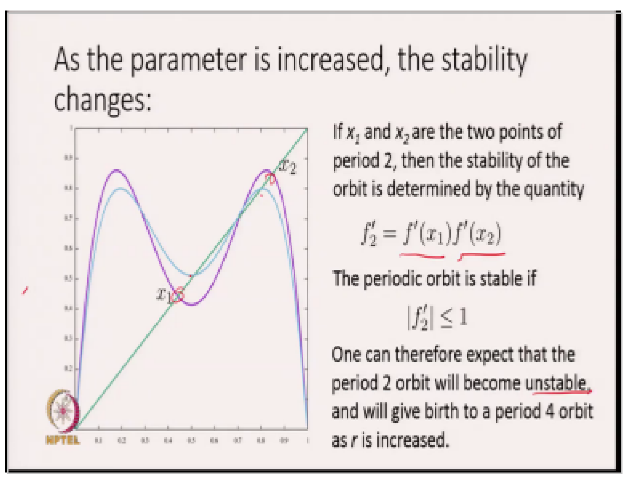
Now, how does this map actually change? As you increase r from let us say 1.8 to 3, notice that the point at which it crosses the diagonal namely the fixed point that keeps changing in location along the map. The local stability which is given by this slope of the map the derivative; the derivative of the map at this fixed point is 2 minus r and so, when r is 1.8 it is actually small and positive. When r is equal to 2 is equal to 0, when r is 2.7 it is 0.7 negative and when r is equal to 3 it is negative 1. So, the orbit itself is born at r is equal to 1 and the stability that is the value of f prime or this derivative over here that keeps decreasing so, the orbit in the sense the fixed point is becoming more and more stable. When r is equal to 2 the slope is equal to 0 and this is the most stable that it could possibly be. So, such behavior is called super stability namely to have a slope 0 because this means that that fixed point attracts everything maximally. When you take the sorry. So, do you just come back over here? At this point you have the period doubling bifurcation the period 1 orbit is no longer stable and instead a period 2 orbit is born. (Refer Slide Time: 02:45)



Now, if you look at the doubled map, the doubled map $f^2(x)$ and as we have discussed in earlier lectures looking at the intersections of the map with the diagonal tells you the fixed points. So, here are the two fixed points of period 2, but if you look at the double map let us say at a value r is equal to 3.4 notice that the inner part of the map if you were to just concentrate on that particular part of the map expand it and flip it around. (Refer Slide Time: 03:21)



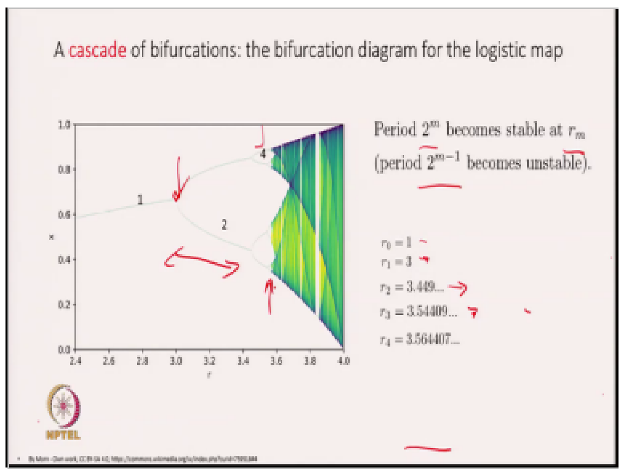
Then, when you flip it around and expand it to assert to the required scale you notice that it looks a lot like the map f(x) at another value of r. So, namely that we try to look for some point in r, where the intersection is on this side so, this is where that intersection looks to be located, notice that it is over here. So, when you look at the map, turn it around expand it, it looks like the same map at another value of r. (Refer Slide Time: 03:59)



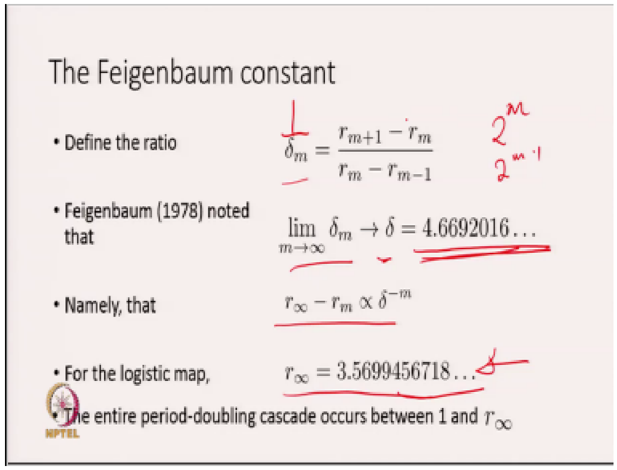
Now, as you keep changing the parameter the stability of the fixed points change. Recall that if x 1 and x 2 are the two points of the period 2 orbit, then the stability of the orbit is determined by this quantity

$$f_2' = f'(x_1)f'(x_2)$$

and the periodic orbit is of course, it is stable if the | $f_2' \leq 1$. Now, these two points of intersection of the period 2 of the period 2 orbit here and here at one value of r. As you change keep changing r, those points migrate and now these two points are over here and over here. And, as you can see the slopes keep changing they are born both slopes are 1 in modulus and then they slowly keep changing and the product over here can first decreases to 0 and then it increases and therefore, one can expect that in the same way as period 1 bifurcated to period 2 orbit, period 2 itself at some point will become unstable and it will give birth to a period 4 orbit as r is increased. (Refer Slide Time: 05:37)



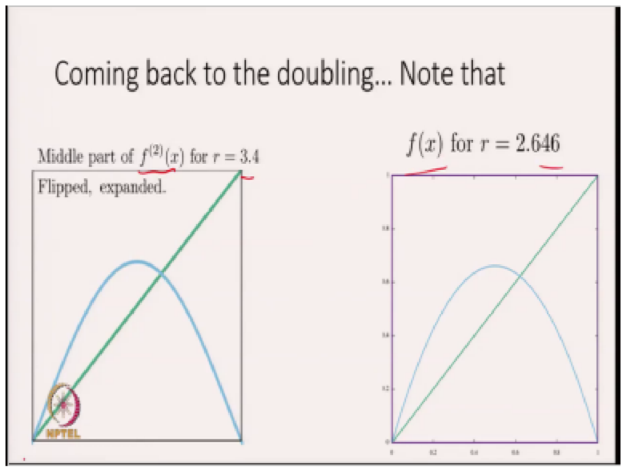
This is really the heart of the period doubling root 2 infinitely long periods period m 2 to the m orbit becomes stable at r sub m. When period 2 to the m becomes stable the period 2 to the m minus 1 necessarily becomes unstable. And, so, period 1 became unstable 2 was stable, 2 becomes unstable 4 is stable, 4 becomes unstable 8 is stable and so on and so forth. The values of these bifurcation points of the parameter at these bifurcation points is as follows: r 0 namely where period 1 becomes stable is 1, period 2 became stable at 3, period 4 becomes stable at 3.449, period 4 sorry, period 8 becomes stable at 3.544 etcetera, period 16 at 3.56 and so on. These are known to much much higher precision, some of them are even known analytically, but as you can see the difference between these between successive members of this sequence namely the width of the windows of stability so to speak those are decreasing. (Refer Slide Time: 07:13)



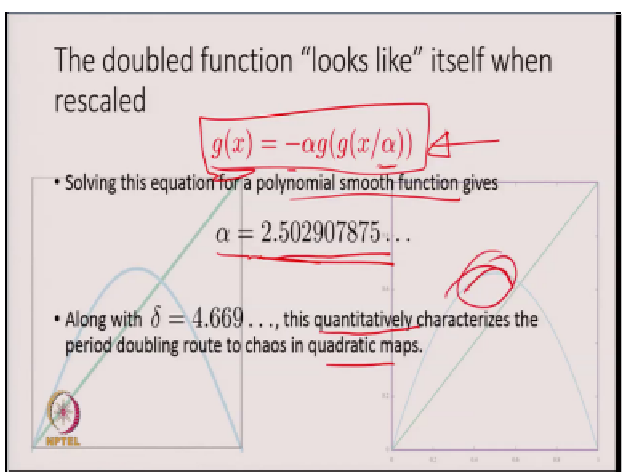
This was investigated extensively by Feigenbaum in 1978 and earlier and he defined this particular ratio, the ratio of the widths of these windows.

$$\frac{1}{\delta_m} = \frac{r_{m+1} - r_m}{r_m - r_{m-1}}$$

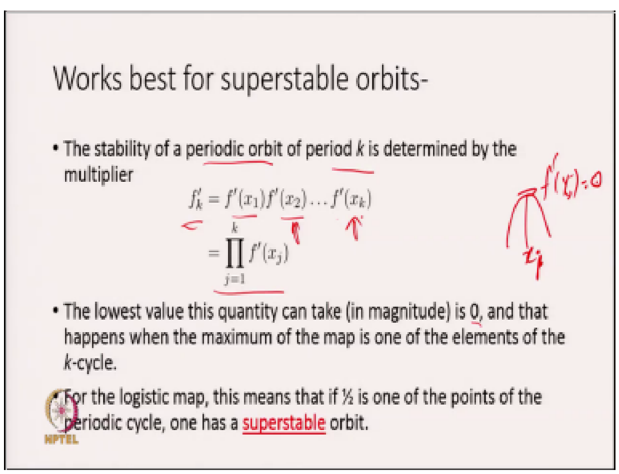
So, what is the width of this window divided by no. So, here is the window where period 2 to the m is stable and this is the window where 2 to the m minus 1 is stable. So, these windows keep shrinking I think there should be this is a sorry this is 1 over delta m over here is defined ok. So, Feigenbaum noticed that these windows not only do they keep shrinking, but they do so at a geometric rate and they and they do so with a very interesting set of properties. The first property is that this ratio decreases approximately by a factor of 5 at each step and as m goes to infinity this reaches the value delta which is this number 4.669. This is an irrational number and it is known to many many more places of decimal. All these period doubling bifurcations they accumulate at a value of r infinity which for the logistic map is 3.5699 etcetera etcetera. So, at between 3 point between 3 and sorry, so, between 3 where the first period doubling bifurcation takes place and 3.5699 which is somewhere over there. All period 2 and all powers of 2 have occurred at that point because each of these windows is shrinking by a factor of about 5, the rate of convergence of this quantity is geometric. This number is very particular to the logistic map, but it turns out that this number delta over here is not and this is a universal number to the first of the universal numbers that was discovered by Feigenbaum and it is an extremely important number because the way in which this period doubling by this period doubling cascade, the widths of the intervals in which these are stable each particular periodic orbit that shrinks at this particular rate 4.669. (Refer Slide Time: 10:03)



Now, coming back to the doubling you note that, we observed that the middle part of $f^2(x)$ at some value of the parameter when you flipped it and you expanded it look like the map itself for another value of the parameter. (Refer Slide Time: 10:25)



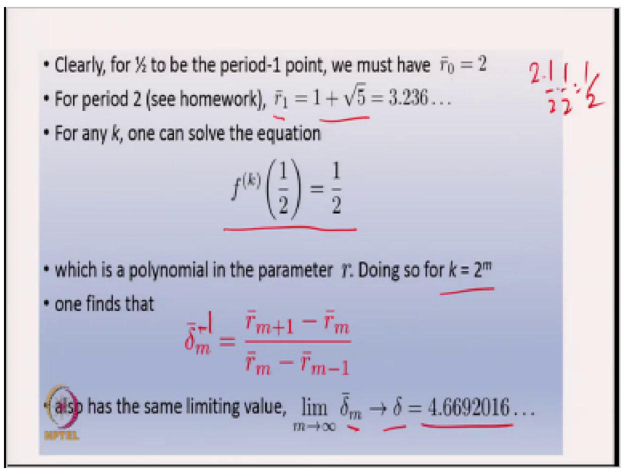
This led Feigenbaum and collaborators to look at the following composition this doubled function looks like itself when it is rescaled. So, Feigenbaum and Cvitanovi wrote down this particular equation which describes this doubling operation. So, if you have a function which when rescaled by a factor of alpha doubled, flipped around then it looks like itself. So, if you can find a fixed point of this equation or namely if you find a function which satisfies this property then you will begin to understand what happens in this particular map. And, you solve this equation for some polynomial smooth function of g and there is a sort of an involved way of doing this which is described in series of very nice papers by Feigenbaum. You find that alpha is a number 2.503 or thereabouts and this is again a universal number which is true for all quadratic maps. So, requiring the doubled function to look like itself and asking for this function g of x to be of a polynomial type that gives you this universal number 2.503 and along with delta which is 4.669 these two numbers quantitatively characterize the qualitative statement of a period doubling root to chaos for quadratic maps ok. Let me just state this again because this is sufficiently important. To say that you reach an infinitely long period by doubling at discrete values of the parameter say period 1 gives rise to period 2, period 2 becomes unstable gives rise to period 4 and so on and so on and so on that is a purely qualitative statement. What Feigenbaum did was to say that not only does this happen as the phenomenon, but the value of the parameter at which this happens these values keeps shrinking at a precise rate which is 4.669 and there is another geometric characterizer which tells you how you must rescale the function at what by what factor and that factor is 2.503 and this is true for all quadratic maps. By quadratic map I mean a map which where the maximum of this map; this maximum is the leading order is quadratic. (Refer Slide Time: 13:31)



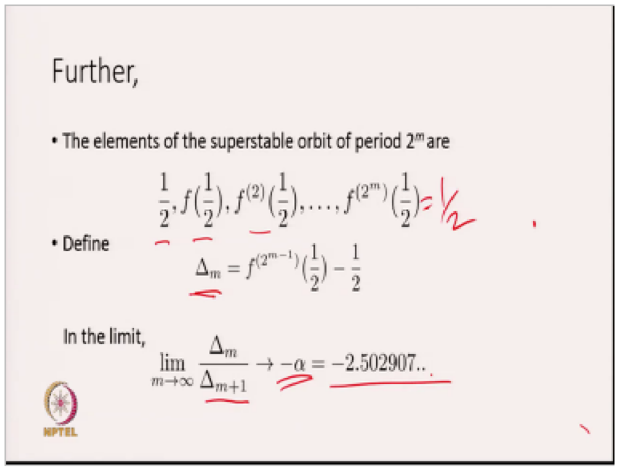
Now, you can do this very accurately for super stable orbits. Remember that a super stable orbit is one where the derivative this f'= 0. If you have a periodic orbit of period k, then it is stability is determined by the following multiplier

$$f_k' = f'(x_1)f'(x_2)...f'(x_k)$$

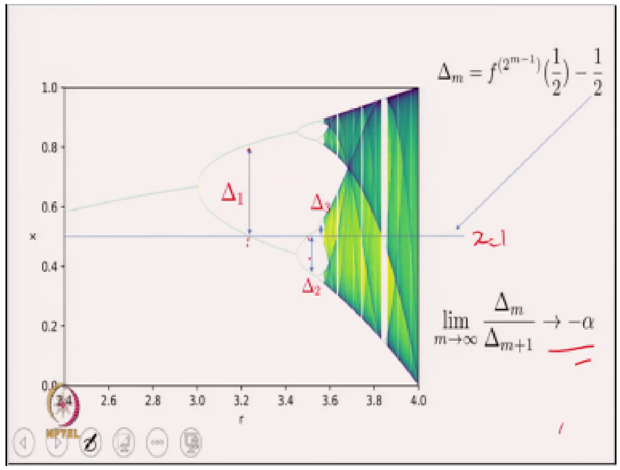
. So, this multiplier over here is just the product of the slope of the function at all the points on the period. The lowest value that this quantity can take in magnitude is 0 and this can happen when one of these one of these points x 1 or x 2 or whatever when one of them is the map maximum if this is some x sub j then f prime of x sub j is equal to 0 and this entire product would vanish. This defines a super stable orbit to the logistic map this means that if the point half is one of the points of the cycle, then one has a super stable orbit. (Refer Slide Time: 15:05)



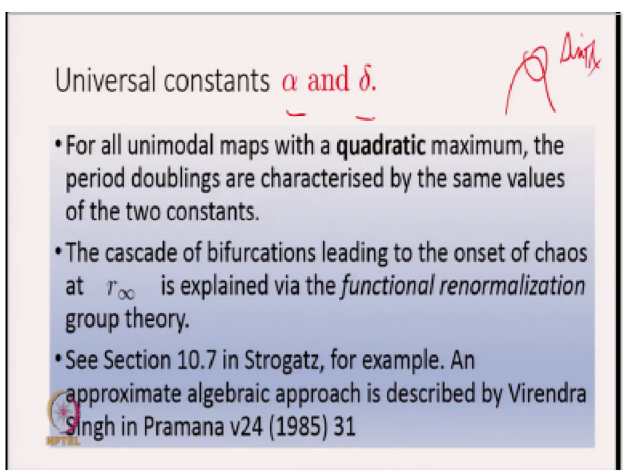
Clearly, for half to be the period-1 point, we must have r is equal to 2 because r x into 1 minus x. So, 2 times a half into a half gives you a half. So, half becomes a period-1 point only when r takes the value 2 and we call this r bar 0 0 to tell you that it was period 1 to the power 0, r bar to say that this is the value where you have super stability. One of the tutorials you have to work out that period 2 is super stable for the value of r 1 which is 1 plus the square root of 5. And, for any k, you can solve this equation f to the k of a half is equal to half; this determines the super stability condition, but this is now a polynomial in r. It is the polynomial in the parameter and this can be solved by Newtons method and this is in fact, what Feigenbaum did in his early calculations. And when you do this for all the powers of 2, you determine that these windows r m plus 1 minus r m divided by r m minus r m minus 1, again this is a inverse over here. This ratio delta also has the same limiting value the delta m goes to delta which is 4.669. So, whether you take the points in the bifurcation or you take the points where the super stable orbits are created the same kind of scaling operates and you find that the ratio is 4.669. (Refer Slide Time: 17:15)



Furthermore, the elements of the super stable orbit of period 2 to the m, what are these? You start at a half; you map f to the half f 2 to the half and so on all the way up till 2 to the m to the half which is equal to a half. Now, if you define delta m to be the distance between the point half and exactly midway along this cycle. So, f to the 2 m minus 1 of a half so that is the distance between the point half and one half the number of iterations on this particular cycle this defines a quantity delta m. And, in the limit delta m divided by delta m plus 1 these points also get closer and closer together they flip from side to side and this as m goes to infinity reaches the ratio of minus alpha which is minus 2.5029 etcetera which is the same quantity that we discovered in the doubling transformation. (Refer Slide Time: 18:31)



Visually, see here is the line x is equal to a half and when you superimpose it on the bifurcation diagram you notice that this is the point where period 2 is super stable, that is the point where period 4 is super stable, then there is a point where period 8 is super stable and so on. The quantity delta 1 is the difference or the distance between the super stable at this point half and it is first iterate. Delta 2 is the difference between the point half and the second iterate because this is a period 4 super stable orbit. Delta 3 is this quantity over here and that is the difference between the point half and it is fourth iterate because it is a period 8 super stable. So, as these values are delta 1, delta 2, delta 3 and so on as you can see that they are shrinking and in the limit they go to the value minus alpha. (Refer Slide Time: 19:41)



These are universal constants and their discovery is been a very important step in this entire study of non-linear dynamics. All unimodal maps that is all maps with a single maximum and a quadratic maximum at the top, the period doublings are character by the same values of the two constants. These constants appear in other points also, but the importance over here is that regardless of what kind of map you have the details do not matter. So, this will apply to the quadratic map. If instead of the quadratic map you will study sine, sine pi x or some map like that that will also have the same numbers and so on. Now, the cascade of bifurcations that you lead that lead to the onset of chaos at r infinity, this requires explanation by the so called functional renormalization group developed again extensively in the 70s and 80s and this helps you to understand why the function when doubled, rescaled, flipped around etcetera looks like itself when it is formalized then from this you can calculate these values of both alpha and with a little more difficulty also delta. Those of you who have a copy of the Strogatz book you would like to see section 10.7 in that text where this is discussed in detail and there is a very nice and sort of very perturbative argument and our algebraic approach without renormalization and so on described by Virendra Singh, there is an article in Pramana in 1985 and I would recommend that strongly for anybody who would like to take it and perhaps even apply it beyond just quadratic maps.